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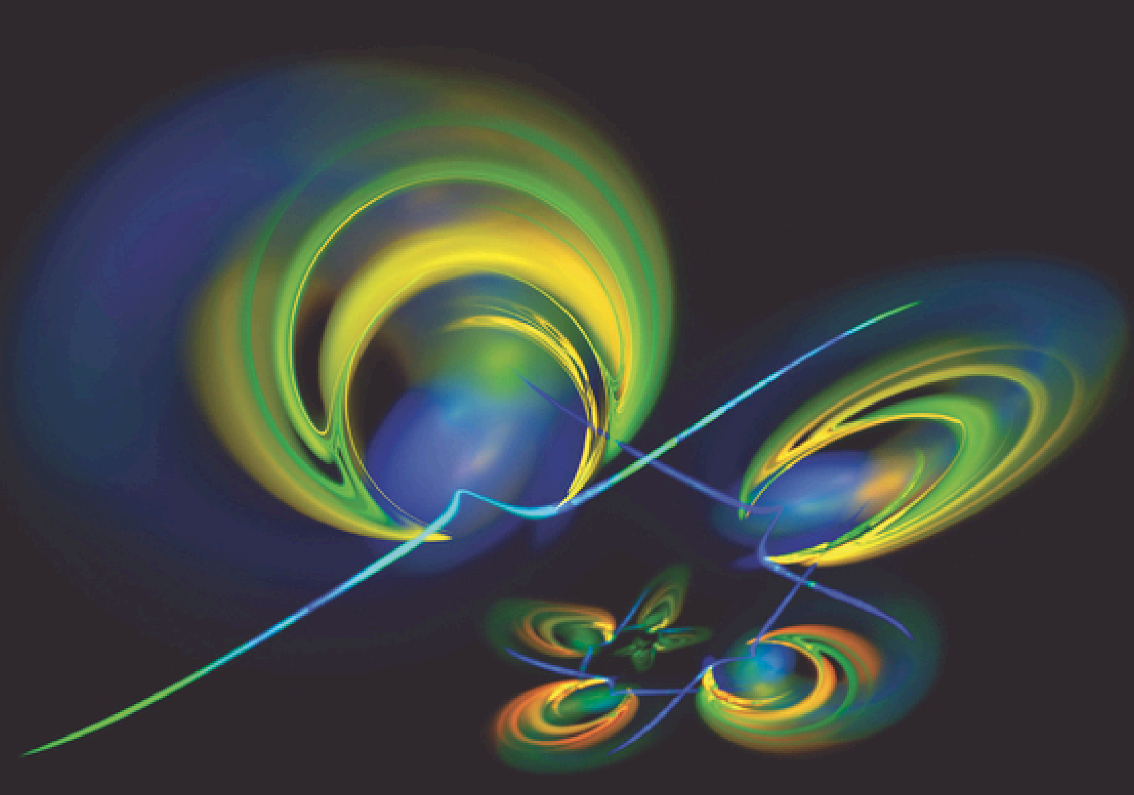
For Nonlinear Differential Equations in Physics



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About the Author

Attilio Maccari is a free-lance researcher in nonlinear physics. He received his PhD from La Sapienza Rome University in 1990. He has published about one hundred papers, mainly on coupled oscillators and nonlinear partial differential equations in physics and engineering.

Among his most important accomplishments, we recall the Maccari system for rogue waves as well as the Hirota–Maccari equation for nonlinear systems. Both equations are integrable and with remarkable nonlinear waves. His recent work has been devoted to vibration control, and he found two new methods in order to perform this very important task, time-delay state feedback control and nonlocal feedback control.

Foreword

This textbook is devoted to nonlinear physics.

The asymptotic perturbation method is used as a mathematical tool and is explained in some detail, and the theory is developed systematically, starting with nonlinear oscillators, limit cycles and their bifurcations, followed by iterated nonlinear maps, continuous systems, nonlinear partial differential equations (NPDEs), and culminating with infinite-period bifurcation in the nonlinear Schrodinger equation and fractal and chaotic solutions in NPDEs.

