Advances in Intelligent Systems and Computing 1440

Manoj Sahni · José M. Merigó ·
Walayat Hussain · Ernesto León-Castro ·
Raj Kumar Verma · Ritu Sahni *Editors*

Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy

Proceedings of the Third International Conference, MMCITRE 2022

Springer

Advances in Intelligent Systems and Computing

Volume 1440

Series Editor

Janusz Kacprzyk, Systems Research Institute, Polish Academy of Sciences,
Warsaw, Poland

Advisory Editors

Nikhil R. Pal, Indian Statistical Institute, Kolkata, India

Rafael Bello Perez, Faculty of Mathematics, Physics and Computing,
Universidad Central de Las Villas, Santa Clara, Cuba

Emilio S. Corchado, University of Salamanca, Salamanca, Spain

Hani Hagras, School of Computer Science and Electronic Engineering,
University of Essex, Colchester, UK

László T. Kóczy, Department of Automation, Széchenyi István University,
Gyor, Hungary

Vladik Kreinovich, Department of Computer Science, University of Texas
at El Paso, El Paso, TX, USA

Chin-Teng Lin, Department of Electrical Engineering, National Chiao
Tung University, Hsinchu, Taiwan

Jie Lu, Faculty of Engineering and Information Technology,
University of Technology Sydney, Sydney, NSW, Australia

Patricia Melin, Graduate Program of Computer Science, Tijuana Institute
of Technology, Tijuana, Mexico

Nadia Nedjah, Department of Electronics Engineering, University of Rio de Janeiro,
Rio de Janeiro, Brazil

Ngoc Thanh Nguyen, Faculty of Computer Science and Management,
Wrocław University of Technology, Wrocław, Poland

Jun Wang, Department of Mechanical and Automation Engineering,
The Chinese University of Hong Kong, Shatin, Hong Kong

The series “Advances in Intelligent Systems and Computing” contains publications on theory, applications, and design methods of Intelligent Systems and Intelligent Computing. Virtually all disciplines such as engineering, natural sciences, computer and information science, ICT, economics, business, e-commerce, environment, healthcare, life science are covered. The list of topics spans all the areas of modern intelligent systems and computing such as: computational intelligence, soft computing including neural networks, fuzzy systems, evolutionary computing and the fusion of these paradigms, social intelligence, ambient intelligence, computational neuroscience, artificial life, virtual worlds and society, cognitive science and systems, Perception and Vision, DNA and immune based systems, self-organizing and adaptive systems, e-Learning and teaching, human-centered and human-centric computing, recommender systems, intelligent control, robotics and mechatronics including human-machine teaming, knowledge-based paradigms, learning paradigms, machine ethics, intelligent data analysis, knowledge management, intelligent agents, intelligent decision making and support, intelligent network security, trust management, interactive entertainment, Web intelligence and multimedia.

The publications within “Advances in Intelligent Systems and Computing” are primarily proceedings of important conferences, symposia and congresses. They cover significant recent developments in the field, both of a foundational and applicable character. An important characteristic feature of the series is the short publication time and world-wide distribution. This permits a rapid and broad dissemination of research results.

Indexed by DBLP, INSPEC, WTI Frankfurt eG, zbMATH, Japanese Science and Technology Agency (JST).

All books published in the series are submitted for consideration in Web of Science.

For proposals from Asia please contact Aninda Bose (aninda.bose@springer.com).

Manoj Sahni · José M. Merigó · Walayat Hussain ·
Ernesto León-Castro · Raj Kumar Verma ·
Ritu Sahni
Editors

Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy

Proceedings of the Third International Conference, MMCITRE 2022

Springer

Editors
Manoj Sahni
Pandit Deendayal Energy University
Gandhinagar, India

José M. Merigó
School of Computer Science
University of Technology Sydney
Ultimo, NSW, Australia

Walayat Hussain
Peter Faber Business School
Australian Catholic University
North Sydney, NSW, Australia

Ernesto León-Castro
Universidad Católica de la Santísima
Concepcion, Chile

Raj Kumar Verma
School of Economics and Business
Universidad de Talca
Talca, Chile

Ritu Sahni
Pandit Deendayal Energy University
Gandhinagar, India

ISSN 2194-5357 ISSN 2194-5365 (electronic)
Advances in Intelligent Systems and Computing
ISBN 978-981-19-9905-5 ISBN 978-981-19-9906-2 (eBook)
https://doi.org/10.1007/978-981-19-9906-2

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Singapore Pte Ltd. 2023, corrected publication 2023

This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

This proceedings features contributions from mathematicians, physicists, and engineers from all over the world including numerous Indian states who attended the Third International Conference on Mathematical Modeling, Computational Intelligence Techniques, and Renewable Energy (MMCITRE 2022). The preceding two conferences were held at Pandit Deendayal Energy University in Gandhinagar, Gujarat, India. This time the conference is held online at the University of Technology Sydney, Australia, located at the southern border of Sydney's central business district, along with the associated universities: Pandit Deendayal Energy University, Gandhinagar, Gujarat, India; Victoria University, Melbourne, Australia; Universidad Autónoma Baja California (UABC), Mexico; Universidad Católica de la Santsima Concepción (UCSC), Chile; and Forum for Interdisciplinary Mathematics (FIM).

The purpose of this conference is the exchange of knowledge, which can lead to research collaboration with scholars and researchers and the development of numerous new ideas for future research. Keeping researchers and scientists up to date on the most recent discoveries and innovations, as well as on the solutions to the practical challenges that are helpful not only for academicians but also for industry persons, is the purpose of this series of MMCITRE conferences, the goal of which is to provide a plethora of usable content that is based on mathematical modeling, artificial intelligence techniques, renewable energy, and a variety of other topics that are interrelated. From the bottom of our hearts, we want to extend a hearty greeting to all of the world's leading scientists, academics, young researchers, business delegates, and students who have participated in this international conference. Our deepest appreciation also extends to everyone who wished us well and provided encouraging notes to help us pull off a smooth event.

