Springer Optimization and Its Applications 201

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Turnpike Phenomenon in Metric Spaces

Springer

Springer Optimization and Its Applications

Volume 201

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ISSN 1931-6828 ISSN 1931-6836 (electronic)
Springer Optimization and Its Applications
ISBN 978-3-031-27210-3 ISBN 978-3-031-27208-0 (eBook)
https://doi.org/10.1007/978-3-031-27208-0

Mathematics Subject Classification: 49J20, 49J99, 49K20, 49K27, 49K40

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

In this book we study the turnpike phenomenon arising in the optimal control theory. The term was first coined by P. Samuelson in 1948 when he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). To have the turnpike property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the objective function and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. The turnpike property discovered by P. Samuelson is well known in the economic literature, where it was studied for various models of economic growth. Usually for these models, a turnpike is a singleton.

In our previous research, it was shown that the turnpike property is a general phenomenon which holds for large classes of finite-dimensional variational and optimal control problems. In that research, using the Baire category (generic) approach, we showed that the turnpike property holds for a generic (typical) variational problem [117] and for a generic optimal control problem [130]. According to the generic approach, we say that a property holds for a generic (typical) element of a complete metric space (or the property holds generically) if the set of all elements of the metric space possessing this property contains a G_δ everywhere dense subset of the metric space which is a countable intersection of open everywhere dense sets. This means that the property holds for most elements of the metric space.

In this book, we are interested in turnpike results, in sufficient and necessary conditions for the turnpike phenomenon and in its stability under small perturbations of objective functions. The most important feature of this book is that we study a large, general class of optimal control problems in metric space. Probably, at the first time in the literature to our knowledge. All the main results obtained in the book are new.

The monograph contains nine chapters. Chapter 1 is an introduction. In Chap. 2 we discuss Banach space valued functions, set-valued mappings in infinite dimensional spaces, and related continuous-time dynamical systems. Some convergence results are obtained. In Chap. 3, we study a discrete-time dynamical system with a Lyapunov function in a metric space induced by a set-valued mapping. For this discrete-time dynamical system, we establish a turnpike result. Chapter 4 is devoted to the study of a class of continuous-time dynamical systems which is an analog of the class of discrete-time dynamical systems considered in Chap. 3. This class also contains the class of dynamical systems studied in Chap. 2. In Chap. 5, we develop a turnpike theory for a class of general dynamical systems in a metric space with a Lyapunov function. This class of dynamical systems contains as particular cases the classes of dynamical systems studied in Chaps. 3 and 4. We obtain several turnpike results and show that the turnpike phenomenon is stable under small perturbations of objective functions. Chapter 6 contains our study of the turnpike phenomenon for discrete-time nonautonomous problems on subintervals of half-axis in metric spaces, which are not necessarily compact. For these optimal control problems, the turnpike is not a singleton. We show that the turnpike phenomenon is stable under small perturbations of an objective function. Using the Baire category approach, we show that for some classes of problems a typical (generic) problem has a turnpike property. Chapter 7 contains preliminaries which we need in order to study turnpike properties of infinite dimensional optimal control problems. We discuss unbounded operators, semigroups of linear operators, evolution equations, and admissible control operators. In Chap. 8, we establish sufficient and necessary conditions for the turnpike phenomenon for continuous-time optimal control problems on subintervals of half-axis in metric spaces. For these optimal control problems, the turnpike is not a singleton. The results of this chapter will be obtained for three large classes of problems which will be treated simultaneously. In Chap. 9, we continue to study the turnpike phenomenon for the continuous-time optimal control problems on subintervals of half-axis in metric spaces which are discussed in Chap. 8. We show that the turnpike phenomenon is stable under small perturbations of an objective function. Using the Baire category approach, we show that for some classes of problems a typical (generic) problem has a turnpike property.

Rishon LeZion, Israel Alexander J. Zaslavski
October 19, 2022

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Chapter 1
Introduction

In this chapter, we discuss turnpike properties in the calculus of variations. To have the turnpike property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the integrand and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. We also consider some classes of optimal control problems studied in the book.

1.1 Turnpike Property for Variational Problems

The study of the existence and structure of solutions of optimal control problems and dynamic games defined on infinite intervals and sufficiently large intervals has been a rapidly growing area of research [9, 10, 24, 31, 39, 43, 44, 46, 60, 63, 64, 76, 88, 117, 119, 121, 124, 129, 133, 138, 139, 142, 144, 147] that has various applications in engineering [6, 31, 70, 117], models of economic growth [7, 11, 29–32, 42, 47, 58, 59, 62, 69, 75, 81, 90, 97, 99, 105, 117, 127, 134, 143, 144], infinite discrete models of solid-state physics related to dislocations in one dimensional crystals [14, 94, 107], model predictive control [38, 50], and the theory of thermodynamical equilibrium for materials [36, 71, 78–80]. Discrete-time optimal control problems were considered in [8, 15, 16, 27, 45, 55, 108, 109, 114, 116, 120, 123, 125, 127, 131, 132, 135, 140, 141, 143], finite dimensional continuous-time problems were analyzed in [23, 25, 26, 28, 33, 69, 72, 74, 77, 91, 112, 115, 118, 130, 145, 146], infinite dimensional optimal control was studied in [31, 32, 51–53, 84, 86, 87, 92, 101, 102, 111, 113, 122], while solutions of dynamic games were discussed in [22, 48, 49, 54, 56, 66, 93, 126, 128, 136, 137]. Sufficient and necessary conditions for the turnpike phenomenon were obtained in our previous research [115, 116, 118, 131].

A. J. Zaslavski, *Turnpike Phenomenon in Metric Spaces*, Springer Optimization and Its Applications 201, https://doi.org/10.1007/978-3-031-27208-0_1

In this section, which is based on [115], we discuss the structure of approximate solutions of variational problems with continuous integrands $f : [0, \infty) \times R^n \times R^n \to R^1$ that belong to a complete metric space of functions. We do not impose any convexity assumption. The main result of this section, obtained in [115], deals with the turnpike property of variational problems.

We consider the variational problems

$$\int_{T_1}^{T_2} f(t, z(t), z'(t))dt \to \min, \ z(T_1) = x, \ z(T_2) = y, \tag{P}$$

$$z : \ [T_1, T_2] \to R^n \text{ is an absolutely continuous function,}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, and $f : [0, \infty) \times R^n \times R^n \to R^1$ belongs to a space of integrands described below.

It is well known that the solutions of the problems (P) exist for integrands f that satisfy two fundamental hypotheses concerning the behavior of the integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex [100]. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals that are crucial in most of the existence proofs, although there are some interesting theorems without convexity [34, 35, 83, 85]. For integrands f that do not satisfy the convexity assumption, the existence of solutions of the problems (P) is not guaranteed, and in this situation, we consider *δ-approximate solutions*.

Let $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, $f : [0, \infty) \times R^n \times R^n \to R^1$ be an integrand, and let δ be a positive number. We say that an absolutely continuous (a.c.) function $u : [T_1, T_2] \to R^n$ satisfying $u(T_1) = x$, $u(T_2) = y$ is a δ-approximate solution of the problem (P) if

$$\int_{T_1}^{T_2} f(t, u(t), u'(t))dt \leq \int_{T_1}^{T_2} f(t, z(t), z'(t))dt + \delta$$

for each a.c. function $z : \ [T_1, T_2] \to R^n$ satisfying $z(T_1) = x, \ z(T_2) = y$.

