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Geometry of the Unit Sphere in Polynomial Spaces

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Contents

1 Introduction ... 1

2 Polynomials of Degree *n* ... 5
- 2.1 On the Real Line ... 5
  - 2.1.1 Polynomials Bounded by a Majorant ... 7
- 2.2 On the Complex Plane ... 8

3 Spaces of Trinomials ... 11
- 3.1 On the Real Line with the Supremum Norm ... 11
  - 3.1.1 The Geometry of $\mathsf{B}_{m,n,\infty}$ for Odd Numbers m, n ... 12
  - 3.1.2 The Geometry of $\mathsf{B}_{m,n,\infty}$ for m Odd and n Even ... 17
  - 3.1.3 The Geometry of $\mathsf{B}_{m,n,\infty}$ for m Even and n Odd ... 18
  - 3.1.4 The Geometry of $\mathsf{B}_{m,n,\infty}$ for Even Numbers m, n ... 22
- 3.2 On the Real Line with the L_p-Norm ... 28
- 3.3 On the Real Plane ... 33
  - 3.3.1 The Geometry of $\mathsf{B}_{2n,n,\infty,2}$ for n Odd ... 34
  - 3.3.2 The Geometry of $\mathsf{B}_{2n,n,\infty,2}$ for n Even ... 34
  - 3.3.3 The Geometry of $\mathsf{B}_{m,n,\infty,2}$ for m Odd ... 34
- 3.4 On the Complex Plane ... 39

4 Polynomials on Non-Balanced Convex Bodies ... 43
- 4.1 The Simplex Δ ... 44
  - 4.1.1 Polynomials of Degree at Most 2 ... 47
- 4.2 The Unit Square ... 49
- 4.3 Circular Sectors ... 51
  - 4.3.1 The Geometry of $\mathsf{B}_{D(\beta)}$ When $\beta \geq \pi$ ... 52
  - 4.3.2 The Geometry of $\mathsf{B}_{D(\beta)}$ When $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ ... 54
  - 4.3.3 The General Case of $\mathsf{B}_{D(\beta)}$ ... 59

5 Sequence Banach Spaces ... 67
- 5.1 The Space ℓ_1^2 ... 67
- 5.2 The Space ℓ_∞^2 ... 70

5.3 The Space ℓ_p^2 when $1 < p < 2$ 73
5.4 The Space ℓ_p^2 when $2 < p < \infty$ 76
5.5 The Space c_0 79
5.6 The Space ℓ_1 82
5.7 The Space ℓ_p when $p > 2$ 84

6 Polynomials with the Hexagonal and Octagonal Norms 87
6.1 Octagonal Norm 87
6.2 Hexagonal Norm 89

7 Hilbert Spaces 93
7.1 The Real and Complex Case for 2-Homogeneous Polynomials 93
7.2 Polynomials of Degree n 97

8 Banach Spaces 99
8.1 Integral and Nuclear Polynomials 99
8.2 Orthogonally Additive Polynomials 101

9 Applications 105
9.1 Bernstein-Markov Type Inequalities 106
9.2 Polarization Constants 118
9.3 Unconditional Constants 121
9.4 Bohnenblust–Hille and Hardy–Littlewood Constants 124
9.4.1 On the Complex Case 128
9.4.2 On the Real Case 128

References 135

Chapter 1
Introduction

This book was completed after the passing of the first named author. The rest of authors would like to dedicate the book to the loving memory of their friend and colleague Jesús Ferrer (1952–2022).