Academicians and professionals from a number of nations, as well as different parts of India, have attended the conference in order to give the academic community the opportunity to learn about their knowledge, research findings, and educational practices. A substantial number of research articles came from all around the world and were submitted to this international conference. It has been taken into account that these articles cover the most recent mathematical techniques that are valuable not only for academics but also for students studying at the undergraduate and postgraduate levels in a variety of fields. Only the finest papers have been selected to be presented orally after undergoing a rigorous peer-review procedure by the experts in their respective fields. Finally, on the basis of the quality of the work of the experts from many fields, young researchers, academics, and students who presented papers, a total of 36 articles have been chosen to be published in this proceedings.

New fundamental mathematical results with a wide range of applications are presented in the proceedings. It helps aspiring engineers and computer scientists develop logic, creativity, and critical thinking skills that apply to almost all of the world's challenges. The proceedings covers recent mathematical breakthroughs, rigorous mathematical methodologies, and unique mathematical modeling of real-life occurrences in education, medical, business, and marketing for society's benefit. We wish this proceedings will help graduate students in mathematics, physics, engineering, and other subjects find new mathematical tools.

Gandhinagar, India	Dr. Manoj Sahni
Ultimo, Australia	Prof. José M. Merigó
Concepcion, Chile	Dr. Ernesto León-Castro
North Sydney, Australia	Dr. Walayat Hussain
Gandhinagar, India	Dr. Ritu Sahni
Talca, Chile	Dr. Raj Kumar Verma

Contents

Theoretical Advancements for Applied Mathematics

Mathematical Modelling

Medical and Statistical Applications

Fluid Modelling

About the Editors

Dr. Manoj Sahni is a dedicated and experienced Mathematics teacher and researcher with more than 18 years of experience and currently serving as an Associate Professor and the head of the Department of Mathematics, School of Technology, Pandit Deendayal Energy University, Gandhinagar, Gujarat, India. He has published more than 70 research papers in peer-reviewed journals, conference proceedings and book chapters with reputed publishers like Springer, Elsevier, etc. He also serves as an advisory board member, a technical Committee member and a reviewer for many international journals of repute. He organized the 1st and 2nd International Conferences on Mathematical Modeling, Computational Techniques, and Renewable Energy on February 21–23, 2020, and February 06–08, 2021, respectively. He has organized many seminars, workshops, and short-term training programs at PDEU and various other universities. In addition, he is a member of many international professional societies, including the American Mathematical Society, Forum for Interdisciplinary Mathematics, IAENG and many more.

Prof. José M. Merigó is a professor in the School of Information, Systems and Modelling at the Faculty of Engineering and Information Technology at the University of Technology Sydney (Australia) and a part-time full professor in the Department of Management Control and Information Systems at the School of Economics and Business at the University of Chile. He holds master's and Ph.D. degrees in Business Administration from the University of Barcelona. He also holds B.Sc. and M.Sc. degrees from Lund University (Sweden). He has published more than 500 articles in journals, books and conference proceedings, including journals such as *Information Sciences*, *IEEE Computational Intelligence Magazine*, *European Journal of Operational Research.* He has also published several books with Springer and with World Scientific. Recently, Thomson and Reuters (Clarivate Analytics) have distinguished him as a highly cited researcher in Computer Science (2015–present).

Dr. Walayat Hussain is a professional academic, a researcher and a practitioner in computer science and information system with over 15 years of academic and industry experience. He worked as a lecturer with the Faculty of Engineering and IT, University of Technology Sydney, Australia, for four years. Walayat's research areas are Computational Intelligence, Soft Computing, Forecasting, Cloud Computing, SLA Management and Decision Support System. His work has been published in different top-ranked reputable journals and conferences such as *Future Generation Computer Systems*, *The Computer Journal* (Oxford University), *Information Systems, Computer and Industrial Engineering*, *IEEE Access*, *Journal of Ambient Intelligence and Humanized Computing*, *MONET*, *IEEE TGCN*, *IEEE TETCI*, *IJCS*, *FUZZ-IEEE*, *ICONIP*. He is the associate editor of *IET-Communications*, an international journal, and has been a guest editor for several other international journals such as—*Computer Networks*, *Forecasting*, *IEEE TETCI*, *WCMC*, *IJCS*.

Dr. Ernesto León-Castro is a full professor at Universidad Autonoma of Baja California (Mexico) and an adjunct research professor at Universidad Católica de la Santísima Concepcion (Chile). He has Ph.D. in Management from University of Occidente. He also has a M.Sc. in Management from University of Occidente and another one in Laws from University Autonoma de Sinaloa. His interest in research is in aggregation operators, decision making, finance, economics, computational intelligence and other related topics. He has published many papers in different journals, books and proceedings, including journals such as *Applied Soft Computing*, *Technological and Economic Development of Economy*, *International Journal of Intelligent Systems*, *International Journal of Machine Learning and Cybernetics* and *Economic Computation* and *Economic Cybernetics Studies and Research*. Also, he has been the guest editor in Special Issues in *Technological and Economic Development of Economy*, *Journal of Intelligent and Fuzzy Systems*, *Computational and Mathematical Organization Theory* and Springer Book Series.

Dr. Raj Kumar Verma received a M.Sc. degree in mathematics from Chaudhary Charan Singh University, Meerut (U.P.), India, in 2006 and a Ph.D. degree in applied mathematics with a specialty in information theory and computational intelligence techniques from Jaypee Institute of Information Technology (Deemed University), Noida (U.P.), India, in 2014. He has authored over 70 research articles published in refereed international journals, including the *International Journal of Intelligent Systems*, *Kybernetika*, *Journal of Intelligent and Fuzzy Systems*, *International Journal of Machine Learning and Cybernetics*, *Neural Computing and Applications*, *Informatica*, *Soft Computing, Cybernetics and Systems*, *Iranian Journal of Fuzzy Systems*, *Granular Computing*, *Applied Intelligence Review*. Dr. Verma has also been a member of the scientific advisory committee of several international conferences and a reviewer in a wide range of refereed international journals. He is currently interested in information measures, aggregation operators, multiple attribute group decision making, computational intelligence techniques and computing with words.