The main result of [115] deals with the turnpike property of the variational problems (P). As usual, to have this property means, roughly speaking, that the approximate solutions of the problems (P) are determined mainly by the integrand and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

In the classical turnpike theory, it was assumed that a cost function (integrand) is convex. The convexity of the cost function played a crucial role there. In [115], we get rid of convexity of integrands and establish necessary and sufficient conditions for the turnpike property for a space of nonconvex integrands $\mathcal{M}$ described below.

Let us now define the space of integrands. Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant, and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$. Denote by $\mathcal{M}$ the set of all continuous functions $f : [0, \infty) \times R^n \times R^n \to R^1$ that satisfy the following assumptions:

A(i) The function f is bounded on $[0, \infty) \times E$ for any bounded set $E \subset R^n \times R^n$.

A(ii) $f(t, x, u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(t, x, u) \in [0, \infty) \times R^n \times R^n$.

A(iii) For each $M, \epsilon > 0$, there exist $\Gamma, \delta > 0$ such that

$$|f(t, x_1, u) - f(t, x_2, u)| \leq \epsilon \max\{f(t, x_1, u), f(t, x_2, u)\}$$

for each $t \in [0, \infty)$ and each $u, x_1, x_2 \in R^n$ that satisfy

$$|x_i| \leq M, \ i = 1, 2, \ |u| \geq \Gamma, \quad |x_1 - x_2| \leq \delta.$$

A (iv) For each $M, \epsilon > 0$, there exists $\delta > 0$ such that $|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon$ for each $t \in [0, \infty)$ and each $u_1, u_2, x_1, x_2 \in R^n$ that satisfy

$$|x_i|, |u_i| \leq M, \ i = 1, 2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta.$$

It is easy to show that an integrand $f = f(t, x, u) \in C^1([0, \infty) \times R^n \times R^n)$ belongs to $\mathcal{M}$ if f satisfies assumption A(ii), and if $\sup\{|f(t, 0, 0)| : t \in [0, \infty)\} < \infty$ and also there exists an increasing function $\psi_0 : [0, \infty) \to [0, \infty)$ such that

$$\sup\{|\partial f/\partial x(t, x, u)|, \ |\partial f/\partial u(t, x, u)|\} \leq \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $t \in [0, \infty)$ and each $x, u \in R^n$.

For the set $\mathcal{M}$, we consider the uniformity that is determined by the following base:

$$E(N, \epsilon, \lambda) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : |f(t, x, u) - g(t, x, u)| \leq \epsilon$$

$$\text{for each } t \in [0, \infty) \text{ and each } x, u \in R^n \text{ satisfying } |x|, |u| \leq N$$

$$\text{and } (|f(t, x, u)| + 1)(|g(t, x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for each } t \in [0, \infty) \text{ and each } x, u \in R^n \text{ satisfying } |x| \leq N\},$$

where $N > 0, \epsilon > 0, \lambda > 1$.

It is not difficult to show that the space $\mathcal{M}$ with this uniformity is metrizable (by a metric ρ_w). It is known (see [115]) that the metric space $(\mathcal{M}, \rho_w)$ is complete. The metric ρ_w induces in $\mathcal{M}$ a topology.

We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(t, x(t), x'(t))dt,$$

where $f \in \mathcal{M}$, $0 \le T_1 < T_2 < \infty$, and $x : [T_1, T_2] \to R^n$ is an a.c. function.

For $f \in \mathcal{M}$, $y, z \in R^n$, and numbers T_1, T_2 satisfying $0 \le T_1 < T_2$, we set

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) : x : [T_1, T_2] \to R^n$$

$$\text{is an a.c. function satisfying } x(T_1) = y,\ x(T_2) = z\}.$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < \infty$ for each $f \in \mathcal{M}$, each $y, z \in R^n$, and all numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathcal{M}$. A locally absolutely continuous (a.c.) function $x : [0, \infty) \to R^n$ is called an (f)-*good function* [117] if for any a.c. function $y : [0, \infty) \to R^n$ there is a number M_y such that

$$I^f(0, T, y) \ge M_y + I^f(0, T, x) \text{ for each } T \in (0, \infty).$$

The following result was proved in [115].

Proposition 1.1 *Let $f \in \mathcal{M}$, and let $x : [0, \infty) \to R^n$ be a bounded a.c. function. Then the function x is (f)-good if and only if there is $M > 0$ such that*

$$I^f(0, T, x) \le U^f(0, T, x(0), x(T)) + M \textit{ for any } T > 0.$$

Let us now give the precise definition of the turnpike property.

Assume that $f \in \mathcal{M}$. We say that f has the turnpike property, or briefly TP, if there exists a bounded continuous function $X_f : [0, \infty) \to R^n$ that satisfies the following condition:

For each $K, \epsilon > 0$, there exist constants $\delta, L > 0$ such that for each $x, y \in R^n$ satisfying $|x|, |y| \le K$, each $T_1 \ge 0$, $T_2 \ge T_1 + 2L$, and each a.c. function $v : [T_1, T_2] \to R^n$ that satisfy

$$v(T_1) = x,\ v(T_2) = y,\ I^f(T_1, T_2, v) \le U^f(T_1, T_2, x, y) + \delta,$$

the inequality $|v(t) - X_f(t)| \le \epsilon$ holds for all $t \in [T_1 + L, T_2 - L]$.

The function X_f is called the turnpike of f.

Assume that $f \in \mathcal{M}$ and $X : [0, \infty) \to R^n$ is a bounded continuous function. How to verify if the integrand f has TP and X is its turnpike? In [115], we introduced two properties (P1) and (P2) and show that f has TP if and only if f possesses the properties (P1) and (P2). The property (P2) means that all (f)-good functions have the same asymptotic behavior, while the property (P1) means that if an a.c. function $v : [0, T] \to R^n$ is an approximate solution and T is large enough, then there is $\tau \in [0, T]$ such that $v(\tau)$ is close to $X(\tau)$.

The next theorem is the main result [115].

Theorem 1.2 *Let $f \in \mathcal{M}$ and $X_f : [0, \infty) \to R^n$ be a bounded absolutely continuous function. Then f has the turnpike property with X_f being the turnpike if and only if the following two properties hold:*

(P1) For each $K, \epsilon > 0$, there exist $\gamma, l > 0$ such that for each $T \geq 0$ and each a.c. function $w : [T, T + l] \to R^n$ that satisfy

$$|w(T)|, |w(T+l)| \leq K, \; I^f(T, T+l, w) \leq U^f(T, T+l, w(T), w(T+l))+\gamma,$$

there is $\tau \in [T, T + l]$ for which $|X_f(\tau) - v(\tau)| \leq \epsilon$.

(P2) For each (f)-good function $v : [0, \infty) \to R^n$,

$$|v(t) - X_f(t)| \to 0 \text{ as } t \to \infty.$$

In [115], we proved the following theorem that is an extension of Theorem 1.2.

