Bhargav Bhatt
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p-adic Hodge Theory, Singular Varieties, and Non-Abelian Aspects

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Preface

This volume contains research articles related to the 2019 Simons Symposium on p-adic Hodge theory. To explain its context, recall that classical formulations of p-adic Hodge theory concern comparisons of cohomology with constant coefficients for smooth proper varieties. As in complex Hodge theory, a more refined understanding can be obtained by investigating the cohomology for varieties which are possibly non-compact or singular, considering cohomology with non-trivial coefficients, and contemplating non-abelian aspects.

In recent years, there have been a number of structural developments that have helped make progress on these directions in p-adic Hodge theory. These include the development of new techniques, such as prismatic cohomology and the saturated de Rham–Witt complex, and the systematic use of adic and perfectoid spaces. The goal of this symposium was to understand some of these developments.

The first article, by Abbes and Gros, gives an overview of their recent work on the Hodge–Tate spectral sequence. This spectral sequence, first introduced in the work of Faltings and later revisited by Scholze, is a key tool in p-adic Hodge theory akin to the Hodge–de Rham spectral sequence in complex geometry. Abbes and Gros develop a relative version of this spectral sequence using the Faltings topos, and a relative variant they develop.

The article of Betts and Litt studies weight-monodromy conjecture for the pro-unipotent completion of the étale fundamental group. This is a certain pro-algebraic completion of the étale fundamental group. While still capturing nonabelian information, it has many properties similar to cohomology; in particular, one can study p-adic Hodge theory for these groups. In this article, the authors prove both weight-monodromy for the fundamental group and the semisimplicity of the Frobenius action.

One feature of modern p-adic Hodge theory, especially after the advent of perfectoid spaces, is that Huber's adic spaces play a key role, akin to the role played by complex analytic spaces in complex geometry. It is thus important to develop many tools in this context. The third article is concerned with a number of foundational problems in developing the theory of log geometry, in the sense of Fontaine, Illusie, and Kato, in the context of adic spaces. This is important for several reasons including the study of open varieties and semistable degenerations. Among the contributions of this article is the development of the Kummer étale and pro-Kummer étale topologies of log adic spaces as well as the primitive comparison theorem in this context.

In recent years, the introduction of prismatic cohomology has substantially improved our understanding of integral p-adic Hodge theory. The formalism of the prismatic site provides not only cohomology but also some natural coefficient systems called crystals. The fourth article, by Gros–Le Stum–Quirós, contributes to our understanding of prismatic crystals: they construct a functor from the category of q-crystals (a slight variant of the notion of a prismatic crystal) to a category of modules of a certain ring of twisted differential operators. One feature of this construction is that it gives a more explicit local description of q-crystals.

The fifth article in the volume by Lurie revisits the moduli spaces of elliptic curves with level structure; a classical topic in arithmetic geometry. It was shown by Scholze that the tower of such moduli spaces naturally defines a perfectoid space over the perfectoid field obtained by adjoining the p-th power roots of unity to $\mathbf{Q}_p$. In the fifth article, Lurie proves an integral version of this statement. An interesting consequence of this is a moduli interpretation of the tilt of the perfectoid space constructed by Scholze.

The book concludes with an article by Ogus on the saturated de Rham–Witt complex, originally introduced by Bhatt, Lurie, and Mathew. In the case of smooth varieties, this complex coincides with the classical de Rham–Witt complex studied by Illusie and others, but in the case of singular varieties it is a new object. Ogus studies the natural question of what information is captured by the saturated de Rham–Witt complex in the case of varieties with toric singularities—a natural first case of mildly singular varieties to consider. Ogus shows that in this case the saturated de Rham–Witt complex has good properties and gives a p-adic version of a complex already familiar in toric geometry mod p.

