

Studies in Choice and Welfare

Michael A. Jones
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Delegate Apportionment in the US Presidential Primaries

A Mathematical Analysis

 Springer

Studies in Choice and Welfare

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Delegate Apportionment in the US Presidential Primaries

A Mathematical Analysis

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To Caroline, for her patience.

MAJ

To my mentors Susan Hermiller, John Meakin, and Brian Raines.

DM

To Loie, for pulling me away just the right amount.

JW

Preface

The purpose of this book is to examine the mathematics of the delegate allocation process in the US presidential primaries. The US presidential primaries are a series of state elections held every four years that determine the Democratic and Republican Parties' nominees for president in the general election. Presidential candidates in each party are awarded delegates based on their shares of the vote in each state contest. At the end of the primary season, each party holds a national convention. The candidate with a majority of state delegates, after one or more rounds of voting, is selected to be the party's nominee for president.

From a mathematical standpoint, the question of how to allocate delegates is largely a problem of apportionment. Apportionment problems arise when a fixed quantity (such as a number of delegates) must be divided among several constituents (such as candidates) in proportion to some attribute (such as vote share), so that each portion is a whole number. Since delegates are individuals and cannot be divided, their allocation is a matter of apportionment. Apportionment or similar allocation problems occur at multiple stages in the delegate selection process from determining how many delegates each state receives, and how they should be divided between statewide and district delegates, to determining what happens after each primary when the delegates must be awarded to the candidates based on how well they do in the election.

Apportionment has been well-studied in the context of allocating state representatives to the US House of Representatives in proportion to state populations. It has also been widely analyzed for its role in proportional representation systems where party seats are awarded to parties based on the vote distribution. It is less well-known in the case of apportioning delegates in the US primaries. Thus the examination of apportionment applied to delegate allocation involves both an identification and analysis of new apportionment methods as well as an evaluation of old and new criteria by which to evaluate their properties.

The goal of this book is to do both. We have spent several years researching delegate allocation in presidential primaries and were inspired to write this volume because of the interesting aspects of apportionment arising in this context. The book's

origin can be traced to 2016, when we observed that the 2016 Georgia Republican party adopted an apportionment method for its state primary that was unique and perhaps the most interesting method we had encountered. This motivated us to investigate the methods used by other state parties, many of which are new, and to consider how the allocation of delegates is affected by elimination thresholds—the minimum levels of support required to receive delegates—and the overall structure of the state delegates selection plans in which delegates are apportioned based on vote distributions in each district and statewide. This led us also to consider how the goals of delegate apportionment differ from that of house or party apportionment—most noticeably in that the state primaries occur over time, and that the end result is not political representation but the declaration of a winner. This means that the usual criteria for evaluating apportionment need to be rethought in an effort to determine what features of traditional apportionment theory are most relevant to delegate apportionment and which features are not.

This book is a work of mathematics, not of political science. We do not attempt to address questions of campaign strategy, conventional voting patterns, or the role of superdelegates at the national convention, etc. Instead, we study the mathematical rules by which delegates are allocated to presidential candidates, combining formal investigation with simulations and empirical research based on data from recent primaries. Our purpose is to provide a rigorous mathematical description and analysis of these rules, similar to the prior work that has been completed on house seat and party seat apportionment. Our hope is that the reader will find the delegate apportionment problem as interesting as we do, and ultimately will appreciate the complexity of the problem and the mathematical creativity of state and national party officials who grapple with how to fairly allocate delegates outside the academic traditions of apportionment theory.

Since the completion of this book in November of 2022, several changes have been proposed for the 2024 Democratic Party presidential primary. On February 3, 2023, the Democratic National Committee approved a reordering of the 2024 state primaries in which South Carolina will hold the first primary on February 3, 2024, followed by New Hampshire and Nevada on February 7, 2024, Georgia on February 13 and Michigan on February 29. (In a separate decision in June 2021, the Nevada Democratic Party voted to switch their contest from a caucus to a primary.) This new ordering represents a significant change from past presidential primaries in which the Iowa caucuses were the crucial first contest, closely followed by New Hampshire, and then Nevada and South Carolina. The intent of these changes is to ensure that, at least in South Carolina, the voters participating in the first contest(s) are more reflective of the diversity of the U.S. and representative of the Democratic Party as a whole.