Theorem 1.3 *Let $f \in \mathcal{M}$, and $X_f : [0, \infty) \to R^n$ be an (f)-good function. Assume that the properties (P1), (P2) hold. Then for each $K, \epsilon > 0$, there exist $\delta, L > 0$, and a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \geq 0$, $T_2 \geq T_1 + 2L$, and each a.c. function $v : [T_1, T_2] \to R^n$ that satisfy*

$$|v(T_1)|, |v(T_2)| \leq K, \; I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta,$$

the inequality $|v(t) - X_f(t)| \leq \epsilon$ holds for all $t \in [T_1 + L, T_2 - L]$.

In [148], we studied sufficient and necessary conditions for the turnpike phenomenon, using the approach developed in [115, 116, 118, 131], for discrete-time optimal control problems in metric spaces, which are not necessarily compact, and for continuous-time infinite dimensional optimal control problems. The main results of [148] have Theorem 1.2 as their prototype.

Since the discovery of the turnpike phenomenon by Paul Samuelson in 1948, different versions of the turnpike property were considered in the literature. In this book as well as in [115, 116, 118, 131, 148], we study the turnpike property introduced and used in our previous research [112, 117, 130, 131, 142]. This turnpike property differs from other versions and has important features. Our turnpike property is a property of approximate solutions, and the turnpike is a nonstationary trajectory. As it was shown in [112, 130], our turnpike property holds for most problems belonging to large classes of variational and optimal control problems.

The main features of this book are that we study optimal control problems in metric spaces, obtain sufficient and necessary conditions for the turnpike phenomenon, and show that the turnpike phenomenon is stable under small perturbations of objective function.

1.2 Notation

In this section, we collect the notation that will be used in the book.

Let (X, ρ_X) be a metric space equipped with the metric ρ_X that induces the topology in X. For each $x \in X$ and each $r > 0$, set

$$B_X(x, r) = \{y \in X : \rho(x, y) \leq r\}.$$

Let (X_i, ρ_{X_i}), $i = 1, 2$, be metric spaces. The set $X_1 \times X_2$ is equipped with the metric

$$\rho_{X_1 \times X_2}((x_1, x_2), (y_1, y_2)) = \rho_{X_1}(x_1, y_1) + \rho_{X_2}(x_2, y_2),$$

$$(x_1, x_2), \ (y_1, y_2) \in X_1 \times X_2.$$

We denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue measurable set $\Omega \subset R^1$ and define

$$\chi_\Omega(x) = 1 \text{ for all } x \in \Omega, \ \chi_\Omega(x) = 0 \text{ for all } x \in R^1 \setminus \Omega.$$

For each function $f : X \to [-\infty, \infty]$, where X is nonempty, we set

$$\inf(f) = \inf\{f(x) : x \in X\}.$$

For each $s \in R^1$, set $s_+ = \max\{s, 0\}$, $s_- = \min\{s, 0\}$, $\lfloor s \rfloor = \max\{z : z \text{ is an integer and } z \leq s\}$ and assume that $s \geq -\infty$, $s > -\infty$.

If $(X, \|\cdot\|)$ is a normed space, then X is equipped with the metric $\rho_X(x, y) = \|x - y\|$, $x, y \in X$.

Let (X, ρ_X) be a metric space and T be a Lebesgue measurable subset of R^1. A function $f : T \to X$ is called Lebesgue measurable if for any Borel set $D \subset X$ the set $f^{-1}(D)$ is a Lebesgue measurable set.

In the sequel, we denote by $\mathrm{Card}(D)$ the cardinality of a set D and suppose that the sum over an empty set is zero, the infimum over an empty set is ∞, and the supremum over an empty set is $-\infty$.

Let $(X, \langle \cdot, \cdot \rangle_X)$ be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle_X$ that induces the norm $\|\cdot\|_X$. If the space X is understood, we use the notation $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_X$ and $\|\cdot\| = \|\cdot\|_X$. Let X be a Banach space equipped with the norm $\|\cdot\|_X$ and X^* be its dual space with the norm $\|\cdot\|_{X^*}$. If the space X is understood, we use the notation $\|\cdot\| = \|\cdot\|_X$. For $x \in X$ and $l \in X^*$, we set $l(x) = \langle l, x \rangle_{X^*,X}$. The symbol $\langle \cdot, \cdot \rangle_{X^*,X}$ is referred to as the duality pairing between X^* and X. When the pair X, X^* is understood, we use the notation $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{X^*,X}$.

1.3 General Dynamical Systems with a Lyapunov Function

In Chap. 5, we develop a turnpike theory for a class of general dynamical systems in a metric space with a Lyapunov function. This class of dynamical systems contains as particular cases the classes of dynamical systems studied in Chaps. 3 and 4. Our goal is to obtain several turnpike results and show that the turnpike phenomenon is stable under small perturbations of objective functions.

Now we describe this class of problems.

Assume that (X, ρ) is a metric space. For each $x \in X$ and each $r > 0$, set

$$B(x, r) = \{y \in X : \ \rho(x, y) \leq r\} \text{ and } B^0(x, r) = \{y \in X : \ \rho(x, y) < r\}.$$

For each $x \in X$ and each pair of nonempty sets $A, B \subset X$, define

$$\rho(x, B) = \inf\{\rho(x, y) : \ y \in B\}$$

and

$$\text{dist}(A, B) = \sup\{\rho(x, B) : \ x \in A\}.$$

Let Γ be either R^1 or the set of all integers $\mathcal{Z} = \{0, \pm 1, \pm 2, \dots\}$.

Assume that for each pair $T_1, T_2 \in \Gamma$ satisfying $T_1 < T_2$, $Y(T_1, T_2)$ is a nonempty set of functions $x : [T_1, T_2] \cap \Gamma \to X$ such that the following properties hold:

(A1) For each pair $T_1, T_2 \in \Gamma$, each pair $S_1, S_2 \in \Gamma$ satisfying $T_1 \leq S_1 < S_2 \leq T_2$, and each $x \in Y(T_1, T_2)$, the restriction of x to $[S_1, S_2]$ belongs to $Y(S_1, S_2)$.

(A2) For each pair $T_1, T_2 \in \Gamma$ satisfying $T_1 < T_2$, each $S \in \Gamma$, and each $x \in Y(T_1, T_2)$, the function $x(t+S)$, $t \in [T_1, T_2] \cap \Gamma$, belongs to $Y(T_1+S, T_2+S)$.

(A3) For each triplet $T_1, T_2, T_3 \in \Gamma$ satisfying $T_1 < T_2 < T_3$, each $x_1 \in Y(T_1, T_2)$, and each $x_2 \in Y(T_2, T_3)$ satisfying $x_1(T_2) = x_2(T_2)$, the function $x_3 : [T_1, T_3] \cap \Gamma \to X$ defined by

$$x_3(t) = x_1(t), \ t \in [T_1, T_2] \cap \Gamma, \ x_3(t) = x_2(t), \ t \in (T_2, T_3] \cap \Gamma$$

belongs to $Y(T_1, T_3)$.