A remarkable feature of the book is the emphasis on applications. There are several examples, and the scientific background is explained at an elementary level and closely integrated with the mathematical theory.

This book is ideal for an introductory course at the senior or first-year graduate level. It is also advisable for a scientist who has not a deep knowledge about nonlinear physics but now wants to begin a complete study.

The prerequisites are multivariable calculus and introductory physics.

Introduction

Nonlinear systems are paramount in engineering and science. Many perturbation methods can be used to study these systems in order to predict remarkable bifurcations (a qualitative change in their behavior). In this book, we will use the asymptotic perturbation (AP) method both for nonlinear ordinary differential equations (NODEs) and nonlinear partial differential equations (NPDEs).

In Chapters 1–4, we will study NODEs and will derive a suitable model system to find the most important nonlinear system characteristics. The main finding is that a nonlinear model system of equations describes their behavior. In particular, in Chapter 2, we will describe an infinite-period bifurcation for a parametrically excited Liènard system and find a peculiar attractor for a weakly nonlinear oscillator with a two-period quasiperiodic forcing.

In Chapter 3, we consider vibration control with time-delay state feedback and perform a successful control strategy. In Chapter 4, we illustrate another vibration-control method based on nonlocal dynamics. Numerical simulation confirms our method's validity.

In Chapter 5, we enlarge our perspective and study nonlinear continuous systems, in particular the vibrations of an Euler–Bernoulli beam resting on a nonlinear elastic foundation and with an external periodic excitation. Frequency–response and external force–response curves can be easily found and compared with numerical simulation.

In Chapter 6, the AP method is used to study NPDEs, and we are able to find two new and integrable nonlinear equations, the Maccari system and the Hirota–Maccari equation.

At the same time, the AP method can be used in order to find approximate solutions to relevant physics problems. In Chapter 7, we study the Zakharov–Kusnetsov equation and show the existence of interacting localized solutions because the ZK equation can be described through a C-integrable (solvable via an appropriate change of variables) system of nonlinear evolution equations. Dromions, lumps, ring solitons, and breathers exist for this remarkable nonlinear equation.

In Chapter 8, we study the connection between the AP method and elementary particle physics.

In Chapter 9, we try to explain the rogue waves appearance in nonlinear systems.

In Chapter 10, we arrive at one of the most important findings in this book, fractal and chaotic solutions are possible for nonlinear systems and perhaps at a very fundamental level we must let the particle concept (i.e. a coherent solution) down because we can state that in general solutions have fractal and chaotic properties.

In Chapter 11, we use the AP method in order to arrive at nonlinear quantum mechanics and achieve the Einstein–de Brogle soliton-particle concept by studying the weakly nonlinear Klein–Gordon equation for a particle confined in a box. In Chapter 12, we illustrate how to modify the Einstein equation so as to explain the accelerating and irreversible evolution of the universe. According to Prigogine's ideas, the entropy increase is connected with matter production.

In Chapter 13, this new theory is used to find how confinement and asymptotic freedom can be explained in a framework where particles are like small black holes. Finally, Chapter 14 is devoted to a reverse infinite-period bifurcation for the nonlinear Schrodinger equation in $2 + 1$ dimensions.

Many teaching years allowed me writing this book, and I would like to thank my students at Foligno in Perugia University, Italy, for their helpful and valuable suggestions.