Dr. Ritu Sahni is a dedicated and experienced Mathematics teacher and a researcher with more than 15 years who is actively involved with both undergraduate and postgraduate-level students. Presently, she is working as a visiting faculty at Pandit Deendayal Energy University, Gandhinagar, Gujarat, India. She worked with her students on various research-related problems and published their work in reputed journals such as Elsevier, Springer. She has published more than 40 research papers in peer-reviewed SCI, ESCI and Scopus indexed international journals and conferences. She has delivered talks and presented many research articles in various international conferences in India and abroad. Her research field includes Fixed Point Theory and its Applications, Numerical Methods, Fuzzy Decision-Making Systems and other allied areas. She is a devoted faculty member, a committed researcher, and enthusiastic about striving to improve the institution's educational offerings and research fields for societal growth.

Theoretical Advancements for Applied Mathematics

A New Result Using Quasi-β-Power Increasing Sequence

Smita Sonker and Rozy Jindal

Abstract Two generalized results were established, concerning the absolute matrix summability. Bor [2–10] worked on many interesting results dealing with $|\overline{N}, p_n|_k$ absolute Riesz summability. Özarslan and many other authors have been worked on matrix summability. In [14–21], they gave new and advanced results on matrix summability and generalize many theorems of Bor. Sonker and Jindal [23, 24] worked on triple product summability means and absolute matrix summability. Özarslan and Yavuz [16] proved two results on $|U, p_l|_q$ summability factors. Here, we generalized both the results for $\varphi - |U, p_l|_q$ matrix summability. Further, we develop new and arbitrary previous findings from the main theorems.

Keywords Matrix summability Abel's theorem · Matrix transformation · Quasi-monotone sequences

1 Introduction

Let $\{s_l\}$ denotes the sequence of partial sum of $\sum a_l$. The lth sequence to sequence transform of $\{s_l\}$ is given by u_l, where

$$u_l = \sum_{k=0}^{\infty} u_{lk} s_k. \tag{1}$$

Definition 1 If

$$\lim_{l\to\infty} u_l = s$$

S. Sonker (✉) · R. Jindal
National Institute of Technology Kurukshetra, Kurukshetra 136119, India
e-mail: smitafma@nitkkr.ac.in

M. Sahni et al. (eds.), *Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy*, Advances in Intelligent Systems and Computing 1440,
https://doi.org/10.1007/978-981-19-9906-2_1

and

$$\sum_{l=1}^{\infty} |u_l - u_{l-1}| < \infty, \tag{2}$$

then $\sum a_l$ is called absolute summable.

Definition 2 [11] If

$$\sum_{l=1}^{\infty} l^{1-k} |t_l - t_{l-1}|^k < \infty, \tag{3}$$

then $\sum a_l$ is s.t.b. summable $|C, 1|_k$.

Definition 3 Let $\{p_s\}$ be of +ive numbers and

$$P_s = \sum_{r=0}^{s} p_r \to \infty, \tag{4}$$

where $(P_{-s} = p_{-s} = 0, s \geq 1)$.

If σ_s defines the $(\overline{N}, p_s)$ mean [12], i.e.,

$$\sigma_s = \frac{1}{P_s} \sum_{q=0}^{s} p_q s_q, \quad P_s \neq 0, \quad s \in N \tag{5}$$

and $\lim_{s\to\infty} \sigma_s = s$, then $\sum a_s$ is s.t.b. $(\overline{N}, p_s)$ summable.

Further, if $\{\sigma_s\}$ is of bounded variation (BV) with $q \geq 1$ [1], i.e.,

$$\sum_{s=1}^{\infty} \left(\frac{P_s}{p_s}\right)^{q-1} |\sigma_s - \sigma_{s-1}|^q < \infty, \tag{6}$$

then $\sum a_s$ is s.t.b. $|\overline{N}, p_s|_q$ summable.

Let $U = (u_{lw})$ be a lower triangular matrix with diagonal terms nonzero, i.e., it is a normal matrix. Then, the transformation of sequence $s = \{s_l\}$ to $Us = \{U_l(s)\}$ by U is given by:

$$U_l(s) = \sum_{w=0}^{l} u_{lw} s_w, \quad n = 0, 1, \ldots \tag{7}$$

If

$$\sum_{l=1}^{\infty} \left(\frac{P_l}{p_l}\right)^{\delta q+q-1} |\overline{\Delta} U_l(s)|^q < \infty, \tag{8}$$

then $\sum a_l$ is s.t.b. $|U, p_l, \delta|_q$ summable [22], $q \geq 1$. Let $\{\varphi_l\}$ be of +ive real numbers. If

$$\sum_{l=1}^{\infty} \varphi_l^{q-1} |\overline{\Delta} U_l(s)|^q < \infty, \tag{9}$$

then $\sum a_l$ is s.t.b. $\varphi - |U, p_l|_q$ summable, $q \geq 1$ and $\overline{\Delta} U_l(s) = U_l(s) - U_{l-1}(s)$.

Taking $\varphi_l = P_l/p_l$ in condition (9), $\varphi - |U, p_l|_q$ changes to $|U, p_l|_q$ summability. Also, if we take $\delta = 0$ in condition (8), then $|U, p_l, \delta|_q$ changes to $|U, p_l|_q$ summability.

Now, we introduce other notations used in the main result as follows.

We are given with a normal matrix $U = (u_{lw})$. Two lower semi-matrices $\overline{U} = (\overline{u}_{lw})$ and $\hat{U} = (\hat{u}_{lw})$ are defined as:

$$\overline{u}_{lw} = \sum_{i=w}^{l} u_{li}; \quad l, w = 0, 1, 2, \ldots \tag{10}$$

and

$$\hat{u}_{00} = \overline{u}_{00} = u_{00}, \quad \hat{u}_{lw} = \overline{u}_{lw} - \overline{u}_{l-1,w}; \quad l = 1, 2, \ldots \tag{11}$$

Then, we have:

$$U_l(s) = \sum_{w=0}^{l} u_{lw} s_w = \sum_{w=0}^{l} \overline{u}_{lw} a_w \tag{12}$$

and

$$\overline{\Delta} U_l(s) = \sum_{w=0}^{l} \hat{u}_{lw} a_w. \tag{13}$$

2 Known Result

Özarslan and Yavuz [16] have proved the two theorems as given below.