If $\Gamma = \mathcal{Z}$, then we set

$$\mathcal{A} = \{(y, z) \in X \times X : \ \text{ there is } x \in Y(0, 1) \text{ such that } x(0) = y, \ x(1) = z\},$$

define

$$a(x) = \{y \in X : \ (x, y) \in \mathcal{A}\}, \ x \in X,$$

and assume that $\mathcal{A}$ is closed in the metric space $X \times X$.

For each pair of numbers $T_1, T_2 \in \Gamma$ satisfying $T_1 < T_2$, elements of the set $Y(T_1, T_2)$ are called trajectories. Let $T_1 \in \Gamma$. A function $x : [T_1, \infty) \cap \Gamma \to X$ is a trajectory if for every $T_2 \in (T_1, \infty) \cap \Gamma$, the restriction of x to $[T_1, T_2] \cap \Gamma$ belongs to $Y(T_1, T_2)$. The set of all trajectories $x : [T_1, \infty) \cap \Gamma \to X$ is denoted by $Y(T_1, \infty)$.

Denote by $Y(\Gamma)$ the set of all functions $x : \Gamma \to X$ such that for each pair of numbers $T_1, T_2 \in \Gamma$ satisfying $T_1 < T_2$ the restriction of x to $[T_1, T_2] \cap \Gamma$ belongs to $Y(T_1, T_2)$. The elements of $Y(\Gamma)$ are called trajectories too.

Fix $\widehat{\theta} \in X$.

Assume that $\phi : X \to R^1$ is a continuous function such that for each pair of numbers $T_1 \in \Gamma$, $T_2 \in (T_1, \infty) \cap \Gamma$, each $x \in Y(T_1, T_2)$, and each $S_1, S_2 \in [T_1, T_2] \cap \Gamma$ satisfying $S_1 < S_2$,

$$\phi(x(S_2)) \le \phi(x(S_1)). \tag{1.1}$$

Assume that X_0 is nonempty, closed subset of X such that for each $r > 0$ the set

$$X_0 \cap B(\widehat{\theta}, r) \text{ is compact} \tag{1.2}$$

and that the following assumptions hold:

(A4) For each pair of numbers $T_1, T_2 \in \Gamma$ satisfying $T_2 > T_1$, each $M > 0$, and each sequence $\{x_n\}_{n=1}^{\infty} \subset Y(T_1, T_2)$ that satisfy

$$\lim_{n\to\infty} \sup\{\rho(x_n(t), X_0 \cap B(\widehat{\theta}, M)) :\ t \in [T_1, T_2] \cap \Gamma\} = 0,$$

there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges uniformly on $[T_1, T_2] \cap \Gamma$ to a trajectory $x \in Y(T_1, T_2)$ as $k \to \infty$.

(A5) For each bounded, nonempty set B, there exists $M > 0$, and for each $\epsilon > 0$, there exists a positive number $T(B, \epsilon) \in \Gamma$ such that for each $T \in \Gamma \cap [T(B, \epsilon), \infty)$ and each $x \in Y(0, T)$ satisfying $x(0) \in B$,

$$\rho(x(t), X_0 \cap B(\widehat{\theta}, M)) \le \epsilon,\ t \in [T(B, \epsilon), T] \cap \Gamma.$$

(Note that in the case $\Gamma = \mathcal{Z}$, (A4) follows from the compactness of the set $X_0 \cap B(\widehat{\theta}, M)$ and the closedness of $\mathcal{A}$.)

We suppose that $Y(0, \infty) \neq \emptyset$. Here we consider a general and abstract dynamical system and treat discrete-time and continuous-time case simultaneously. For examples of such systems, see Sects. 2.8 and 3.1. There are also infinite dimensional examples considered in [18, 82].

Define

$$\Omega = \{z \in X :\ \text{ for very } \epsilon > 0 \text{ there exists } x \in Y(0, \infty)$$

$$\text{for which } \liminf_{t\to\infty} \rho(z, x(t)) \le \epsilon\}.$$

Clearly, Ω is a closed subset of the metric space (X, ρ). We will show that

$$\Omega \ne \emptyset,\ \ \Omega \subset X_0,$$

and for each $x \in Y(0, \infty)$,

$$\lim_{t\to\infty} \rho(x(t), \Omega) = 0.$$

We also assume that the following assumption holds:

(A6) If $x \in Y(\Gamma)$ is bounded and for all $t \in \Gamma$,

$$x(t) \in X_0 \text{ and } \phi(x(t)) = \phi(x(0)),$$

then $x(t) \in \Omega$ for every $t \in \Gamma$.

If $\Gamma = \mathcal{Z}$ and $E \subset \Gamma$, then we set

$$\mathrm{mes}(E) = \mathrm{Card}(E).$$

For each $x \in X$, set

$$\pi(x) = \lim_{T\to\infty} \sup\{\phi(x(T)) :\ x \in Y(0, T) \text{ and } x(0) = x\}.$$

(We assume that the supremum over an empty set is $-\infty$.)

It is clear that

$$\pi(x) \le \phi(x),\ \ x \in X.$$

The function π plays an important role in our study.

Evidently, the following result is true.

Proposition 1.4 *Let $T > 0$ and $x \in Y(0, T)$. Then $\pi(x(s)) \le \pi(x(0))$ for all $s \in [0, T] \cap \Gamma$.*

The following result is proved in Sect. 5.3.

Proposition 1.5 *Let $x \in X$. The value $\pi(x)$ is finite if and only if for every $T \in \Gamma \cap (0, \infty)$, there exists $y \in Y(0, T)$ satisfying $y(0) = x$.*

In Sect. 5.4, we prove the following result.

Proposition 1.6 *Let $M_0 > 0$. Then there exists $M > 0$ such that for each $x \in B(\widehat{\theta}, M_0)$ satisfying $\pi(x) > -\infty$,*

$$B(\widehat{\theta}, M) \cap \Omega \cap \phi^{-1}(\pi(x)) \ne \emptyset.$$

The following turnpike result is proved in Sect. 5.5.

Theorem 1.7 *Let $\epsilon > 0$ and $\widehat{x} \in X$ satisfy $\pi(\widehat{x}) > -\infty$. Then there exist $\delta > 0$ and $L > 0$ such that for every $T \in \Gamma \cap (2L, \infty)$ and every $x \in Y(0, T)$ that satisfy*

$$x(0) = \widehat{x} \text{ and } \phi(x(T)) \geq \pi(\widehat{x}) - \delta,$$

the inequality

$$\rho(x(t), \Omega) \leq \epsilon$$

holds for all $t \in [L, T - L] \cap \Gamma$.

Clearly, the theorem above establishes the turnpike property for approximate optimal trajectories that have a fixed starting point with respect to the objective function ϕ. The turnpike is the set Ω, and δ, L depend on ϵ and the initial point $\widehat{x}$.

For each $x \in X$ satisfying $\pi(x) > -\infty$, set

$$F(x) = \Omega \cap \phi^{-1}(\pi(x)),$$

which is nonempty.