1

The Asymptotic Perturbation Method for Nonlinear Oscillators

1.1 Introduction

Oscillations are a fundamental topic in physics. When a system is near its equilibrium point, it begins to oscillate, but if the displacement increases, then the nonlinear terms are not negligible. The starting point is the differential equation for the harmonic oscillator

$$\frac{d^2X}{dt} + \omega^2 X(t) = 0 \tag{1.1}$$

where $X(t)$ is the displacement and ω the circular frequency. The most general solution is

$$X(t) = 2\rho\cos(-\omega t + \theta) \tag{1.2}$$

where ρ and θ are fixed by the initial conditions (the Cauchy problem)

if $X(0) = X_0$ for the displacement
and $\dot{X}(0) = \dot{X}_0$ for the initial velocity

then we easily get

$$2\rho = \sqrt{\left((X_0)^2 + \left(\frac{\dot{X}_0}{\omega}\right)^2\right)} \tag{1.3}$$

and

$$\tan\theta = \left(\frac{\dot{X}_0}{\omega X_0}\right) \tag{1.4}$$

Now, we can consider a weakly nonlinear part in the differential Eq. (1.1) or, on the contrary, a strongly nonlinear part but with small solutions. The first consequence is that the amplitude and the phase are slowly varying with time, so we can introduce another slow time

$$\tau = \varepsilon^q t \tag{1.5}$$

where ε is a bookkeeping device and q is a rational number that will be chosen afterwards. If we want to study the asymptotic solution behavior ($t \to \infty$) and $\varepsilon \to 0$, then

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τ must assume finite values. So, we assume that an approximate solution is given by

$$X(t) = 2\rho(\tau)\cos(-\omega t + \theta(\tau)) = (\rho(\tau)\exp(-i\omega t + i\theta) + c.c.) \tag{1.6}$$

or better

$$X(t) = \varepsilon^{(1+r)}\Psi_0 + (\varepsilon\Psi_1 \exp(-i\omega t) + \varepsilon^2\Psi_2 \exp(-2\omega t) + \varepsilon^3\Psi_3 \exp(-3i\omega t) + c.c. + h.o.t.) \tag{1.7}$$

where $c.\,c.$ stands for complex conjugate and $h.\,o.\,t.$ for higher order terms.

Following this path, we are mixing the most important features of two well-known perturbation methods, the harmonic balance and the multiple scale methods (for more details about these two perturbation methods, see Refs. [202, 203, 249]).

If we consider a weakly nonlinear differential equation

$$\frac{d^2X}{dt} + \omega^2 X(t) = NL \tag{1.8}$$

where NL stands for the nonlinear part, for instance,

$$aX(t)^2 + bX(t)^3 \tag{1.9}$$

we can insert the solution (1.7) in the nonlinear Eq. (1.8) and with some algebra manipulation, we get for $n = 0$

$$\omega^2\varepsilon^{(1+r)}\Psi_0 = 2a\varepsilon^2|\Psi|^2 \tag{1.10}$$

then $r = 1$, for $n = 2$

$$-3\omega^2\varepsilon^2\Psi_2 = a\varepsilon^2\Psi^2 \tag{1.11}$$

and for $n = 1$

$$-2i\omega\varepsilon^q\psi_\tau = 2a(\varepsilon^{82} + r)\Psi_0\Psi + \varepsilon^2\Psi_2(c.c.\Psi) + 3b\varepsilon^2|\Psi|^2\Psi \tag{1.12}$$

then, $q = 2$ for the proper nonlinear term balance and with some algebra manipulation

$$\frac{d\Psi}{d\tau} = \frac{iA}{2\omega}|\Psi|^2\Psi \tag{1.13}$$

where

$$A = \frac{10a^2}{3\omega^2} + b \tag{1.14}$$

$$\frac{d\rho}{d\tau} = 0 \quad \frac{d\theta}{d\tau} = \frac{A}{2\omega}\rho^2 \tag{1.15}$$

We observe that the variable change (1.5) implies that

$$\frac{d}{dt} \to -in\omega + \varepsilon^q\frac{d}{d\tau} \tag{1.16}$$

when the temporal differential operator acts on the function

$$\Psi_n(\tau)\exp(-in\omega t) \tag{1.17}$$

From Eq. (1.10), we can see that the approximate solution is always periodic, the amplitude is constant, but the period changes and becomes

$$T = \frac{2\pi}{\Omega} \quad \text{where} \quad \Omega = \omega - \frac{A}{2\omega}\rho^2 \tag{1.18}$$

However, if

$$b = -\left(\frac{10a^2}{3\omega^2}\right) \tag{1.19}$$

the period does not change and is equal to the linear case period.