2.1 Theorem [16]

Let $U = (u_{lw})$ be a +ive normal matrix with

$$\overline{u}_{l0} = 1, \quad \text{where } l = 0, 1, 2, \ldots \tag{14}$$

$$u_{l-1,w} \geq u_{lw} \quad \text{for } l \geq w + 1, \tag{15}$$

and

$$u_{ll} = \mathcal{O}\left(\frac{p_l}{P_l}\right). \tag{16}$$

Let $\{Y_l\}$ be quasi-β-power increasing and $\exists$ $\{B_l\}$ and $\{\lambda_l\}$ satisfying:

$$|\Delta\lambda_l| \leq B_l, \quad \forall\, l, \tag{17}$$

$$B_l \to 0 \quad \text{as} \quad l \to \infty, \tag{18}$$

$$\sum_{l=1}^{\infty} l|\Delta B_l| Y_l < \infty, \tag{19}$$

$$|\lambda_l| Y_l = \mathcal{O}(1), \tag{20}$$

$$(\lambda_l) \in BV, \tag{21}$$

$$\sum_{w=1}^{l} \frac{|z_w|^q}{w} = \mathcal{O}(Y_l), \tag{22}$$

and

$$\sum_{l=1}^{m} \left(\frac{p_l}{P_l}\right) |z_l|^q = \mathcal{O}(Y_m). \tag{23}$$

Then, $\sum a_l \lambda_l$ is $|U, p_l|_q$ summable, $q \geq 1$.

2.2 *Theorem [16]*

Let $U = (u_{lw})$ and $\{Y_l\}$ be as defined in Theorem 2.1, and satisfying (17)–(21), (23) with

$$\sum_{l=1}^{\infty} P_l |\Delta B_l| Y_l < \infty \tag{24}$$

and

$$\sum_{l=1}^{m} \frac{|z_l|^q}{P_l} = \mathcal{O}(Y_m). \tag{25}$$

Then, $\sum a_l \lambda_l$ is $|U, p_l|_q$ summable, $q \geq 1$.

3 Main Result

A sequence $\{\lambda_l\}$ is of bounded variation, if

$$\sum_{l=1}^{\infty} |\Delta\lambda_l| = |\lambda_l - \lambda_{l-1}| < \infty.$$

Our purpose is to generalize both the known results one by one by finding the least conditions. So, the theorems were stated as follows.

3.1 Theorem

Let the matrix U, $\{Y_l\}$ and $\{B_l\}$ be as defined in the known result such that the conditions (14)–(21) with

$$\sum_{w=1}^{l} \varphi_w^{q-1} \left(\frac{p_w}{P_w}\right)^{q-1} \frac{|z_w|^q}{w} = \mathcal{O}(Y_l), \tag{26}$$

$$\sum_{l=1}^{m} \varphi_l^{q-1} \left(\frac{p_l}{P_l}\right)^{q-1} |z_l|^q = \mathcal{O}(Y_m), \tag{27}$$

and

$$\sum_{l=w+1}^{m+1} \varphi_l^{q} \left(\frac{p_l}{P_l}\right)^{q} |\Delta_w \hat{u}_{lw}| = \mathcal{O}\left(\frac{p_w}{P_w}\right)^{q-1} \quad \text{as} \quad m \to \infty, \tag{28}$$

are satisfied. Then, $\sum a_l\lambda_l$ is $\varphi - |U, p_l|_q$ summable, $q \geq 1$.

3.2 Theorem

Let the matrix U, $\{Y_l\}$ and $\{B_l\}$ be as defined in the known result such that the conditions (14)–(21), (23)–(24) with

$$\sum_{l=1}^{m} \varphi_l^{q-1} \left(\frac{p_l}{P_l}\right)^{q-1} \frac{|z_l|^q}{P_l} = \mathcal{O}(Y_m) \quad \text{as} \quad m \to \infty, \tag{29}$$

are satisfied. Then, $\sum a_l\lambda_l$ is $\varphi - |U, p_l|_q$ summable, $q \geq 1$.

4 Lemmas

The following lemmas are needed to prove the main result.

4.1 *Lemma [13]*

By using the conditions of result 3.1, we have:

$$l B_l Y_l = \mathcal{O}(1) \quad \text{as} \quad l \to \infty \tag{30}$$

and

$$\sum_{l=1}^{\infty} B_l Y_l < \infty. \tag{31}$$

4.2 *Lemma*

Let $\{Y_l\}$ and $\{B_l\}$ be as defined in Theorem 3.2. Then, we have:

$$P_l B_l Y_l = \mathcal{O}(1) \tag{32}$$

and

$$\sum_{l=1}^{\infty} p_l Y_l B_l \ < \ \infty. \tag{33}$$

5 Proof of the Main Results

Now, we prove all these theorems one by one.