The following result, which is a generalization of the previous theorem, is proved in Sect. 5.7. According to the previous theorem, for all $t \in [L, T - L] \cap \Gamma$, the distance between $x(t)$ and the turnpike Ω does not exceed ϵ. According to our next result, for all $t \in [L, T - L] \cap \Gamma$, the distance between $x(t)$ and the set $F(\widehat{x}) \cap B(\widehat{\theta}, M)$ does not exceed ϵ, where a positive constant M depends on ϵ and the initial point $\widehat{x}$ and $F(\widehat{x})$ is the set of all points of the turnpike Ω where the value of the Lyapunov function ϕ is $\pi(\widehat{x})$.

Theorem 1.8 *Let $\epsilon > 0$ and $\widehat{x} \in X$ satisfy $\pi(\widehat{x}) > -\infty$. Then there exist $M, L, \delta > 0$ such that for every $T \in \Gamma \cap (2L, \infty)$ and every $x \in Y(0, T)$ that satisfy*

$$x(0) = \widehat{x} \text{ and } \phi(x(T)) \geq \pi(\widehat{x}) - \delta,$$

the inequality

$$\rho(x(t), F(\widehat{x}) \cap B(\widehat{\theta}, M)) \leq \epsilon$$

holds for all $t \in [L, T - L] \cap \Gamma$.

In Chap. 5, we obtain several extensions and generalizations of these two turnpike results. In particular, we show that if the function π is continuous on the turnpike set Ω, then the constants L, δ do not depend on the starting point $\widehat{x}$. We also show that the turnpike phenomenon is stable under small perturbations of objective functions.

1.4 Continuous-Time Nonautonomous Problems on Half-Axis

In Chapter 8, we establish sufficient and necessary conditions for the turnpike phenomenon for continuous-time optimal control problems on subintervals of half-axis in metric spaces. For these optimal control problems, the turnpike is not a singleton. The results of this chapter are obtained for three large classes of problems that will be treated simultaneously. In Chap. 9, for the same classes of problems, we show that the turnpike phenomenon is stable under small perturbations of objective functions.

Let us describe these classes of problems.

Let (E, ρ_E) be a complete metric space and (F, ρ_F) be a metric space. We suppose that $\mathcal{A}$ is a nonempty subset of $[0, \infty) \times E$, and $\mathcal{U} : \mathcal{A} \to 2^F$ is a point to set mapping with a graph

$$\mathcal{M} = \{(t, x, u) : \ (t, x) \in \mathcal{A}, \ u \in \mathcal{U}(t, x)\}.$$

We suppose that $\mathcal{M}$ is a Borel measurable subset of $[0, \infty) \times E \times F$. Assume that for each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2 < \infty$, we are given a set $X(T_1, T_2)$ of pairs (x, u) (called as trajectory–control pairs) such that $x : [T_1, T_2] \to E$ is a continuous function, and $u : [T_1, T_2] \to F$ is a Lebesgue function satisfying

$$(t, x(t)) \in \mathcal{A}, \ t \in [T_1, T_2],$$

$$u(t) \in \mathcal{U}(t, x(t)), \ t \in [T_1, T_2] \text{ almost everywhere (a.e.).}$$

We suppose that the following property holds:

if $0 \le T_1 \le S_1 < S_2 \le T_2 < \infty$, $(x, u) \in X(T_1, T_2)$, $\tilde{x}$ is the restriction of x to $[S_1, S_2]$, and $\tilde{u}$ is the restriction of u to $[S_1, S_2]$, then $(\tilde{x}, \tilde{u}) \in X(S_1, S_2)$;

for each triplet of nonnegative numbers $T_1 < T_2 < T_3$, each $(x_1, u_1) \in X(T_1, T_2)$, and each $(x_2, u_2) \in X(T_2, T_3)$ satisfying

$$x_1(T_2) = x_2(T_2)$$

a pair $(x_3, u_3) \in X(T_1, T_3)$, where

$$x_3(t) = x_1(t), \ t \in [T_1, T_2], \ x_3(t) = x_2(t), \ t \in (T_2, T_3],$$

$$u_3(t) = x_1(t), \ t \in [T_1, T_2], \ u_3(t) = u_2(t), \ t \in (T_2, T_3].$$

Let $a_0 > 0$, $\theta_0 \in E$, $\theta_1 \in F$, and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\psi(t) \to \infty \text{ as } t \to \infty,$$

and let $\psi_1 : \mathcal{M} \to [0, \infty)$ satisfy

$$\psi_1(t, x, u) \geq \psi(\rho_E(x, \theta_0)), \ (t, x, u) \in \mathcal{M}.$$

Denote by $\mathfrak{M}_\psi$ the set of all Borelian functions $g : \mathcal{M} \to R^1$ such that for each $(t, x, u) \in \mathcal{M}$,

$$g(t, x, u) \geq \psi_1(t, x, u) - a_0, \ (t, x, u) \in \mathcal{M}.$$

For each pair of numbers $T_2 > T_1 \geq 0$, each $(x, u) \in X(T_1, T_2)$, and each $g \in \mathfrak{M}_\psi$, define

$$I^g(T_1, T_2, x, u) = \int_{T_1}^{T_2} g(t, x(t), u(t))dt \in (-\infty, \infty].$$

We consider functionals of the form $I^g(T_1, T_2, x, u)$, where $0 \leq T_1 < T_2$, $(x, u) \in X(T_1, T_2)$, and $g \in \mathfrak{M}_\psi$. Let $g \in \mathfrak{M}_\psi$. For each pair of numbers $T_2 > T_1 \geq 0$ and each pair of points $(T_1, y), (T_2, z) \in \mathcal{A}$, we define

$$U^g(T_1, T_2, y, z) = \inf\{I^g(T_1, T_2, x, u) :$$

$$(x, u) \in X(T_1, T_2), \ x(T_1) = y, \ x(T_2) = z\},$$

$$U^g(T_1, T_2, y) = \inf\{I^g(T_1, T_2, x, u) : (x, u) \in X(T_1, T_2), \ x(T_1) = y\},$$

$$\widehat{U}^g(T_1, T_2, z) = \inf\{I^g(T_1, T_2, x, u) : (x, u) \in X(T_1, T_2), \ x(T_2) = z\},$$

$$U^g(T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) : (x, u) \in X(T_1, T_2)\}.$$

We equip the space $\mathfrak{M}_\psi$ with the uniformity determined by the base

$$E(N, \epsilon, \lambda) = \{(f, g) \in \mathfrak{M}_\psi \times \mathfrak{M}_\psi : |f(t, x, u) - g(t, x.u)| \leq \epsilon$$

$$\text{for each } (t, x, u) \in \mathcal{M} \text{ satisfying } \rho_E(x, \theta_0), \ \rho(u, \theta_1) \leq N\}$$

$$\cap \{(f, g) \in \mathfrak{M}_\psi \times \mathfrak{M}_\psi : (|f(t, x, u)| + 1)(|g(t, x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for each } (t, x, u) \in \mathcal{M} \text{ satisfying } \rho_E(x, \theta_0) \leq N\},$$

where $N, \epsilon > 0, \lambda > 1$. Clearly, the uniform space $\mathfrak{M}_\psi$ is Hausdorff and has a countable base. Therefore, $\mathfrak{M}_\psi$ is metrizable (by a metric d_ψ). It is not difficult to see that the uniform space $\mathfrak{M}_\psi$ is complete. It is equipped with a topology induced by the uniformity.