The Democratic National Committee's Rules and Bylaws Committee has given the New Hampshire Democratic Party until early June to work out the details for the change in date. However, state officials in both Iowa and New Hampshire have indicated that they will continue to hold their primaries at the beginning of the season. There will, no doubt, be further changes both to the calendar and to the Delegate

Selection Plans which, for both parties, determine how delegates are allocated to presidential candidates leading to the selection of the parties' presidential nominee. The analysis in this book, therefore, provides a thorough investigation of past primaries and a starting point for the analysis of future elections.

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Abbreviations and Acronyms

CD	Congressional district
cedm	Close-election delegate minimum Definition 5.14 in Sect. 5.3.4
CEDM	Close-election delegate maximum Definition 5.14 in Sect. 5.3.4
DEM_DATA	Database of Democratic primary data Appendix B
DNC	Democratic National Committee
HAM	Hamilton’s method Definition 1.1 in Sect. 1.3
HAM*	Hamilton’s method, conceptualized using nearest integer rounding Definition 1.2 in Sect. 1.3
HNH	Hamilton-NIE hybrid method Definition 4.4 in Sect. 4.3.1
ILQ	Iterated lower quota method Definition 4.7 in Sect. 4.3.2
LAR	Large method Definition 4.6 in Sect. 4.3.2
LEAD	Leader criterion Definition 5.16 in Sect. 5.5
MAJ	Majority criterion Definition 5.15 in Sect. 5.5
NIE	Nearest Integer Extremes method Definition 4.1 in Sect. 4.3.1
NIS	Nearest Integer Sequential method Definition 4.2 in Sect. 4.3.1
PLEO	Party Leaders and Elected Officials A type of delegate in a Democratic primary, allocated based on the statewide vote distribution
REP_DATA	Database of Republican primary data Appendix B
RNC	Republican National Committee
SUQ	Sequential upper quota method Definition 4.8 in Sect. 4.3.3

Part I

Description of Delegate Allocation Rules

The subject of this book is the mathematics behind the delegate allocation process in the US presidential primaries. The primaries consist of a series of state elections held every four years that determine the Democratic Party and Republican Party nominees for president in the general election. Each party creates a delegate selection plan which determines how many delegates each state receives and how the delegates are allocated to each candidate after the state primary based on the candidates' shares of the vote. At the end of the process, each party holds a national convention. The candidate with a majority of state delegates is nominated at the convention to be the party's nominee for president.

The structure of the US presidential primaries has undergone intense scrutiny, both within the sphere of public debate and in the academic literature. Previous work has focused on the history and political consequences of the current primary system, the role of the primary calendar, the changing demographics of voters, the differences between primaries and caucuses, and the effect of the primaries on political campaigns and on the selection of winners. But there has been little detailed analysis of the actual rules that govern how delegates are awarded, rules that in modern primaries are almost entirely mathematical.

The allocation of delegates is dictated by each party's *delegate selection plan*, which determines how many delegates each state receives, and how they should be divided into the two categories of statewide and district delegates. It also provides guidance for how delegates should be awarded to the candidates after the election, based on the resulting votes. Significant differences exist between Democratic and Republican parties: the Democratic party prescribes uniform rules for each state primary while the Republican party allows each state to determine their own rules, within some general parameters. For both parties, the allocation of delegates at all stages is closely tied to apportionment. Thus, we analyze the delegate selection rules through the lens of apportionment methods (which, in a few cases, requires some interpretation), and we evaluate and compare the rules in comparison to how apportionment is applied in other contexts.

In Part I we lay out the landscape, providing a complete mathematical description for how delegates are allocated in the presidential primaries and identifying the

different apportionment methods used at various stages of the allocation process. Chapter 1 introduces the subject of delegate apportionment, including key definitions and notation, and provides some historical context for the selection of party nominees in the US primaries. We also discuss why the analysis of the apportionment methods used for delegate allocation may require different approaches from those used in other settings such as apportionment in the US House of Representatives.

In Chap. 2, we look at delegate allocation in the Democratic primary, describing how delegates are awarded at each stage of the process. We also present simplicial geometry as a means to visualize and to analyze apportionment methods. In Chap. 3, we describe delegate allocation in the Democratic caucuses, focusing particularly on the 2020 Iowa Democratic caucuses. The Iowa caucuses are especially interesting because it is the first state contest to occur in the primary season. However, the delegate selection process for the Iowa Democratic caucuses is also uniquely complex. For comparison, this chapter concludes with an examination of the 2020 Nevada Democratic caucuses, whose simpler rules more closely mirror those of a typical Democratic caucus election.

In Chap. 4, we turn our attention to how delegates are awarded in the Republican Party's state primaries, a process which differs significantly from that of the Democratic Party. Throughout Part I, we illustrate the delegate selection process using data from several recent primaries, especially those from 2016 and 2020. Part I is largely descriptive; in-depth analysis of the delegate allocation process is reserved for Part II.