Proof of Theorem 3.1 Let K_l represents the A-transform of $\sum a_l \lambda_l$. Then, we obtain

$$\begin{aligned}
\overline{\Delta} K_l &= \sum_{w=0}^{l} \hat{u}_{lw} \lambda_w a_w \\
&= \sum_{w=1}^{l-1} \Delta_w \hat{u}_{lw} \lambda_w \sum_{k=1}^{w} a_k + \hat{u}_{ll} \lambda_l \sum_{w=1}^{l} a_w
\end{aligned}$$

$$\begin{aligned}
&= \sum_{w=1}^{l-1} (\hat{u}_{lw}\lambda_w - \hat{u}_{l,w+1}\lambda_{w+1}) z_w + u_{ll}\lambda_l z_l \\
&= \sum_{w=1}^{l-1} (\hat{u}_{lw}\lambda_w - \hat{u}_{l,w+1}\lambda_{w+1} - \hat{u}_{l,w+1}\lambda_w + \hat{u}_{l,w+1}\lambda_w) z_w + u_{ll}\lambda_l z_l \\
&= \sum_{w=1}^{l-1} \Delta_w(\hat{u}_{lw})\lambda_w z_w + \sum_{w=1}^{l-1} \hat{u}_{l,w+1}\lambda_w z_w + u_{ll}\lambda_l z_l \\
&= K_l^{(1)} + K_l^{(2)} + K_l^{(3)}. \qquad (34)
\end{aligned}$$

To prove the main result, it is enough to show that:

$$|K_l^{(1)} + K_l^{(2)} + K_l^{(3)}|^q \le 3^q \left(|K_l^{(1)}|^q + |K_l^{(2)}|^q + |K_l^{(3)}|^q \right),$$

then Eq. (34) reduces to:

$$\sum_{l=1}^{\infty} \varphi_l^{q-1} |K_l^{(v)}|^q = J_v < \infty \quad \text{for} \quad v = 1, 2, 3. \qquad (35)$$

Now, we have

$$\begin{aligned}
J_1 &= \mathcal{O}(1) \sum_{l=2}^{m+1} \varphi_l^{q-1} \times \left(\sum_{w=1}^{l-1} |\Delta_w(\hat{u}_{lw})||\lambda_w||z_w| \right)^q \\
&= \mathcal{O}(1) \sum_{l=2}^{m+1} \varphi_l^{q-1} \left(\sum_{w=1}^{l-1} |\Delta_w(\hat{u}_{lw})||\lambda_w|^q |z_w|^q \right) \times \left(\sum_{w=1}^{l-1} |\Delta_w(\hat{u}_{lw})| \right)^{q-1} \\
&= \mathcal{O}(1) \sum_{w=1}^{m} |\lambda_w||\lambda_w|^{q-1} |z_w|^q \times \sum_{l=w+1}^{m+1} \left(\frac{\varphi_l p_l}{P_l} \right)^{q-1} |\Delta_w(\hat{u}_{lw})| \\
&= \mathcal{O}(1) \sum_{w=1}^{m} \left(\frac{\varphi_w p_w}{P_w} \right)^{q-1} |\lambda_w||\lambda_w|^{q-1} |z_w|^q \times \sum_{l=w+1}^{m+1} |\Delta_w(\hat{u}_{lw})| \\
&= \mathcal{O}(1) \sum_{w=1}^{m} \varphi_w^{q-1} \left(\frac{p_w}{P_w} \right)^q |\lambda_w||z_w|^q \\
&= \mathcal{O}(1) \sum_{w=1}^{m-1} \Delta|\lambda_w| \sum_{i=1}^{w} \varphi_i^{q-1} \left(\frac{p_i}{P_i} \right)^q |z_i|^q \\
&\quad + \mathcal{O}(1)|\lambda_m| \sum_{w=1}^{m} \varphi_w^{q-1} \left(\frac{p_w}{P_w} \right)^q |z_w|^q
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}(1)\sum_{w=1}^{m-1} B_w Y_w + \mathcal{O}(1) Y_m |\lambda_m| \\
&= \mathcal{O}(1), \qquad (36)
\end{aligned}
$$

using the conditions of result 3.1 and Lemma 4.1.

Now, using $(\lambda_w) \in BV$, we have

$$
\begin{aligned}
J_2 &= \mathcal{O}(1)\sum_{l=2}^{m+1} \varphi_l^{q-1} \times \left(\sum_{w=1}^{l-1} |\Delta\lambda_w||(\hat{u}_{l,w+1})||z_w|\right)^q \\
&= \mathcal{O}(1)\sum_{l=2}^{m+1} \varphi_l^{q-1} \left(\sum_{w=1}^{l-1} |(\hat{u}_{l,w+1})||\Delta\lambda_w||z_w|^q\right) \\
&\quad \times \left(\sum_{w=1}^{l-1} |(\hat{u}_{l,w+1})||\Delta\lambda_w|\right)^{q-1} \\
&= \mathcal{O}(1)\sum_{l=2}^{m+1} \left(\frac{\varphi_l p_l}{P_l}\right)^{q-1} \times \left(\sum_{w=1}^{l-1} w|\Delta_w(\hat{u}_{lw})|B_w|z_w|^q\right) \\
&= \mathcal{O}(1)\sum_{w=1}^{m} wB_w|z_w|^q \sum_{l=w+1}^{m+1} \left(\frac{\varphi_l p_l}{P_l}\right)^{q-1} |\Delta_w(\hat{u}_{lw})| \\
&= \mathcal{O}(1)\sum_{w=1}^{m} vB_w|z_w|^q \left(\frac{\varphi_w p_w}{P_w}\right)^{q-1} \sum_{l=w+1}^{m+1} |\Delta_w(\hat{u}_{lw})| \\
&= \mathcal{O}(1)\sum_{w=1}^{m} wB_w\varphi_w^{q-1} \left(\frac{p_w}{P_w}\right)^{q} |z_w|^q \\
&= \mathcal{O}(1)\sum_{w=1}^{m-1} \Delta(wB_w) \sum_{i=1}^{w} \varphi_i^{q-1} \left(\frac{p_i}{P_i}\right)^{q} |z_i|^q \\
&\quad + \mathcal{O}(1) mB_m \sum_{w=1}^{m} \varphi_w^{q-1} \left(\frac{p_w}{P_w}\right)^{\delta q-1} \frac{|z_w|^q}{w} \\
&= \mathcal{O}(1)\sum_{w=1}^{m-1} w|\Delta B_w|Y_w + \mathcal{O}(1)\sum_{w=1}^{m-1} B_{w+1}Y_{w+1} \\
&\quad + \mathcal{O}(1) mB_m Y_m \\
&= \mathcal{O}(1), \qquad (37)
\end{aligned}
$$

using the conditions of result 3.1 and Lemma 4.1.