In this chapter, we assume that the following assumption holds:

(A0) For each $L_0 > 0$, there exists $L_1 > 0$ such that for each $T_1 \geq 0$, each $T_2 \in (T_1, T_1 + L_0]$, each $(x, u) \in X(T_1, T_2)$, and each $g \in \mathfrak{M}_\psi$ that satisfy

$$\rho_E(x(T_1), \theta_0) \leq L_0, \ I^g(T_1, T_2, x, u) \leq L_0,$$

the inequality

$$\rho_E(\theta_0, x(t)) \leq L_1$$

is true for all $t \in [T_1, T_2]$.

Let $f \in \mathfrak{M}_\psi$. In some cases, we will use the following assumption:

(A0)' For each $L_0 > 0$, there exists $L_1 > 0$ such that for each $T_1 \geq 0$, each $T_2 \in (T_1, T_1 + L_0]$, and each $(x, u) \in X(T_1, T_2)$ that satisfy

$$I^f(T_1, T_2, x, u) \leq L_0$$

and

$$\min\{\rho_E(\theta_0, x(t)) : \ t \in [T_1, T_2]\} \leq L_0,$$

the inequality

$$\rho_E(\theta_0, x(t)) \leq L_1$$

is true for all $t \in [T_1, T_2]$.

Assumption (A0) easily implies the following result.

Proposition 1.9 *Assume that $M_1 > 0$, $0 < \tau < \tau_0 < \tau_1$. Then there exists $M_2 > 0$ such that for each $T_1 \geq 0$, each $T_2 \in [T_1 + \tau_0, T_1 + \tau_1]$, each $g \in \mathfrak{M}_\psi$, and each $(x, u) \in X(T_1, T_2)$ that satisfy*

$$I^g(T_1, T_2, x, u) \leq M_1,$$

the inequality

$$\rho_E(\theta_0, x(t)) \leq M_2$$

is true for all $t \in [T_1 + \tau, T_2]$.

Usually, a control system is governed by a differential equation, and its trajectories are solutions of this equation. Therefore, properties of trajectories depend on the equation. In this book, we use a different framework. We assume that sets of trajectory–control pairs are given and that they satisfy some natural assumptions that hold in the control theory. No equation is involved in the definition of our control system. Therefore, our results can be implied to different classes of control systems.

Now we describe three classes of optimal control problems that are particular cases of the class of problems introduced above.

The First Class of Problems
We begin with the description of the first class of problems. Let $(E, \|\cdot\|)$ be a Banach space and E^* be its dual. Let $\{A(t) : t \in [0, \infty)\}$ be the family of closed densely defined linear operators with the domain and range in the Banach space E. Let (F, ρ_F) be a metric space,

$$\theta_0 = 0,\ \theta_1 \in F,$$

$\rho_E(x, y) = \|x - y\|,\ x, y \in E$. We suppose that $\mathcal{A}$ is a nonempty subset of $[0, \infty) \times E$, and $\mathcal{U} : \mathcal{A} \to 2^F$ is a point to set mapping with a graph

$$\mathcal{M} = \{(t, x, u) :\ (t, x) \in \mathcal{A},\ u \in \mathcal{U}(t, x)\}.$$

We suppose that $\mathcal{M}$ is a Borel measurable subset of $[0, \infty)\times E\times F$ and $G : \mathcal{M} \to E$ is a Borelian function. We consider the homogeneous Cauchy problem

$$x'(t) = A(t)x(t),\ t \in [0, \infty).$$

We assume that there exists a function $U : \{(t, s) \in R^2 :\ 0 \le s \le t < \infty\} \to L(E)$ that has the following properties [21]:

(i) For each $x_0 \in E$, the function $(t, s) \to U(t, s)x_0$ is continuous on the set $\{(t, s) \in R^2 :\ 0 \le s \le t < \infty\}$.
(ii) $U(s, s) = Id$ for all $s \in [0, \infty)$, where Id is the identity operator.
(iii) $U(t, s)U(s, \tau) = U(t, \tau)$ for all numbers $t \ge s \ge \tau \ge 0$.
(iv) For each $s \ge 0$, there exists a densely linear subspace E_s of E such that for each $x_0 \in E_s$ the function $t \to U(t, s)x_0$ is continuously differentiable on $[s, \infty)$ and

$$(\partial/\partial t)U(t, s)x_0 = A(t)U(t, s)x_0,\ t \in [s, \infty).$$

(v) There exists an increasing function $\tau \to \Delta_\tau > 0,\ \tau > 0$ such that for each $\tau > 0$, each $s \ge 0$, and each $t \in [s, s + \tau]$,

$$\|U(t, s)\| \le \Delta_\tau.$$

In this case problem, homogeneous Cauchy problem is called well-posed [21].

Let $0 \le T_1 < T_2$, and consider the following equation:

$$x'(t) = A(t)x(t) + f(t),\ t \in [T_1, T_2],\ x(0) = x_{T_1},$$

where $x_{T_1} \in E$ and $f \in L^1(T_1, T_2; E)$.

A continuous function $x : [T_1, T_2] \to E$ is a solution of the equation above if

$$x(t) = U(t, T_1)x(T_1) + \int_{T_1}^{t} U(t, s)f(s)ds, \ t \in [T_1, T_2].$$

Assume that the equation above holds and $\tau \in [T_1, T_2]$. It is not difficult to see (p. 386, [148]) that $x : [\tau, T_2] \to E$ is a solution of the equation

$$y'(t) = A(t)y(t) + f(t), \ t \in [\tau, T_2], \ y(\tau) = x(\tau).$$

Let $0 \le T_1 < T_2 < T_3$, $z_0 \in E$, $f \in L^1(T_1, T_3; E)$, a continuous function $x_1 : [T_1, T_2] \to E$ is a solution of the equation

$$x'(t) = A(t)x(t) + f(t), \ t \in [T_1, T_2], \ x(T_1) = z_0,$$

and a continuous function $x_2 : [T_2, T_3] \to E$ is a solution of the equation

$$x'(t) = A(t)x(t) + f(t), \ t \in [T_2, T_3], \ x(T_2) = x_1(T_2).$$

Set

$$x(t) = x_1(t), \ t \in [T_1, T_2], \ x(t) = x_2(t), \ t \in [T_2, T_3].$$

Clearly, the function $x : [T_1, T_3] \to E$ is continuous. It is not difficult to see (p. 387, [148]) that $x(\cdot)$ is a solution of the equation

$$x'(t) = A(t)x(t) + f(t), \ t \in [T_1, T_3], \ x(T_1) = z_0.$$

Let $0 \le T_1 < T_2$. We consider the following equation:

$$x'(t) = A(t)x(t) + G(t, x(t), u(t)), \ t \in [T_1, T_2].$$

A pair of functions $x : [T_1, T_2] \to E$, $u : [T_1, T_2] \to F$ is called a (mild) solution of the equation above if $x : [T_1, T_2] \to E$ is a continuous function, $u : [T_1, T_2] \to F$ is a Lebesgue measurable function,

$$(t, x(t)) \in \mathcal{A}, \ t \in [T_1, T_2],$$

$$u(t) \in \mathcal{U}(t, x(t)), \ t \in [T_1, T_2] \text{ almost everywhere (a.e.)},$$

$G(s, x(s), u(s)), \ s \in [T_1, T_2]$ is Bochner integrable, and for every $t \in [T_1, T_2]$,

$$x(t) = U(t, T_1)x(T_1) + \int_{T_1}^{t} U(t, s)G(s, x(s), u(s))ds.$$

The set of all solutions (x, u) is denoted by $X(T_1, T_2)$.