In this chapter, we want to extend this method and study a generalized Van der Pol–Duffing oscillator in resonance with a periodic excitation

$$\ddot{X}(t) + X(t) + f_2X^2(t) + f_3X^3(t) = g_0\dot{X}(t) + g_1X(t)\dot{X}(t) + g_2X^2(t)\dot{X}(t) + 2F\cos(\Omega t) \tag{1.20}$$

We use the asymptotic perturbation (AP) method based on Fourier expansion and time rescaling (see above) and demonstrate through a second-order perturbation analysis the existence of one or two limit cycles. Moreover, we identify a sufficient condition to obtain a doubly periodic motion when a second low frequency appears, in addition to the forcing frequency. The comparison with the solution obtained by the numerical integration confirms the validity of our analysis.

1.2 Nonlinear Dynamical Systems

The study of nonlinear dynamical systems has interested many researchers, and various methods have been used. Historically, the AP method was first applied in order to study the most important characteristics of a nonlocal oscillator [112, 113, 118].

We now devote our attention to the following type of nonlinear equation

$$\ddot{X}(t) + f(X(t)) = g(X(t), \dot{X}(t)) \tag{1.21}$$

where the dot denotes differentiation with respect to the time and the functions $f(x)$ and $g(x, y)$ are supposed to be analytic.

The limit cycles of the modified Van der Pol equation

$$\ddot{X}(t) + X(t) + X^3(t) = \varepsilon(1 - X^2(t))\dot{X}(t) \tag{1.22}$$

have been studied in Ref. [23] by means of a time transformation method.

Phase portraits and dynamical properties of the equation

$$\ddot{X}(t) + (\alpha + \beta X^2(t))\dot{X}(t) + \gamma X(t) + \delta X^3(t) = 0 \tag{1.23}$$

have been investigated with the methods of differentiable dynamics [74] and the equation

$$\ddot{X}(t) + X(t) = \varepsilon f(X(t), \dot{X}(t)) \tag{1.24}$$

with the method of averaging, the KBM method, the method of multiple scales, and the Poincaré–Lindstedt method [202, 203].

Note that Eqs. (1.22)–(1.24) belong to the general class (1.21) and are characterized by the fact that $f(x)$ is an odd function of x.

We restrict our study to the following particular case of Eq. (1.21)

$$\ddot{X}(t) + X(t) + f_2X^2(t) + f_3X^3(t) = g_0\dot{X}(t) + g_1X(t)\dot{X}(t) + g_2X^2(t)\dot{X}(t) \tag{1.25}$$

Eq. (1.5) can be considered a generalized Van der Pol–Duffing equation because it includes as particular cases the Van der Pol oscillator ($f_2, f_3, g_1 = 0$ and $g_0 = -g_2 \neq 0$) and the Duffing equation ($f_2 = g_1 = g_2 = 0$ and $g_0 = f_3 \neq 0$). Many authors have studied the problem of approximating the limit cycle of the Van der Pol equation. Stokes [249] used the nonlinear Galerkin method and developed a series representation; Deprit and Schmidt [47] utilized the Poincaré–Lindstedt method to find the amplitude and frequency of the limit cycle; and Garcia-Magallo and Bejarano [57] considered a generalized Van der Pol equation by means of the harmonic balance method. The steady-state behavior of the Van der Pol oscillator has also been studied by integral manifold methods and symbolic manipulation packages by Gilsinn [59, 61]. Mehri and Ghorashi [195] considered the periodically forced Duffing equation in order to establish sufficient conditions to have a periodic solution, and Qaisi [233] studied a similar problem using an analytical approach based on the power series method. In a series of papers [69–71], Hassan used the higher order method of multiple scales with reconstitution and the harmonic balance method to determine the periodic state response of the Duffing oscillator.