Again, using the concept in J_1, finally we have

$$\begin{aligned}
J_3 &= \mathcal{O}(1)\sum_{l=1}^{m}\varphi_l^{q-1}|u_{ll}|^q|\lambda_l|^q|z_l|^q\\
&= \mathcal{O}(1)\sum_{l=1}^{m}\varphi_l^{q-1}\left(\frac{p_l}{P_l}\right)^q|\lambda_l||\lambda_l|^{q-1}|z_l|^q\\
&= \mathcal{O}(1)\sum_{w=1}^{m}\varphi_w^{q-1}\left(\frac{p_w}{P_w}\right)^q|\lambda_w||z_w|^q\\
&= \mathcal{O}(1)\sum_{w=1}^{m-1}\Delta|\lambda_w|\sum_{i=1}^{w}\varphi_i^{q-1}\left(\frac{p_i}{P_i}\right)^q|z_i|^q\\
&\quad+\mathcal{O}(1)|\lambda_m|\sum_{w=1}^{m}\varphi_w^{q-1}\left(\frac{p_w}{P_w}\right)^q|z_w|^q\\
&= \mathcal{O}(1) \quad \text{as} \quad m\to\infty,
\end{aligned} \tag{38}$$

using the conditions of result 3.1 and Lemma 4.1.

So, collecting conditions (36)–(38), condition (35) holds.

Hence, Theorem 3.1 has been proved.

Proof of Theorem 3.2 Using Lemma 4.2 and doing as in Theorem 3.1 and put back $\sum_{v=1}^{m} B_v P_v\,(|z_v|^q/P_v)$ in place of $\sum_{v=1}^{m} B_v|z_v|^q$, Theorem 3.2 can be easily proved.

6 Corollaries

Here, we derive some new results and previous known results as follows.

Corollary 1 *If we take $\varphi_l = P_l/p_l$ in the main result 3.1 and 3.2, then known results can be easily derived.*

Corollary 2 *If we take $p_n = 1$ and in the main result $\varphi_l = P_l/p_l$ 3.1 and 3.2, then we get two new results on $|U|_q$ summability factors.*

Corollary 3 *Finally, if we take (Y_l) and $\varphi_l = P_l/p_l$ as almost increasing sequence in the main result 3.1 and 3.2, we get two different results on $|U, p_l|_q$ summability.*

7 Conclusion

The main objective is to get negligible set of equations for absolute matrix summability. Here, we get two generalized results on $\varphi - |U, p_l|_q$ summability, from which several previous results can be derived.

Acknowledgements The authors were highly thankful for the financial support to the Science and Engineering Research Board through Project No.: EEQ/2018/000393.

References

1. Bor, H.: On two summability methods. Math. Proc. Camb. Philos. Soc. **97**, 147–149 (1985)
2. Bor, H.: On the relative strength of two absolute summability methods. Proc. Am. Math. Soc. **113**, 1009–1012 (1991)
3. Bor, H.: A note on $|N, p_n|_k$ summability factors. Bull. Calcutta Math. Soc. **82**, 357–362 (1990)
4. Bor, H.: Absolute summability factors for infinite series. Math. Jpn. **36**, 215–219 (1991)
5. Bor, H.: Factors for $|N, p_n|_k$ summability of infinite series. Ann. Acad. Sci. Fenn. Ser. A 1 Math. **16**, 151–154 (1991)
6. Bor, H.: On absolute summability factors for $|N, p_n|_k$ summability. Comment. Math. Univ. Carol. **32**(3), 435–439 (1991)
7. Bor, H.: On the $|N, p_n|_k$ summability factors of infinite series. Proc. Indian Acad. Sci. **101**, 143–146 (1991)
8. Bor, H.: A note on $|N, p_n|_k$ summability factors. Rend. Mat. Appl. **12**(7), 937–942 (1992)
9. Bor, H.: On absolute summability factors. Proc. Am. Math. Soc. **118**, 71–75 (1993)
10. Bor, H.: On absolute Riesz summability factors. Rocky Mt. J. Math. **24**, 1263–1271 (1994)
11. Flett, T.M.: On an extension of absolute summability and some theorems of Littlewood and Paley. Proc. Lond. Math. Sci. **3**(1), 113–141 (1957)
12. Hardy, G.H.: Divergent Series. Oxford University Press, Oxford (1949)
13. Leindler, L.: A new application of quasi-power increasing sequences. Publ. Math. (Debr.) **58**, 791–796 (2001)
14. Özarslan, H.S., Keten, A.: On a new application of almost increasing sequences. J. Inequal. Appl. **2013**, 13 (2013)
15. Özarslan, H.S., Kartal, B.: A generalization of a theorem of Bor. J. Inequal. Appl. **2017**, 179 (2017)
16. Özarslan, H.S., Yavuz, E.: A new note on absolute matrix summability. J. Inequal. Appl. **2013**, 474 (2013)
17. Özarslan, H.S., Ari, T.: Absolute matrix summability methods. Appl. Math. Lett. **24**, 2102–2106 (2011)
18. Özarslan, H.S.: A new study on generalized absolute matrix summability methods. Maejo Int. J. Sci. Technol.
19. Özarslan, H.S.: A new study on generalized absolute matrix summability. Commun. Math. Appl. **7**(4), 303–309 (2016)
20. Özarslan, H.S.: An application of δ-quasi monotone sequence. Int. J. Appl. **14**(2), 134–139 (2017)
21. Özarslan, H.S., Sakar, M.Ö.: A new application of absolute matrix summability. Math. Sci. Appl. E-Notes **3**, 36–43 (2003)
22. Sulaiman, W.T.: Inclusion theorems for absolute matrix summability methods of an infinite series (IV). Indian J. Pure Appl. Math. **34**(11), 1547–1557 (2003)
23. Sonker, S., Jindal, R.: Approximation of signals by the triple product summability means of the Fourier series. Soft Comput. Theor. Appl. **425**, 169–179 (2022)
24. Sonker, S., Jena, B.B., Jindal, R., Paikray, S.K.: A generalized theorem on double absolute factorable matrix summability. Appl. Math. Inf. Sci. **16**(2), 315–322 (2022)

(C, 1, 1)-Quasinormal Convergence of Double Sequence of Functions

Smita Sonker and Priyanka

Abstract In this present paper, we have considered the extension of the concept of strong Cesàro-type quasinormal convergence, strong lacunary-quasinormal convergence, statistical-quasinormal convergence, lacunary-statistical-quasinormal convergence for double sequence of functions and derived some inclusive relation between these notions.