Let $T_1 \geq 0$. A pair of functions $x : [T_1, \infty) \to E$, $u : [T_1, \infty) \to F$ is called a (mild) solution of the system

$$x'(t) = A(t)x(t) + G(t, x(t), u(t)),\ t \in [T_1, \infty)$$

if for every $T_2 > T_1$, the pair $x : [T_1, T_2] \to E$, $u : [T_1, T_2] \to F$ belongs to $X(T_1, T_2)$. The set of all such pairs (x, u), which are solutions of the equation above, is denoted by $X(T_1, \infty)$.

A function $x : I \to E$, where I is either $[T_1, T_2]$ or $[T_1, \infty)$ $(0 \leq T_1 < T_2)$, is called a trajectory if there exists a Lebesgue measurable function $u : I \to F$ (referred to as a control) such that $(x, u) \in X(T_1, T_2)$ or $(x, u) \in X(T_1, \infty)$, respectively.

For this class of problem, for each $(t, x, u) \in \mathcal{M}$, set

$$\psi_1(t, x, u) = \max\{\psi(\|x\|),\ \psi(\|u\|),$$

$$\psi((\|G(t, x, u)\| - a_0\|x\|)_+)(\|G(t, x, u)\| - a_0\|x\|)_+\}.$$

In Sect. 8.4, we show that assumption (A0) holds for this class of problems.

The Second Class of Problems

Let $(E, \langle\cdot,\cdot\rangle)_E$ be a Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle_E$ that induces the norm $\|\cdot\|_E$, and let $(F, \langle\cdot,\cdot\rangle_F)$ be a Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle_F$ that induces the norm $\|\cdot\|_F$. For simplicity, we set $\langle\cdot,\cdot\rangle_E = \langle\cdot,\cdot\rangle$, $\|\cdot\|_E = \|\cdot\|$, $\langle\cdot,\cdot\rangle_F = \langle\cdot,\cdot\rangle$, $\|\cdot\|_F = \|\cdot\|$, if E, F are understood. Set

$$\theta_0 = 0,\ \theta_1 = 0,$$

$\rho_E(x, y) = \|x - y\|,\ x, y \in E$, $\rho_F(x, y) = \|x - y\|,\ x, y \in F$.

We suppose that $\mathcal{A}_0$ is a nonempty subset of E, and $\mathcal{U}_0 : \mathcal{A}_0 \to 2^F$ is a point to set mapping with a graph

$$\mathcal{M}_0 = \{(x, u) :\ x \in \mathcal{A}_0,\ u \in \mathcal{U}_0(x)\}.$$

We suppose that $\mathcal{M}_0$ is a Borel measurable subset of $E \times F$. Define

$$\mathcal{A} = [0, \infty) \times \mathcal{A}_0,$$

$\mathcal{U} : \mathcal{A} \to 2^F$ by

$$\mathcal{U}(t, x) = \mathcal{U}_0(x),\ (t, x) \in \mathcal{A},$$

$$\mathcal{M} = [0, \infty) \times \mathcal{M}_0.$$

Let a linear operator $A : \mathcal{D}(A) \to E$ generate a C_0 semigroup $S(t) = e^{At}$, $t \in [0, \infty)$ on E. As usual, we denote by $S(t)^*$ the adjoint of $S(t)$. Then $S(t)^*,\ t \in [0, \infty)$, is C_0 semigroup, and its generator is the adjoint A^* of A. The domain $\mathcal{D}(A^*)$ is a Hilbert space equipped with the graph norm $\| \cdot \|_{\mathcal{D}(A^*)}$:

$$\|z\|^2_{\mathcal{D}(A^*)} = \|z\|^2_E + \|A^*z\|^2_E,\ z \in \mathcal{D}(A^*).$$

Let $\mathcal{D}(A^*)'$ be the dual of $\mathcal{D}(A^*)$ with the pivot space E. In particular,

$$E_1^d := \mathcal{D}(A^*) \subset E \subset \mathcal{D}(A^*)' = E_{-1}.$$

(Here we use the notation of Chap. 7.)

Let $G : \mathcal{M} \to \mathcal{D}(A^*)' = E_{-1}$, $B \in \mathcal{L}(F, E_{-1})$ is an admissible control operator for e^{At}, $t \geq 0$, and for all $(t, x, u) \in \mathcal{M}$,

$$G(t, x, u) = Bu.$$

Let $0 \leq T_1 < T_2$. We consider the following equation:

$$x'(t) = Ax(t) + Bu(t),\ t \in [T_1, T_2] \text{ a.e.}$$

A pair of functions $x : [T_1, T_2] \to E$, $u : [T_1, T_2] \to F$ is called a (mild) solution of the equation above if $x : [T_1, T_2] \to E$ is a continuous function, $u : [T_1, T_2] \to F$ is a Lebesgue measurable function, $u \in L^2(T_1, T_2; F)$,

$$(t, x(t)) \in \mathcal{A},\ t \in [T_1, T_2],$$

$$u(t) \in \mathcal{U}(t, x(t)),\ t \in [T_1, T_2] \text{ a.e.,}$$

and for each $t \in [T_1, T_2]$,

$$x(t) = e^{A(t-T_1)}x(T_1) + \int_{T_1}^{t} e^{A(t-s)}Bu(s)ds$$

in E_{-1}. The set of all solutions (x, u) is denoted by $X(T_1, T_2, A, G)$. In the sequel for simplicity, we use the notation $X(T_1, T_2) = X(T_1, T_2, A, G)$ if the pair (A, G) is understood.

Let $T_1 \geq 0$. A pair of functions $x : [T_1, \infty) \to E$, $u : [T_1, \infty) \to F$ is called a (mild) solution of the system

$$x'(t) = Ax(t) + Bu(t),\ t \in [T_1, \infty))$$

if for every $T_2 > T_1$, $x : [T_1, T_2] \to E$, $u : [T_1, T_2] \to F$ belongs to $X(T_1, T_2)$. The set of all such pairs (x, u), which are solutions of the equation above, is denoted by $X(T_1, \infty)$.

A function $x : I \to E$, where I is either $[T_1, T_2]$ or $[T_1, \infty)$ $(0 \le T_1 < T_2)$, is called a trajectory if there exists a Lebesgue measurable function $u : I \to F$ (referred to as a control) such that $(x, u) \in X(T_1, T_2)$ or $(x, u) \in X(T_1, \infty)$, respectively.

For this class of problem for each $(t, x, u) \in \mathcal{M}$, set

$$\psi_1(t, x, u) = \max\{\psi(\|x\|),\ K_0\|u\|^2\} - a_0,$$

where K, a_0 are positive constants.