The study and classification of the extreme points of the unit ball of a Banach space is a classical problem in functional analysis. This question is particularly interesting in the case of Banach spaces of polynomials. The case of integral, nuclear or orthogonally additive polynomials in Banach spaces have been studied, for instance, in [10, 11, 17, 20]. We devote Chap. 8 to show a selection of results where extreme integral, nuclear or orthogonally additive polynomials have been characterized in several different settings. As a matter of fact the geometry of the unit ball of polynomial spaces has been studied intensively for decades. Special attention has to be given to polynomial spaces of finite dimension. The case of polynomials on the real line of degree at most n endowed with the norm

$$\|P\| = \sup\{|P(x)| : x \in [-1, 1]\},$$

which we will represent by $\mathcal{P}_n(\mathbb{R})$, was solved by Konheim and Rivlin in [40] as early as in 1966 providing a characterization of the extreme polynomials of the unit ball B_n of $\mathcal{P}_n(\mathbb{R})$. The search for characterizations of the extreme polynomials of other finite dimensional polynomial spaces has been intensified since the late 90's of the twentieth century, motivating dozens of publications. In this paper we present a thorough revision of the most relevant results in this topic with special emphasis in the polynomial spaces of dimension 3. The fact that in dimension three we are able to provide a visual representation of the unit ball of a polynomial space is in itself a powerful tool in the study of the geometry of polynomial spaces. Although Konheim and Rivlin characterization of the extreme polynomials in B_n is not explicit, we will see in the next chapters that in many finite-dimensional Banach spaces of polynomials extreme polynomials can be fully described. Some

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representative examples of the spaces which have been studied so far are listed below:

- The subspaces $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R})$ of $\mathcal{P}_n(\mathbb{R})$ (see [4, 5]).
- The space of the quadratic polynomials on the complex plane with real coefficients, $\mathcal{P}_2(\mathbb{C})$, endowed with the sup norm over the unit disk $\mathbb{D}$ (see [5]).
- The subspace $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ $(m > n)$ of $\mathcal{P}_m(\mathbb{R})$ consisting of all the trinomials of the form $ax^m + bx^n + c$ (see [50]).
- The trinomials $ax^m + bx^n y^{m-n} + cy^m$ $(m > n)$ on $\mathbb{R}^2$, represented by $\mathcal{P}^h_{m,n,\infty}(\mathbb{R}^2)$, endowed with the sup norm on the unit ball of $\ell^2_\infty(\mathbb{R})$ (see [35]).
- The spaces of quadratic forms on $\ell^2_p(\mathbb{R})$ $(1 \le p \le \infty)$, namely $\mathcal{P}({}^2\ell^2_p)$, endowed with the sup norm over the unit ball of $\ell^2_p(\mathbb{R})$ (see for instance [14–16, 25–28]).
- The space $\mathcal{P}({}^3\ell^2_2)$, of 3-homogeneous polynomials on $\mathbb{R}^2$ endowed with the sup norm over the unit ball of $\ell^2_2(\mathbb{R})$ (see [29]).
- The spaces $\mathcal{P}({}^2\Delta)$ and $\mathcal{P}({}^2\square)$ of the quadratic forms on $\mathbb{R}^2$ endowed with the sup norm over the simplex Δ and the square $\square = [0, 1]^2$ respectively (see [23, 46]).
- The space $\mathcal{P}_2(\Delta)$ of polynomials of degree at most 2 on $\mathbb{R}^2$ endowed with the supremum norm over the simplex Δ (see[43]).
- The space $\mathcal{P}({}^2D(\alpha, \beta))$ with $\alpha \le \beta$ (see [6, 45]) of the quadratic forms on $\mathbb{R}^2$ endowed with the sup norm on the sectors
$$D(\alpha, \beta) = \{re^{i\theta} : r \in [0, 1] \text{ and } \theta \in [\alpha, \beta]\}.$$
- The space $\mathcal{P}({}^2\mathcal{O}^2_w)$ of the quadratic forms on $\mathbb{R}^2$ endowed with the norm
$$\|P\|_{\mathcal{O}^2_w} = \sup\{|P(x, y)| \colon \|(x, y)\|_{\mathrm{oct}(w)} \le 1\},$$
where
$$\|(x, y)\|_{\mathrm{oct}(w)} = \max\left\{|x|, |y|, \frac{|x| + |y|}{1 + w}\right\}$$
for a fixed $w \in [0, 1]$ (see [38]).