Chapter 1

Apportionment in the US Presidential Primaries



The US presidential primaries are a series of state elections held every four years to determine the Democratic Party and Republican Party candidates for president. During the roughly nine-month period preceding the general presidential election in November, each state holds a primary or caucus in which voters select their preference for one of their party's presidential candidates. Based on the results of these contests, delegates are awarded to each candidate. The candidate with a majority of delegates at the end of the process is officially endorsed as the party's nominee for president at the party's national convention held during the summer prior to the November presidential election. If no candidate receives a majority of the delegates, a further process is enacted during the national conventions to determine the party's final nominee.

In this book, we examine how this delegate allocation process occurs, from the initial division of delegates among states, to the classification of state and local delegates, to the awarding of delegates to the presidential candidates. Throughout, we focus on the role that apportionment plays in the many stages of delegate allocation.

Apportionment problems arise any time a finite resource, such as a set of delegates, must be divided among several constituencies based on some criterion of proportionality, under the restriction that each share must be a nonnegative whole number. In primaries, apportionment problems occur when states are initially allocated delegates based on a combination of their population and previous voting patterns and when these delegates are then split among smaller geographic regions, such as congressional districts. Apportionment problems also arise after ballots are tallied for a state primary election, when candidates receive delegates based on their percentage of the vote share.

Mathematically, the allocation of delegates is similar to other well-known apportionment problems such as determining representation in the US House of Representatives or party seats in a legislative body using proportional representation. However, the implementation of apportionment methods for delegate allocation in the primaries, as well as the criteria by which methods should be evaluated, are sufficiently different to warrant a close examination.

In this chapter, we provide an introduction to the primary process. Section 1.1 gives a brief overview of how delegates are allocated in the US presidential primaries. In Sect. 1.2, the apportionment problem is described more generally, and some of the historical considerations for determining the number of representatives each state receives in the US House of Representatives are summarized. In Sect. 1.3, we characterize formally the apportionment problem and define several well-known apportionment methods. Section 1.4 provides a brief history of the delegate selection process in the US presidential primaries; this section also includes a description of how the primaries evolved into their current structure. In Sect. 1.5, we summarize some of the ways in which delegate apportionment raises different questions from apportionment in other contexts.

1.1 Apportionment and the Primaries

The modern system by which the Republican Party and Democratic Party select their presidential candidates dates roughly from the late 1960s. Prior to this, decisions were largely made during national conventions, where compromises were brokered among party officials and leaders who had influence over large blocs of state delegates (Coleman 2012). This process reached a head during the 1968 Democratic Convention when party officials nominated former Vice President and Minnesota Senator Hubert Humphrey despite wide-spread support for former Attorney General Robert Kennedy. The ensuing protests are largely credited with initiating widespread changes, leading ultimately to the system currently in place.

Since the 1960s, first the Democratic Party, and then the Republican Party, enacted a series of reforms designed broadly to increase public input into the candidate selection process while ensuring that the eventual nominee had the best chance of winning the general election. Overall, these reforms have had the effect of standardizing the timeline and the way in which states hold contests, and how delegates are selected. While some of these reforms concern internal party structure or procedural rules, many of them affect not only how delegates are allocated among states and districts before the election, but also how delegates are distributed among the candidates following the primary elections. The current delegate allocation process is a product of these reforms. While changes continue to be made, the last several primaries have been sufficiently stable to allow an analysis of the current rules.

Prior to the start of the primary season, the national parties determine how many delegates to award each state party. A state party's number of delegates is generally based on a combination of how many party members reside in the state, the number

of registered voters, and prior election results. In the Democratic primary, a portion of these delegates is allocated to each congressional district (CD) based on similar factors within the district. In the Republican primary, each CD automatically receives three delegates. After a state's primary or caucus is held, an apportionment method is used to award delegates to each candidate based on the proportion of the vote the candidate received. Delegates are awarded at both the state and district level. In the Republican primary, state parties decide on what apportionment method to use, often creating their own methods. In the Democratic primary, each state party uses the same apportionment method, but there is variability in how delegates are awarded in the caucuses.

Apportionment methods are widely studied for their roles in determining political and party representation. In the US, the 435 members of the House of Representatives are redistributed among the 50 states every ten years following the decennial census based on each state's share of the population. Apportionment methods are also used to allocate members of the European Parliament among member countries. Both of these examples illustrate how apportionment is used to determine representation across different geographic constituencies. In a different setting, apportionment methods are also used to allocate seats in legislative bodies among different political parties based on their shares of the vote. In this context, apportionment methods determine representation across different political constituencies.