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The Relative Hodge–Tate Spectral Sequence: An Overview

Ahmed Abbes and Michel Gros

1 Introduction

1.1. Let K be a complete discrete valuation field of characteristic 0, with *algebraically closed* residue field of characteristic $p > 0$, $\mathscr{O}_K$ the valuation ring of K, $\overline{K}$ an algebraic closure of K, $\mathscr{O}_{\overline{K}}$ the integral closure of $\mathscr{O}_K$ in $\overline{K}$. We denote by G_K the Galois group of $\overline{K}$ over K, by $\mathscr{O}_C$ the p-adic completion of $\mathscr{O}_{\overline{K}}$, by $\mathfrak{m}_C$ the maximal ideal of $\mathscr{O}_C$ and by C its field of fractions. We set $S = \mathrm{Spec}(\mathscr{O}_K)$ and $\overline{S} = \mathrm{Spec}(\mathscr{O}_{\overline{K}})$ and we denote by s (resp. η, resp. $\overline{\eta}$) the closed point of S (resp. generic point of S, resp. generic point of $\overline{S}$). For any integer $n \geq 0$, we set $S_n = \mathrm{Spec}(\mathscr{O}_K/p^n\mathscr{O}_K)$. For any S-scheme X, we set

$$\overline{X} = X \times_S \overline{S} \quad \text{and} \quad X_n = X \times_S S_n. \tag{1}$$

The following statement, called the *Hodge–Tate decomposition*, was conjectured by Tate ([16] Remark page 180) and proved independently by Faltings [8, 9] and Tsuji [17, 18].

Theorem 1.2 *For any proper and smooth η-scheme X and any integer $n \geq 0$, there exists a canonical functorial G_K-equivariant decomposition*

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$$\mathrm{H}^n_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \xrightarrow{\sim} \bigoplus_{i=0}^{n} \mathrm{H}^i(X, \Omega^{n-i}_{X/\eta}) \otimes_K C(i-n). \tag{2}$$

The Hodge–Tate decomposition is equivalent to the existence of a canonical functorial G_K-equivariant spectral sequence, the *Hodge–Tate spectral sequence*,

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(X, \Omega^j_{X/\eta}) \otimes_K C(-j) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C. \tag{3}$$

The two statements are equivalent by a theorem of Tate ([16] theo. 2). Indeed, the cohomology group $\mathrm{H}^0(G_K, C(1))$ vanishes, which implies that the spectral sequence degenerates at E_2. The cohomology group $\mathrm{H}^1(G_K, C(1))$ also vanishes, which implies that the abutment filtration splits.

This Hodge–Tate spectral sequence, which one can guess implicitly in the work of Faltings [9], has been explicitly formulated only later by Scholze [14].

1.3. We give in this note an overview of a recent work [2] leading to a generalization of the Hodge–Tate spectral sequence to morphisms. The latter takes place in Faltings topos. Its construction requires the introduction of a relative variant of this topos which is the main novelty of our work. Using a different approach, Caraiani and Scholze ([7] 2.2.4) have constructed a relative Hodge–Tate filtration for proper smooth morphisms of adic spaces. Hyodo has also considered earlier a special case for abelian schemes [11].

Beyond the Hodge–Tate spectral sequences, we give in [2] complete proofs of Faltings' main p-adic comparison theorems. The latter are essential in the construction of these spectral sequences. Although the absolute version of these theorems is rather well-understood, the relative version which was very roughly sketched by Faltings in the appendix of [9], has remained little studied. Scholze has proved similar results ([15] 1.3 and 5.12) in his setting of adic spaces and pro-étale topos.

In a work in progress, we extend the relative Hodge–Tate spectral sequence to more general coefficients in relation with the p-adic Simpson correspondence [3]. This sheds new lights on the functoriality of the p-adic Simpson correspondence by proper (log)smooth pushforward (for a related result see the work of Liu and Zhu [13]).

2 The Local Version of the Relative Hodge–Tate Spectral Sequence

2.1. Let $X = \mathrm{Spec}(R)$ be an affine smooth[1] S-scheme satisfying the following two conditions:

[1] We treat in [2] schemes with toric singularities using logarithmic geometry, but for simplicity, we consider in this overview only the smooth case.

(i) X is *small* in the sense of Faltings, that is to say it admits an étale S-morphism to a S-torus, $X \to \mathbb{G}^d_{m,S} = \mathrm{Spec}(\mathscr{O}_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}])$, for an integer $d \geq 0$.
(ii) X_s is non-empty.