Keywords Statistical convergence · Quasinormal convergence · Lacunary strong quasinormal convergence · (C, 1, 1)-quasinormal convergence

1 Introduction

Fast [9] introduced a generalization of ordinary convergences of sequences named as statistical convergence. And later some basic properties of statistical convergence were established by Schoenberg [20] for real and complex sequences. Gökhan and Güngör [12] introduced the notion of pointwise statistical convergence. Also the notion of pointwise and uniform statistical convergence of α order, λ-statistical convergence of α order and pointwise lacunary statistical convergence of α order are studied in [4, 7, 8], respectively. The concept of statistical convergence of double sequences is introduced by Mursaleen and Edely in [17]. Let's define some basic definitions from the literature.

For a subset X of IN × IN, let $X(i, j)$ be the numbers of (k, l) in X such that $k \leq i$ and $l \leq j$. Then double natural density of $X \subset \text{IN} \times \text{IN}$ is defined as

$$\delta_2(X) = \lim_{i,j\to\infty} \frac{X(i, j)}{ij},$$

provided $\frac{X(i,j)}{ij}$ has a limit in Pringsheim's sense.

S. Sonker (✉) · Priyanka
National Institute of Technology Kurukshetra, Kurukshetra, Haryana 136119, India
e-mail: smitafma@nitkkr.ac.in

M. Sahni et al. (eds.), *Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy*, Advances in Intelligent Systems and Computing 1440,
https://doi.org/10.1007/978-981-19-9906-2_2

The sequence $\{x_{rs}\}$ of reals is known as statistically convergent [10] to the number x_0 if for each $\delta > 0$, the set

$$\delta_2(\{(r, s) : |x_{rs} - x_0| \geq \delta\}) = 0$$

A double sequence $\{g_{rs}(u)\}$ of functions is known as statistically convergent to $g(u)$ if for every $\delta > 0$ and $u \in E$,

$$\lim_{r,s\to\infty} \frac{1}{rs} \left|\{(i, j) : i \leq r, j \leq s, |g_{ij}(u) - g(u)| > \delta\}\right| = 0$$

A real sequence x_s is called Cesàro summable [11] to a number x_0 if

$$\lim_{s\to\infty} \frac{1}{s} \sum_{k=1}^{s} x_k = x_0$$

and we will write this as (C, 1)-lim $x_s = x_0$.

Bukovskà [1] introduced the notion of quasinormal convergence. Also the same concept with the name as equal convergence is introduced by Csàszàr and Laczkovich in [5, 6]. More details about this are given in [2, 3, 19]. In the present work, we extended the concept to double sequences of functions.

For the real valued mappings g_{rs}, g defined on a non empty set E, the double sequence $\{g_{rs}(u)\}$ is known as quasinormal convergent to $g(u)$ if we have a double sequence δ_{rs} of positive reals tending to 0 so that for every $u \in E$, $\exists\, n_0 = n_0(u)$ with

$$|g_{rs}(u) - g(u)| < \delta_{rs} \quad \forall\, n \geq n_0.$$

The uniform convergence of double sequence $\{g_{rs}\}$ to g implies the quasinormal convergence.

2 Preliminaries

From [18], we have the following results:

Theorem 1 *For real valued mappings g_k, g defined on a non empty set E, the following inclusive relation holds:*

1. *If $g_k(u)$ is strongly Cesàro convergent to $g(u)$, then $g_k(u)$ is statistically quasinormal convergent to $g(u)$.*
2. *If $g_k(u)$ is uniformly bounded and statistical-quasinormal convergent to $g(u)$, $g_k(u)$ is strongly Cesàro convergent to $g(u)$.*

Theorem 2 *For real valued functions g_k, g on a non empty domain E. θ be the lacunary sequence, we have*

1. *If $\{g_k(u)\}$ is strong lacunary convergent to $g(u)$, then $\{g_k(u)\}$ is lacunary-quasinormal statistical convergent to $g(u)$.*
2. *If $\{g_k(u)\}$ is uniformly bounded and statistically quasinormal convergent to $g(u)$, then $\{g_k(u)\}$ is lacunary strongly convergent to $g(u)$.*

3 Main Result

Throughout this paper let $g_{rs}(u)$, $g(u)$ be real valued mappings on a non empty set E and the limit of double sequences is taken in Pringsheim's sense.

3.1 (C, 1, 1)-Quasinormal Convergence of Double Sequence of Functions

As defined in [21], the sequence $\{g_{rs}(u)\}$ is called Cesàro (C, 1, 1)-summable to $g(u)$ if for every $u \in E$,

$$\lim_{r,s\to\infty} \frac{1}{rs} \sum_{k=1,l=1}^{r,s} g_{kl}(u) = g(u)$$

Also the sequence $\{g_{rs}(u)\}$ is known as strongly Cesàro (C, 1, 1)-summable to $g(u)$ if for $u \in E$,

$$\lim_{r,s\to\infty} \frac{1}{rs} \sum_{k=1,l=1}^{r,s} |g_{kl}(u) - g(u)| = 0$$

Definition 1 The double sequence $\{g_{rs}(u)\}$ is known as (C, 1, 1)-Quasinormal summable to $g(u)$ if we have a sequence $\{\delta_{rs}\}$ of positive real numbers tending to 0 as $r, s \to \infty$ such that for every $u \in E$, $\exists\, n_0 = n_0(u)$ with

$$\frac{1}{rs} \sum_{k=1,l=1}^{r,s} g_{kl}(u) - g(u) < \delta_{rs} \quad \forall\, r, s \geq n_0$$