Having an explicit description of the extreme points of the unit ball of a polynomial space has many interesting applications. The Krein-Milman approach allows us to prove many sharp polynomial inequalities. Recall that, as a direct consequence of the Krein-Milman theorem, any convex function on a convex body of a finite dimensional Banach space attains its maximum at an extreme point. Using this idea combined with a description of the extreme points of a polynomial space one can derive a number of polynomial inequalities. Sharp Bernstein and Markov inequalities are among the applications of the Krein-Milman approach. Other problems of interest where the geometry of the unit ball of polynomial spaces yield excellent results are the calculation of exact unconditional constants in polynomial spaces, the calculation of polarization constants or the calculation of sharp Bohnenblust-Hile and Hardy-Littlewood constants. Chapter 9 is devoted to present a selection of the many achievements that can be obtained by using the Krein-Milman approach.

In this book we pursuit three main achievements. The first is to provide the reader with a visual perspective of each of the Banach spaces of polynomials we study by representing their unit spheres. To this end the following steps are implemented in most of the cases:

1. First we give an explicit formula to calculate the polynomial norm.
2. Then we parametrize the unit sphere of the space, for which it might be of help to calculate the projection of the unit ball onto a plane.
3. The parametrization of the unit sphere is a valuable source of information that allows us to identify and classify the extreme points of the unit ball of each polynomial space.

The third point above accomplishes the second of the main objective of this monograph, providing the reader with explicit characterizations of the extreme polynomials in several Banach spaces of polynomials. The third objective is to highlight the many applications of having an explicit classification of the extreme points of the unit ball of a space of polynomials. In particular, we will show a number of interesting sharp Bernstein and Markov type inequalities and Bohnenblust-Hille inequalities obtained using the already mentioned Krein-Milman approach. We can also obtain exact unconditional constants, polarization constants and other related results.

This book is arranged as follows: In Chap. 2 we study the spaces $\mathcal{P}_n(\mathbb{R})$ with $n = 2, 3$. Polynomials with majorants are also considered. In Chap. 3 we study several spaces of trinomials including $\mathcal{P}_{m,n}(\mathbb{R})$ and $\mathcal{P}^h_{m,n}(\mathbb{R}^2)$ $(m > n)$ defined above, but also other related problems. Trinomials with the L_p-norm or trinomials on the complex plane are studied as well. In Chap. 4 we consider several polynomial spaces where the norm is calculated as the supremum over a non-symmetric convex body. In particular, Chap. 4 comprises the spaces $\mathcal{P}(^2\Delta)$, $\mathcal{P}(^2\square)$ and $\mathcal{P}(^2D(\alpha, \beta))$. In Chap. 5 we treat the case of polynomials defined on several ℓ_p-spaces. More specifically we investigate the spaces $\mathcal{P}(^2\ell_p^2)$ for all $p \in [1, \infty]$ and the spaces of quadratic forms on c_0, ℓ_1 and ℓ_2 for $p > 2$. In Chap. 6 we consider the space of quadratic forms in $\mathbb{R}^2$ with the sup norm over an octagon, represented as $\mathcal{P}(^2O_w^2)$ above, and with the sup norm over the hexagon defined by

$$\|(x, y)\|_{\mathrm{hex}(w)} := \max\{|y|, |x| + (1 - w)|y|\} = 1$$

for $w \in [0, 1]$. In Chap. 7 we study polynomials on real or complex Hilbert spaces. In Chap. 8 extreme integral, nuclear or orthogonally additive polynomials are regarded. Finally, in Chap. 9 we gather a number of applications of the geometrical results included in Chaps. 2–8.

Chapter 2
Polynomials of Degree *n*

Abstract This chapter focuses on the study of the geometry of the unit ball of the space of polynomials in one variable of degree at most $n \in \mathbb{N}$ endowed with the supremum norm defined on the interval $[-1, 1]$ (when the polynomial is defined over $\mathbb{R}$) or on the unit disk (when the polynomial is defined over $\mathbb{C}$). More precisely, we are interested on the parametrization of the unit ball as well as the extreme points when we are dealing with the space of polynomials of degree at most 2. For the space of polynomials of arbitrary degree with the supremum norm defined on $[-1, 1]$, we are only interested on the extreme polynomials of the unit ball.