The apportionment problems that arise in the presidential primaries share commonalities with these situations, but also exhibit differences. Historically, apportionment methods used to allocate seats in the US House of Representatives have been evaluated for their degree of *bias*—the degree to which they favor either large or small states. Observations from 200 years of data from the census have given rise to an understanding of different methods' bias, and their strengths and weaknesses more generally. The strengths include desirable properties, such as population monotonicity, where states which grow quickly in relative terms cannot lose seats to states that grow more slowly. The weaknesses are often framed as counterintuitive properties (or paradoxes), such as the Alabama paradox, where the number of House seats a state receives decreases despite the number of total representatives to be allocated increasing (while each state's population remains fixed).

In contrast to geographic-based apportionment, studies of apportionment for proportional representation have been less concerned with bias than with questions about how seat allocation impacts whether parties are able to form a majority government or to build stable party coalitions. In this situation, bias may be viewed favorably—whether in favor of large parties to encourage coalition building, or in favor of small parties to encourage a diversity of opinions. Apportionment methods are also compared based on the percentage of votes necessary or sufficient to receive a certain number of seats. This latter idea is embodied in the thresholds of inclusion (the vote shares below which a party cannot possibly receive a certain number of seats) and exclusion (the vote shares above which a party cannot fail to receive a certain number of seats). Other comparisons between methods are complicated by the fact that many proportional representation systems also employ a *cutoff*—a minimal percentage of votes a party must receive to have their votes count.

For the delegate allocation process in the presidential primaries, the first step of allocating delegates among the states is similar in spirit to apportionment in the US House of Representatives, with the proportionality criterion based not on state population but on compound measures related to the strength of the party in the state. In this step, a lack of bias is preferable. However, the final step in which delegates are allocated to candidates is more similar to apportionment in a legislative body, where bias is less of a concern. Similar to the case of party representation, bias may actually be desirable in delegate apportionment.

Bias toward stronger candidates in the primaries helps narrow the field of candidates and builds consensus in the selection of a party's final nominee, while bias toward weaker candidates encourages a broad range of party members' views. In addition, presidential primaries often apply cutoffs to keep inconsequential votes (sometimes for a candidate from another party) from affecting the calculations. Applications of apportionment methods to delegate allocation also have some unique features. One concerns the effects of aggregation, as when delegates are awarded to candidates in each CD in a state and then summed, rather than awarded in a single state-wide contest. Another feature is the use of apportionment at different points in the process, resulting in a composition or nesting of apportionment methods whose outcome may deviate greatly from true proportionality.

The evaluation of apportionment methods in presidential primaries also differs in their susceptibility to paradoxes. Indeed, many of the well-known paradoxes discovered when apportioning the US House of Representatives are less relevant when assigning delegates to candidates. In the context of delegate allocation, the Alabama paradox corresponds to the situation when an increase in the total number of delegates results in a candidate receiving fewer delegates. This situation does not arise because the number of delegates is fixed during the primary season. On the other hand, the incorporation of cutoffs for delegate allocation may cause some apportionment methods to suffer the elimination paradox—when the elimination of a weaker candidate results in a stronger candidate receiving fewer delegates. We explore these ideas further in Chap. 6.

A comprehensive historical and mathematical account of apportionment to the US House can be found in Balinski and Young (2001). A thorough analysis of apportionment in European parliaments is considered in Pukelsheim (2017). While the use of apportionment both in the US House of Representatives and in proportional representation has been well-studied, there has been little analysis of the applications of apportionment in the US presidential primaries or in the mathematical framework of delegate allocation more broadly.

Geist et al. (2010) examine the effect of using Hamilton's method in the Democratic Party Delegate Selection Rules. Jones, McCune, and Wilson look at the effect of cutoffs on the Democratic selection process (Jones et al. 2019) and analyze the methods used in the state Republican primaries (Jones et al. 2020). This book takes a more comprehensive approach, examining the allocation of delegates in both parties throughout the whole primary process.

1.2 What is the Apportionment Problem?

In the US, apportionment is best known for its role in determining the composition of the House of Representatives. The number of House seats has increased from 65 representatives in 1789 to the current 435 representatives, fixed as part of the Apportionment Act of 1911. This act set the cap at 433 seats but included increases of one seat each for when Alaska and Hawaii were granted statehood. Each state receives a number of representatives based on its proportion of the country's population. Mathematically, this is the same problem as allocating delegates to candidates based on their shares of the popular vote. In either case, an apportionment problem arises because the number of seats or delegates awarded must be integers.