In our treatment of Eq. (1.25), no conditions are imposed on the coefficients f_2, f_3, g_1, and g_2, which can be of order 1. Only the dissipative coefficient g_0 is supposed to be of order e^2. Eq. (1.25) transforms into

$$\ddot{X}(t) + X(t) + f_2X^2(t) + f_3X^3(t) = \varepsilon^2 g_0\dot{X}(t) + g_1X(t)\dot{X}(t) + g_2X^2(t)\dot{X}(t) \tag{1.26}$$

In the second section, we calculate the approximate solution good to the order of ε^4 and construct accurate expressions for the limit cycle of Eq. (1.26). Moreover, we demonstrate that, in the first approximation, the behavior of the solution can be described by means of a model system of differential equations, which represents the characteristics of Eq. (1.26) by means of a reduced set of parameters.

Usually, perturbation analysis is carried out only to the first order because, in many cases, a second order-calculation does not change the qualitative behavior of the solution. However, in Section 1.2, we demonstrate that if the parameters are appropriately chosen, we can find two limit cycles and can calculate their positions only by a second-order perturbation analysis.

In Section 1.3, a comparison with the results of the numerical integration permits discussion of the validity of the AP method.

In Section 1.4, we treat an extension of Eq. (1.26) that is a nonlinear oscillator forced by a small periodic excitation, of order e^2, in resonance with the natural frequency of the oscillator

$$\ddot{X}(t) + X(t) + f_2X^2(t) + f_3X^3(t) = \varepsilon^2 g_0\dot{X}(t) + g_1X(t)\dot{X}(t) + g_2X^2(t)\dot{X}(t) + 2\varepsilon^2 f\cos(t) \tag{1.27}$$

We demonstrate that, under appropriate conditions, a stable limit cycle appears and calculate the relative approximate solution. Moreover, we derive sufficient conditions for the existence of a doubly periodic motion when the fundamental oscillation is subjected to a slight modulation, with an amplitude proportional to the magnitude of the periodic excitation.

Finally, in the last section, we briefly recapitulate the most important results and indicate some possible generalizations of the present study.

1.3 The Approximate Solution

The AP method we use to calculate the approximate solution was first developed in Refs. [1, 2], and then in this section, we sketch the main steps of this perturbation technique.

First of all, we now introduce a rational number

$$q = \text{rational number} \tag{1.28}$$

the temporal rescaling

$$t = e^q t \tag{1.29}$$

where the rational number q will be fixed afterwards because it establishes to what extent we can push the temporal asymptotic limit in such a way that the nonlinear effects become consistent and not negligible. If $t \to \infty$, then $\varepsilon \to 0$, when τ assumes a finite value.

If we take $\varepsilon = 0$ in Eqs. (1.26) and neglect nonlinear terms, we see that it admits simple harmonic solutions $X(t) = A\exp(-\text{it}) + c.c.$, where A is a constant depending on initial conditions and $c.c.$ stands for complex conjugate. Nonlinear effects induce a modulation of the amplitude A and the appearance of higher harmonics. The modulation is best described in terms of the rescaled variable t that accounts for the need to look on larger time scales, to obtain a nonnegligible contribution from the nonlinear term.

The assumed solution $X(t)$ of (1.26) can be expressed by means of a power series in the expansion parameter ε, we formally write

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n} \Psi_n(t,\varepsilon)\exp(-\text{int}) \tag{1.30a}$$

with $\gamma_n = |n|$ for $n \neq 0$, and $\gamma_0 = r$ is a positive number, which will be fixed later on; in consequence of the reality of (1.30a)

$$\Psi_n(t,\varepsilon) = c.c.(\Psi_{(-n)}(t,\varepsilon)) \tag{1.30b}$$

The assumed solution (1.30a) can be considered a combination of the different harmonics, solutions of the linear equation, i.e. of the equation obtained after neglecting all the nonlinear terms, and the coefficients of this combination depend on τ and ε.