2.1 On the Real Line

Let us endow the vector space of real polynomials of the degree at most $n \in \mathbb{N}$, that is, of the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$ where $a_i \in \mathbb{R}$ for every $i \in \{1, \ldots, n\}$ and $x \in \mathbb{R}$, with the supremum norm

$$\|P\|_{\mathbb{R}} = \max\{|P(x)| : x \in [-1, 1]\}.$$

We denote this normed space by $\mathcal{P}_n(\mathbb{R})$. Now consider the following construction: let us define the mapping T from $\mathcal{P}_n(\mathbb{R})$ to $\mathbb{R}^{n+1}$ that assigns to each polynomial $a_n x^n + \cdots + a_1 x + a_0$ the vector $(a_n, \ldots, a_1, a_0)$, i.e., each polynomial is mapped into the vector formed by its coefficients. This mapping T is a topological isomorphism between $\mathcal{P}_n(\mathbb{R})$ and $\mathbb{R}^{n+1}$ when we endow $\mathbb{R}^{n+1}$ with the norm

$$\|(a_n, \ldots, a_1, a_0)\|_{\mathbb{R}} := \|a_n x^n + \cdots + a_1 x + a_0\|_{\mathbb{R}}.$$

Let us denote the unit ball and the unit sphere of $(\mathbb{R}^{n+1}, \|\cdot\|_{\mathbb{R}})$ by $\mathsf{B}_n(\mathbb{R})$ and $\mathsf{S}_n(\mathbb{R})$, respectively. Thus, in particular, on the space $\mathcal{P}_2(\mathbb{R})$, we can give a visual representation of the unit ball.

The geometry of $\mathcal{P}_n(\mathbb{R})$ was already studied by A. G. Konheim and T. J. Rivlin in 1966 [40]. They were able to characterize when a polynomial of degree at most $n \in$

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$\mathbb{N}$ that belongs to the unit ball is an extreme polynomial based on the multiplicity of intersection of the polynomial with 1 and -1.

Definition 2.1 Let P be a real polynomial of degree at most n. We denote by $N(P, y)$ the total multiplicity with which the value y is assumed by P and, in particular, let us define the multiplicity of P by the number $N(P) := N(P, 1) + N(P, -1)$.

Theorem 2.1 (Konheim and Rivlin [40]) *Let $P \in \mathcal{P}_n(\mathbb{R})$ with $\|P\| \leq 1$. We have that P is an extreme polynomial if, and only if, $N(P) > n$.*

Although Konheim and Rivlin gave a characterization of the extreme polynomials of the unit ball of $\mathcal{P}_n(\mathbb{R})$, they do not give an explicit formula for the values of the extreme polynomials. However, R. M. Aron and M. Klimek [5] were able to obtain an explicit formula for the extreme polynomials in the unit ball of $\mathcal{P}_2(\mathbb{R})$ by using an approach that will appear in many results of this survey. Firstly, they gave an explicit formula for the norm of a polynomial of degree at most 2. Secondly, they found the projection of the unit ball onto a plane. And finally, using this information, they were able to parametrize the unit ball and, in the process, find the extreme polynomials of the unit ball. The results that Aron and Klimek provided are shown below.

Theorem 2.2 (Aron and Klimek [5]) *Let $P(x) = ax^2 + bx + c$. We have*

$$\|(a, b, c)\|_{\mathbb{R}} = \begin{cases} \left|\frac{b^2}{4a} - c\right| & \text{if } |b| < 2|a|t \text{ and } \frac{c}{a} + 1 < \frac{1}{2}\left(\left|\frac{b}{2a}\right| - 1\right)^2, \\ |a + c| + |b| & \text{otherwise.} \end{cases}$$