There are multiple approaches to solving these apportionment problems. Because the question of congressional apportionment is familiar in the US, we use this setting to explore some of these approaches. House apportionment traces its origin to the following passage from Article I Section 2 of the US Constitution:

Representatives [to the US House] and direct taxes shall be apportioned among the several states which may be included in this union, according to their respective numbers . . .

In simpler terms, if a state has $p\%$ of the nation's population then that state should receive $p\%$ of the seats in the US House of Representatives. Proportional representation as a model for fair representation is seemingly universal: governmental structures across the world are built on this principle. But while the concept is intuitive and easy to understand, it is not always easy to implement. At the heart of the problem is that representation is usually based on individuals, who are indivisible.

To illustrate the kind of difficulties that can arise, consider the following example.

Example 1.1 Suppose a small country has 100 seats in its house of representatives and contains five states A , B , C , D , and E , whose populations are shown in Table 1.1, totaling 100,000 people. The second row identifies each state's *quota*—the percentage of a state's population multiplied by the total number of seats. A state's quota represents the number of seats that each state would receive in a purely proportional allocation if house seats were divisible. Given these quotas, how should the 100 seats be distributed to the states?

The intuitive solution to the question posed in Example 1.1 is to round each quota to the nearest integer. This leads to an apportionment of (54, 19, 17, 6, 3), resulting in an under allocation of 1 seat. Thus, apportioning seats are not merely a matter

Table 1.1 Example with 100 seats to apportion among 5 states with a total population of 100,000

State	A	B	C	D	E
Population	54,440	19,380	17,290	6370	2520
Quota	54.44	19.38	17.29	6.37	2.52

of standard rounding. What if we accepted the apportionment of (54, 19, 17, 6, 3), thereby changing the size of the house to 99? There are two problems with this solution. First, the size of the house is often prescribed by law. Second, and more importantly, if the house size changes, then the quotas do as well. In this example, if the house size were changed to 99 then the quota of state *E* falls to 2.4948, in which case standard rounding allocates only 2 seats to *E*, for a total of 98 seats. This kind of reasoning can result in a non-ending cycle where changing the house size to accommodate nearest integer rounding leads to a change in quotas which leads in turn to another potential change in the house size. Although this is probably not the reason why the size of the house is fixed in practice, it certainly supports that the house size should be fixed in apportionment theory.

Assuming then that we must allocate all 100 seats in Example 1.1, there are several possible apportionments that might be appropriate depending on the context including the following:

- (55, 19, 17, 6, 3): This apportionment would be advocated by Alexander Hamilton, who proposed a method to allocate seats in the first US House following the 1790 census. He stipulated that each state should be apportioned the integer part of its quota, and the remaining k seats should be allocated one at a time to the k states with largest fractional parts. In this example, rounding down the states' quotas allocates a total of 98 seats. States *A* and *E* have the two largest fractional parts and thus each is awarded an additional seat. As an added bonus, (55, 19, 17, 6, 3) is the integer lattice point in $\mathbb{R}^5$ that is closest to (54.44, 19.38, 17.29, 6.37, 2.52) in the standard Euclidean metric, and thus there is a natural distance argument for this apportionment because it is the closest apportionment to the vector of quotas. Hence, one could argue that (55, 19, 17, 6, 3) is the most fair or most proportional apportionment.
- (54, 19, 17, 7, 3): Beginning with each state receiving the integer value of their quotas, this allocation gives the two remaining seats one each to the states with the smallest populations. For state *A*, having 55 versus 54 seats probably does not make much of a difference in terms of the state's representation. For a smaller state like *E*, on the other hand, the difference between 2 and 3 seats creates a sizable difference in the state's quality of representation, an argument for smaller states to be given a boost at the expense of larger states.
- (55, 20, 17, 6, 2): As before, each state receives initially the integer value of its quota, leaving two seats to be allocated. In this allocation, the two remaining seats are awarded to the states with the largest populations, ensuring that the proportional advantage given to any state by rounding up is minimized. This approach may be reasonable if this country also has a senate-type body that operates like the US Senate where each state receives 2 senators regardless of size. Since that body has an explicit bias in favor of small states, it might make sense to favor the large states in the house of representatives as this apportionment does.
- (56, 19, 17, 6, 2): This apportionment takes a different approach, based on the idea that each representative should represent roughly $10,000/100 = 1000$ people. Thus, we consider each state's population divided by 1000. Of course these