Eq. (1.30a) can be written more explicitly

$$X(t) = \varepsilon^r \Psi_0(t;\varepsilon) + \varepsilon\Psi_1(t;\varepsilon)\exp(-\text{it}) + \varepsilon^2\Psi_2(t;\varepsilon)\exp(-2\text{it}) + \varepsilon^3\Psi_3(t;\varepsilon)\exp(-3\text{it})o(\varepsilon^4) \tag{1.30c}$$

The functions $\Psi_n(t, \varepsilon)$ depend on the parameter ε, and we suppose that Ψ_n's limit for $\varepsilon \to 0$ exists and is finite and, moreover, they can be expanded in power series of ε, i.e.

$$\Psi_n(\tau; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \Psi_n^{(i)}(\tau) \tag{1.31}$$

In the following, for simplicity, we use the abbreviations $\Psi_n^{(0)} = \Psi_n$ for $n \neq 1$ and $\Psi_1^{(0)} = \Psi$ for $n = 1$.

Note that the variable change (1.29) implies that

$$\frac{(\partial \psi_n \exp(-\text{int}))}{\partial t} = \left(-\text{in}\psi_n + \varepsilon^q \frac{\partial \psi_n}{\partial t}\right) \exp(-\text{int}) \tag{1.32}$$

After inserting this expansion into Eq. (1.26), we obtain equations for every harmonic and for a fixed order of approximation, which are right for the purpose of determining the coefficients.

For $n = 0$, we obtain

$$\varepsilon^r \Psi_0 + 2f_2|\Psi|^2\varepsilon^2 + o(\varepsilon^4, \varepsilon^{r+2}) = 0 \tag{1.33a}$$

A correct balance of terms shows $r = 2$, and then we derive the following relation

$$\varepsilon^2 \Psi_0 = -2\varepsilon^2 f_2(|\Psi|^2) + O(\varepsilon^4) \tag{1.33b}$$

For $n = 2$, taking into account Eq. (1.32), we have

$$-3\varepsilon^2 \Psi_2 + f_2 \Psi^2 \varepsilon^2 = -ig_1 \varepsilon^2 \Psi^2 + o(\varepsilon^4, \varepsilon^{2+q}) \tag{1.34a}$$

and then

$$\varepsilon^2 \Psi_2 = \frac{f_2 + ig_1}{3} \varepsilon^2 \Psi^2 + o(\varepsilon^4) \tag{1.34b}$$

For $n = 1$, Eq. (1.26) yields for the right-hand side

$$-2i \frac{d\Psi}{dt} \varepsilon^{1+q} + 2f_2(\Psi_0 \Psi \varepsilon^3 + \Psi_2(c.c.(\Psi))\varepsilon^3) + 3f_3|\Psi|^2\Psi\varepsilon^3 \tag{1.35a}$$

and for the left-hand side

$$i\varepsilon^3 g_0 \Psi - ig_1(\Psi_0 \Psi \varepsilon^3 + \Psi \Psi_2(c.c.\Psi)\varepsilon^3) - i\varepsilon^3 g_2 |\Psi|^2 \Psi + o(\varepsilon^5, \varepsilon^{1+2q}) \tag{1.35b}$$

If $q = 2$, the first term has the same magnitude order of nonlinear terms.