Let us define the sets

$$\begin{aligned} U &= \left\{(a, b) \in \mathbb{R}^2 \colon a \leq 0 \text{ and } |b| \leq \min\left\{2|a|, 2\left(\sqrt{2|a|} - |a|\right)\right\}\right\}, \\ V &= \left\{(a, b) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1] \colon |b| \geq 2|a|\right\}, \\ W &= \left\{(a, b) \in \mathbb{R}^2 \colon a \geq 0 \text{ and } |b| \leq \min\left\{2|a|, 2\left(\sqrt{2|a|} - |a|\right)\right\}\right\}. \end{aligned}$$

Theorem 2.3 (Aron and Klimek [5]) *The projection of $\mathsf{B}_2(\mathbb{R})$ onto the ab-plane is the set $U \cup V \cup W$ (see Fig. 3.5 for a representation of $U \cup V \cup W$ with $n = 1$).*

Theorem 2.4 (Aron and Klimek [5]) *Let us define the functions*

$$\begin{aligned} f_+(a, b) &= 1 - |b| - |a|, \\ g_+(a, b) &= \frac{b^2}{4a} - 1, \end{aligned}$$

and also the functions $f_-(a, b) = -f_+(-a, b)$ and $g_-(a, b) = -g_+(-a, b)$. We have *(i)* $\mathsf{S}_2(\mathbb{R}) = \operatorname{graph}\left(f_+|_{(V\cup W)}\right) \cup \operatorname{graph}\left(f_-|_{(U\cup V)}\right) \cup \operatorname{graph}(g_+|_W) \cup \operatorname{graph}(g_-|_U)$ *(see Fig. 3.6 for a representation of* $\mathsf{B}_2(\mathbb{R})$ *with* $n = 1$*).*

(ii) The set of extreme points (denoted by ext*) is*

$$\operatorname{ext}(\mathsf{B}_2(\mathbb{R})) = \left\{\pm\left(t, \pm 2(\sqrt{2t} - t), 1 + t - 2\sqrt{2t}\right) : t \in \left[\frac{1}{2}, 2\right]\right\} \bigcup \{\pm(0, 0, 1)\}.$$

The following results of this section are devoted to the study of extreme polynomials of degree at most 3.

Theorem 2.5 (Araújo et al. [4]) *The extreme polynomials of the unit ball of* $\mathcal{P}_3(\mathbb{R})$ *are given by*

(i) $P_1(x) = \pm 1$*;*

(ii) $P_2(x) = \pm\left[1 - \frac{1}{4}(\pm x + 1)^3\right]$*;*

(iii) $P_3(x) = \pm(2x^2 - 1)$*;*

(iv) $P_4(x) = \pm\left[1 - \frac{1}{(1-q^2)^2}(x - q)^2(4qx + 2 + 2q^2)\right]$ *and* $P_5(x) = \pm\left[1 + \frac{1}{(1-q^2)^2}(x + q)^2(4qx - 2 - 2q^2)\right]$*, for every* $q \in \left(-\frac{1}{3}, 0\right)$*;*

(v) $P_6(x) = \pm\left[1 + \frac{1}{(1+t)^2}(x - t)^2(x - 1)\right]$ *and* $P_7(x) = \pm\left[1 - \frac{1}{(1+t)^2}(x + t)^2(x + 1)\right]$*, for every* $t \in \left(-\frac{1}{2}, 1\right)$*;*

(vi) $P_8(x) = \pm\left[1 + \frac{4}{(s-r)^3}(x - r)^2\left(x - \frac{3s-r}{2}\right)\right]$ *and* $P_9(x) = \pm\left[1 - \frac{4}{(s-r)^3}(x + r)^2\left(x + \frac{3s-r}{2}\right)\right]$*, for every* $-1 \le r < s \le 1$ *such that* $s \ge \min\left\{3r + 2, \frac{r+2}{3}\right\}$*.*

2.1.1 Polynomials Bounded by a Majorant

Assume that P is a polynomial of degree at most n such that P is constrained on the interval $[-1, 1]$ by a mapping $\varphi\colon [-1, 1] \to [0, +\infty)$ called the majorant, i.e., $|P(x)| \le \varphi(x)$ for every $x \in [-1, 1]$. We will denote by $\mathcal{P}_n^{\varphi}(\mathbb{R})$ the space of polynomials on the real line of degree at most n that are bounded by a majorant φ endowed with the supremum norm over the interval $[-1, 1]$. In this section we are interested in studying the extreme points of the unit ball of the space $\mathcal{P}_3^{\varphi}(\mathbb{R})$ when φ is a circular majorant, that is, $\varphi(x) = \sqrt{1 - x^2}$ for any $x \in [-1, 1]$.