Definition 2 The double sequence $\{g_{rs}(u)\}$ is known as strongly Cesàro (C, 1, 1)-quasinormal summable to $g(u)$ if there is a positive sequence $\{\delta_{rs}\}$ of real numbers tending to zero as $r, s \to \infty$ such that for every $u \in E$, $\exists\, n_0 = n_0(u)$ with

$$\frac{1}{rs} \sum_{k=1,l=1}^{r,s} |g_{kl}(u) - g(u)| < \delta_{rs} \quad \forall\, r, s \geq n_0.$$

3.2 Statistical-Quasinormal Convergence of Double Sequences of Functions

Definition 3 The double sequence $\{g_{rs}(u)\}$ is said to be statistical-quasinormal convergent to a real valued function $g(u)$ if there is a double sequence $\delta_{rs} \to 0$ (as $r, s \to \infty$) such that for each $u \in E$,

$$\lim_{r,s\to\infty} \frac{1}{rs} |\{(k,l) : k \leq r, l \leq s, |g_{kl}(u) - g(u)| \geq \delta_{kl}\}| = 0.$$

Let QNS^2 be the space of all statistical-quasinormal convergent double sequences of functions.

Theorem 3 *For the double sequence $\{g_{rs}(u)\}$, following relations holds*

1. *If $\{g_{rs}(u)\}$ is strongly Cesàro (C, 1, 1) convergent to $g(u)$, then $\{g_{rs}(u)\}$ is statistically quasinormal convergent to $g(u)$.*
2. *If $\{g_{rs}(u)\}$ is uniformly bounded and statistical-quasinormal convergent to $g(u)$, then $\{g_{rs}(u)\}$ is strongly Cesàro (C, 1, 1) convergent to $g(u)$.*

Proof 1. As $\{g_{rs}(u)\}$ is strongly Cesàro (C, 1, 1) convergent to $g(u)$, so for $\delta_{rs} \to 0$ and every u in E, we've

$$\begin{aligned}\frac{1}{rs}\sum_{p=1,q=1}^{r,s} |g_{pq}(u) - g(u)| &= \frac{1}{rs}\sum_{p=1,q=1,|g_{pq}(u)-g(u)|\geq\delta_{pq}}^{r,s} |g_{pq}(u) - g(u)| \\ &+ \frac{1}{rs}\sum_{p=1,q=1,|g_{pq}(u)-g(u)|<\delta_{pq}}^{r,s} |g_{pq}(u) - g(u)| \\ &\geq \frac{1}{rs}\sum_{p=1,q=1,|g_{pq}(u)-g(u)|\geq\delta_{pq}}^{r,s} \delta,\end{aligned}$$

where $\delta = \min\{\delta_{pq} : p \leq r, q \leq s\}$. Now

$$\begin{aligned}&\frac{1}{rs}\sum_{p=1,q=1}^{r,s} |g_{pq}(u) - g(u)| \\ &\quad = \frac{1}{rs}\left|\{(p,q) : p \leq r,\ q \leq s, |g_{pq}(u) - g(u)| \geq \delta_{pq}\}\right|\end{aligned}$$

Thus from strong (C, 1, 1) convergence of $\{g_{rs}(u)\}$, we have

$$\lim_{r,s\to\infty} \frac{1}{rs}\left|\{(p,q) : p \leq r, q \leq s, |g_{pq}(u) - g(u)| \geq \delta_{pq}\}\right| = 0$$

i.e. $\{g_{rs}(u)\}$ is statistically quasinormal convergent to $g(u)$.

2. Let $\{g_{rs}(u)\}$ be uniformly bounded and statistical-quasinormal convergent to $g(u)$. Since $\{g_{rs}(u)\}$ is bounded, so there is some constant K for which

$$|g_{rs}(u) - g(u)| \leq K$$

for all r, s and $\forall\ u \in E$

$$\begin{aligned}\frac{1}{rs}\sum_{p=1,q=1}^{r,s}|g_{pq}(u)-g(u)| &= \frac{1}{rs}\sum_{p=1,q=1,|g_{pq}(u)-g(u)|\geq\delta_{pq}}^{r,s}|g_{pq}(u)-g(u)| \\ &\quad + \frac{1}{rs}\sum_{p=1,q=1,|g_{pq}(u)-g(u)|<\delta_{pq}}^{r,s}|g_{pq}(u)-g(u)| \\ &\leq K\frac{1}{rs}|\{(p,q) : p \leq r, q \leq s, \\ &\quad |g_{pq}(u)-g(u)| \geq \delta_{pq}\}| \\ &\quad + \frac{1}{rs}\sum_{p=1,q=1}^{r,s}\delta_{pq}\end{aligned}$$

Taking the limit $r, s \to \infty$, we obtain

$$\lim_{r,s\to\infty}\frac{1}{rs}\sum_{p=1,q=1}^{r,s}|g_{pq}(u)-g(u)| = 0$$

i.e. $\{g_{rs}(u)\}$ is strongly Cesàro (C, 1, 1) convergent to g. ☐

3.3 Lacunary-Quasinormal-Statistical Convergence of Double Sequences of Functions

The concept of Lacunary strong summability and Lacunary statistical convergence of function sequences are discussed in [13–16]. An increasing sequence $\theta = \{k_r\}$ is known as if $\{k_r\}$ is increasing sequence such that $k_0 = 0$ and m_r tends to ∞ as r tends to ∞, where $m_r = k_r - k_{r-1}$. Let $\phi = \{l_s\}$ be another lacunary sequence. Denote $n_s = l_s - l_{s-1}$ and $I_r = (k_{r-1}, k_r]$ and $J_s = (l_{s-1}, l_s]$. θ and ϕ be lacunary sequences. Then the double sequence $\{g_{rs}(u)\}$ is called lacunary summable to the function g if

$$\lim_{r,s\to\infty}\frac{1}{m_r n_s}\sum_{k\in I_r, l\in J_s} g_{kl}(u) = g(u)$$

And the double sequence $\{g_{rs}(u)\}$ of functions is called strongly lacunary summable to g if