Taking into account Eqs. (1.33b) and (1.34b), we can derive a differential equation, which involves only Ψ,

$$\frac{d\Psi}{dt} = \alpha_1 \Psi + (\beta_1 + i\beta_2)|\Psi|^2 \Psi \tag{1.36}$$

with

$$\alpha_1 = \frac{g_0}{2} \tag{1.37}$$

$$\beta_1 = \frac{g_2}{2} - \frac{g_1 f_2}{2} \tag{1.38}$$

$$\beta_2 = \frac{g_1^2}{6} - \frac{3}{2} f_3 + \frac{5}{3} f_2^2 \tag{1.39}$$

Substituting the polar form

$$\Psi(\tau) = \rho(\tau)\exp(i\theta(\tau)) \tag{1.40}$$

into Eq. (1.36), and separating real and imaginary parts, we arrive at the following model system:

$$\frac{d\rho}{dt} = \alpha_1\rho + \beta_1\rho^3 \tag{1.41}$$

$$\frac{dJ}{dt} = \beta_2\rho^2 \tag{1.42}$$

As we can see from Eqs. (1.30c), (1.31), and (1.40), the approximate solution of Eq. (1.26) can be written as a sum of a contribution of order ε and a contribution of order ε^2

$$\begin{aligned}
&X(t) = \varepsilon X_1(t) + \varepsilon^2 X_2(t) + o(\varepsilon^3),\\
&X_1(t) = 2\rho(\tau)\cos(-t+\theta(\tau)),\\
&X_2(t) = -2f_2\rho^2(\tau) + \frac{2}{3}f_2\rho^2(\tau)\cos(-2t+2\theta(\tau)) + \frac{2}{3}g_1\rho^2(\tau)\sin(-2t+2\theta(\tau))
\end{aligned} \tag{1.43}$$

By inspection of Eq. (1.41), which can be easily integrated, we conclude that a stable steady-state response is possible if $\alpha_1 > 0$ and $\beta_1 < 0$. In this case, we obtain a stable equilibrium point, which corresponds to a stable limit cycle for Eq. (1.26), and its approximate expression is given by (1.43), with

$$\rho(t) = \rho_E = \sqrt{-\frac{\alpha_1}{\beta_1}} = \text{constant}, \theta(t) = \beta_2\rho_E t \tag{1.44}$$

The natural frequency of the oscillator is subject to a slight modification and becomes

$$\omega_E = \omega - \beta_2\rho_E \tag{1.45}$$

If we want to improve the validity of the approximate solution, we must include higher order terms. However, we can easily conclude that $\Psi_0^{(1)} = \Psi_1^{(1)} = \Psi_2^{(1)} = 0$ (for their definition, see Eq. (1.31)). Indeed, we consider Eq. (1.26) for $n = 0$ and Eqs. (1.33b) and (1.34a) for $n = 0$ and $n = 2$ in such a way to obtain

$$\Psi_0^{(1)} = -2f_2\left(\Psi_1^{(1)}(c.c.\Psi) + \Psi c.c.\left(\Psi_1^{(1)}\right)\right),\quad \Psi_2^{(1)} = \frac{2}{3}(f_2 + ig_1)\Psi_1^{(1)}\Psi \tag{1.46}$$

After inserting (1.26b) into (1.26a), we see that the resulting equation is satisfied if $\Psi_1^{(1)} = 0$. Recall that we can always assume that the initial condition is $\Psi_1^{(1)}(0) = 0$, because the initial conditions associated with equation (1.25), $X(0) = X_0$ and $\dot{X}(0) = \dot{X}_0$, can be used to determine $\Psi(0) = \rho(0)\exp(i\theta(0))$.

A valid higher order approximation can be derived only if we take into account $\Psi_1^{(2)}, \Psi_2^{(2)}, \Psi_0^{(2)}$.

For $n = 0$, we derive the following relation

$$\begin{aligned}
\varepsilon^2\Psi_0 + \varepsilon^4\Psi_0^{(2)} = &\Big((-2f_2\varepsilon^2 + A_1\varepsilon^4)|\Psi|^2 - 2f_2\varepsilon^4\Big(\Psi_1^{(2)}(c.c.\Psi)\\
&+ \left(c.c.\Psi_1^{(2)}\right)\Psi\Big) + A_2\varepsilon^4|\Psi|^4\Big) + h.o.t.
\end{aligned} \tag{1.47a}$$