Notice that if a polynomial P belongs to $\mathcal{P}_3^{\varphi}(\mathbb{R})$, where φ is a circular majorant, then P has roots at ± 1. Hence all polynomials of degree not greater than 3 bounded by a circular majorant are of the form $P_{a,b}(x) = (1-x^2)(ax+b)$ for some $a, b \in \mathbb{R}$. Thus, in fact, we have the following inequality $|(1-x^2)(ax+b)| \le \sqrt{1 - x^2}$ for any

$x \in [-1,1]$, which is equivalent to $\left|\sqrt{1-x^2}(ax+b)\right| \le 1$ for any $x \in [-1,1]$. The latter shows that we can study the unit ball of the space $\mathcal{P}_3^{\varphi}(\mathbb{R})$, when φ is a circular majorant, by studying the unit ball of the norm space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$, where

$$\|(a,b)\|_{\infty,\varphi} = \sup\left\{\left|\sqrt{1-x^2}(ax+b)\right| : x \in [-1,1]\right\}.$$

We begin by showing an explicit formula for the norm $\|\cdot\|_{\infty,\varphi}$.

Theorem 2.6 (Muñoz et al. [47]) *If $\varphi\colon [-1,1] \to [0,+\infty)$ is defined by $\varphi(x) = \sqrt{1-x^2}$, then for every $(a,b) \in \mathbb{R}^2$ we have*

$$\|(a,b)\|_{\infty,\varphi} = \begin{cases} \frac{(3|b|+\sqrt{8a^2+b^2})\sqrt{4a^2-b^2+|b|\sqrt{8a^2+b^2}}}{8\sqrt{2}|a|} & \text{if } a \neq 0, \\ |b| & \text{if } a = 0. \end{cases}$$

As an easy consequence of Theorem 2.6 we have the following characterization of the unit ball of $\mathcal{P}_3^{\varphi}(\mathbb{R})$.

Theorem 2.7 (Muñoz et al. [47]) *Let $\varphi\colon [-1,1] \to [0,+\infty)$ be defined by $\varphi(x) = \sqrt{1-x^2}$. If $(a,b) \in \mathbb{R}^2$, then $\|(a,b)\|_{\infty,\varphi} \le 1$ if, and only if,*

$$\left(\sqrt{8a^2+b^2} + 3|b|\right)^3 \le 32\left(\sqrt{8a^2+b^2} + |b|\right),$$

where equality is satisfied if, and only if, $\|(a,b)\|_{\infty,\varphi} = 1$. Moreover, the set of extreme points of the unit ball of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$ are the points of the unit sphere.

Figure 2.1 shows an approximate representation of the unit sphere of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$.

2.2 On the Complex Plane

Let us consider now the vector space of complex polynomials with real coefficients of degree at most $n \in \mathbb{N}$, that is, we have polynomials of the form $P(z) = a_n z^n + \ldots + a_1 z + a_0$ where $a_i \in \mathbb{R}$ and $z \in \mathbb{C}$, endowed with the following norm

$$\|P\|_{\mathbb{C}} = \sup_{|z|\le 1} |P(z)|.$$

We denote this normed space by $\mathcal{P}_{\mathbb{R},n}(\mathbb{C})$. Using the mapping T defined on Sect. 2.1, there is a topological isomorphism between the space $\mathcal{P}_{\mathbb{R},n}(\mathbb{C})$ and $\mathbb{R}^{n+1}$ endowed with the norm