In Sect. 8.4, we show that assumption (A0) holds for this class of problems.

The Third Class of Problems

Let (E, ρ_E) be a complete metric space, (F, ρ_F) be a metric space,

$$\theta_0 \in E,\ \theta_1 \in F.$$

We suppose that $\mathcal{A}$ is a nonempty subset of $[0, \infty) \times E$, and $\mathcal{U} : \mathcal{A} \to 2^F$ is a point to set mapping with a graph

$$\mathcal{M} = \{(t, x, u) :\ (t, x) \in \mathcal{A},\ u \in \mathcal{U}(t, x)\}.$$

We suppose that $\mathcal{M}$ is a Borel measurable subset of $[0, \infty) \times E \times F$ and $G : \mathcal{M} \to [0, \infty)$ is a Borelian function.

Let $0 \le T_1 < T_2 < \infty$. Denote by $X(T_1, T_2)$ the set of all pairs of functions $x : [T_1, T_2] \to E$ and $u : [T_1, T_2] \to F$ such that x is continuous on $[T_1, T_2]$, u is Lebesgue measurable on function $[T_1, T_2]$,

$$(t, x(t)) \in \mathcal{A},\ t \in [T_1, T_2],$$

$$u(t) \in \mathcal{U}(t, x(t)),\ t \in [T_1, T_2] \text{ almost everywhere (a.e.)},$$

$G(s, x(s), u(s))$, $s \in [T_1, T_2]$ is Bochner integrable, and for every pair $S_1, S_2 \in [T_1, T_2]$ satisfying $S_1 < S_2$,

$$\rho_E(x(S_1), x(S_2)) \le \int_{S_1}^{S_2} G(t, x(t), u(t))dt.$$

(See Section 1.1 of [5].) Pairs $(x, u) \in X(T_1, T_2)$ are called trajectory–control pairs.

Let $T_1 \ge 0$. Denoted by $X(T_1, \infty)$ the set of all pairs of functions $x : [T_1, \infty) \to E$, $u : [T_1, \infty) \to F$ such that for every $T_2 > T_1$, the pair $x : [T_1, T_2] \to E$, $u : [T_1, T_2] \to F$ belongs to $X(T_1, T_2)$. Pairs $(x, u) \in X(T_1, \infty)$ are called trajectory–control pairs.

A function $x : I \to E$, where I is either $[T_1, T_2]$ or $[T_1, \infty)$ $(0 \le T_1 < T_2)$, is called a trajectory if there exists a Lebesgue measurable function $u : I \to F$ (referred to as a control) such that $(x, u) \in X(T_1, T_2)$ or $(x, u) \in X(T_1, \infty)$, respectively).

For this class of problem for each $(t, x, u) \in \mathcal{M}$, set

$$\psi_1(t, x, u) = \max\{\psi(\rho_E(x, \theta_0)),\ \psi(\rho_F(u, \theta_1)),$$

$$\psi((G(t, x, u) - a_0\rho_E(x, \theta_0))_+)(G(t, x, u) - a_0\rho_E(x, \theta_0))_+\}.$$

For each pair of numbers $T_2 > T_1 \geq 0$, each $(x, u) \in X(T_1, T_2)$, and each $g \in \mathfrak{M}_\psi$, define

$$I^g(T_1, T_2, x, u) = \int_{T_1}^{T_2} g(t, x(t), u(t))dt \in (-\infty, \infty].$$

In Chap. 8, we show now that assumption (A0) holds for this class of problems.

Let us consider our general class of problems.

We say that the integrand f possesses the turnpike property (or TP for short) if for each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and $L > 0$ such that for each $T_1 \geq 0$, each $T_2 \geq T_1 + 2L$, and each $(x, u) \in X(T_1, T_2)$ that satisfy

$$I^f(T_1, T_2, x, u) \leq \min\{U^f(T_1, T_2) + M,\ U^f(T_1, T_2, x(T_1), x(T_2)) + \delta\}$$

there exist $\tau_1, \tau_2 \in [0, L]$ such that

$$\rho_E(x(t), x_f(t)) \leq \epsilon \text{ for all } t \in [T_1 + \tau_1, T_2 - \tau_2].$$

Moreover, if $\rho_E(x(T_2), x_f(T_2)) \leq \delta$, then $\tau_2 = 0$, and if $T_1 \geq L$ and $\rho_E(x(T_1), x_f(T_1)) \leq \delta$, then $\tau_1 = 0$.

The next theorem is the main result of Chap. 8. It is proved in Sect. 8.13.

Theorem 1.10 *f has TP if and only if the following properties hold:*

(P1) For each (f)-good pair $(x, u) \in X(0, \infty)$,

$$\lim_{t \to \infty} \rho_E(x(t), x_f(t)) = 0.$$

(P2) For each $\epsilon > 0$ and each $M > 0$, there exist $\delta > 0$ and $L > 0$ such that for each $T \geq 0$ and each $(x, u) \in X(T, T + L)$ that satisfy

$$I^f(T, T + L, x, u) \leq \min\{U^f(T, T + L, x(T), x(T + L)) + \delta,$$

$$I^f(T, T + L, x_f, u_f) + M\}$$

there exists $s \in [T, T + L]$ such that $\rho_E(x(s), x_f(s)) \leq \epsilon$.

In Chap. 9, we study the stability of the turnpike phenomenon. In particular, it is shown that if TP holds, then it is stable under small perturbations of the objective function.

Chapter 2
Differential Inclusions

In this chapter, we discuss Banach space-valued functions, set-valued mappings in infinite dimensional spaces, and related continuous-time dynamical systems. Some convergence results are obtained.

2.1 Banach Space-Valued Functions

Let $(X, \|\cdot\|)$ be a Banach space and $a < b$ be real numbers. For any set $E \subset R^1$, define

$$\chi_E(t) = 1 \text{ for all } t \in E \text{ and } \chi_E(t) = 0 \text{ for all } t \in R^1 \setminus E.$$

If a set $E \subset R^1$ is Lebesgue measurable, then its Lebesgue measure is denoted by $|E|$ or by $\mathrm{mes}(E)$.

A function $f : [a, b] \to X$ is called a simple function if there exists a finite collection of Lebesgue measurable sets $E_i \subset [a, b],\ i \in I$, mutually disjoint, and $x_i \in X, i \in I$ such that

$$f(t) = \sum_{i \in I} \chi_{E_i}(t)x_i,\ t \in [a, b].$$

A function $f : [a, b] \to X$ is strongly measurable if there exists a sequence of simple functions $\phi_k : [a, b] \to X, k = 1, 2, \dots,$ such that

$$\lim_{k\to\infty} \|\phi_k(t) - f(t)\| = 0,\ t \in [a, b] \text{ almost everywhere (a.e.).} \tag{2.1}$$

For every simple function $f(\cdot) = \sum_{i\in I} \chi_{E_i}(\cdot)x_i$, where the set I is finite, define its Bochner integral by

A. J. Zaslavski, *Turnpike Phenomenon in Metric Spaces*, Springer Optimization and Its Applications 201, https://doi.org/10.1007/978-3-031-27208-0_2