Let $\overline{y}$ a geometric point of $X_{\overline{\eta}}$. We denote by $X^\star_\eta$ (resp. $X^*_{\overline{\eta}}$) the connected component of X_η (resp. $X_{\overline{\eta}}$) containing the image of $\overline{y}$ and by $(V_i)_{i\in I}$ the universal cover of $X^*_{\overline{\eta}}$ at $\overline{y}$ ([3] VI.9.7.3). We set $\Gamma = \pi_1(X^\star_\eta, \overline{y})$ and $\Delta = \pi_1(X^*_{\overline{\eta}}, \overline{y})$. For every $i \in I$, we denote by $X_i = \mathrm{Spec}(R_i)$ the normalization of $\overline{X} = X \times_S \overline{S}$ in V_i.

$$\begin{array}{ccc} V_i & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ X_{\overline{\eta}} & \longrightarrow & \overline{X} \end{array} \tag{4}$$

The $\mathscr{O}_{\overline{K}}$-algebras $(R_i)_{i\in I}$ form naturally an inductive system. We denote by $\overline{R}$ its inductive limit,

$$\overline{R} = \varinjlim_{i\in I} R_i, \tag{5}$$

and by $\widehat{\overline{R}}$ its p-adic completion, that we equip with the natural actions of Γ. The Γ-representation $\widehat{\overline{R}}$ is an analog of the G_K-representation $\mathscr{O}_C$.

Theorem 2.2 (Abbes and Gros [2] 6.9.6) *Under the assumption of 2.1, for any projective and smooth morphism $g\colon X' \to X$, and every integer $q \geq 0$, there exists $(\mathrm{fil}^q_r)_{0\leq r\leq q+1}$, a canonical exhaustive decreasing filtration of $\mathrm{H}^q_{\text{ét}}(X'_{\overline{y}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}[\frac{1}{p}]$ by $\widehat{\overline{R}}[\frac{1}{p}]$-representations of Γ, such that fil^q_{q+1} is zero and such that for every integer $0 \leq r \leq q$, we have a canonical Γ-equivariant exact sequence*

$$0 \to \mathrm{fil}^q_{r+1} \to \mathrm{fil}^q_r \to \mathrm{H}^r(X', \Omega^{q-r}_{X'/X}) \otimes_R \widehat{\overline{R}}[\frac{1}{p}](r-q) \to 0. \tag{6}$$

It amounts to saying that there exists a canonical Γ-equivariant spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(X', \Omega^j_{X'/X}) \otimes_R \widehat{\overline{R}}[\frac{1}{p}](-j) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(X'_{\overline{y}}, \mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}. \tag{7}$$

Indeed, it follows from Faltings' almost-purity theorem that, for every $j \neq 0$, the group $\mathrm{H}^0(\Gamma, \widehat{\overline{R}}[\frac{1}{p}](j))$ is zero. Hence the spectral sequence (7) degenerates at E_2. However, as the group $\mathrm{H}^1(\Gamma, \widehat{\overline{R}}[\frac{1}{p}](1))$ doesn't vanish in general, the abutment filtration doesn't split in general.

2.3. We conjectured the existence of the spectral sequence (7) in a first version of this work. Scholze immediately told us that he knew how to construct such a spectral sequence by using the relative Hodge–Tate filtration associated to a proper smooth morphism of adic spaces he developed with Caraiani ([7] 2.2.4). We also learned from Bhatt that he has a strategy to deduce the spectral sequence (7) from the general formalism of prismatic cohomology.

He [10] constructed the relative Hodge–Tate filtration Theorem 2.2 in an even more general setting than that of 2.1. He deduced it from the global variant of our Hodge–Tate relative spectral sequence Theorem 3.5 and a cohomological descent result for Faltings' topos he established. Our proof of Theorem 2.2, similar in spirit to his proof, has independently been suggested by the referee of [2]. To do this, we prove a cohomological descent result ([2], 4.6.30) that turns out to be a particular case of that of He [10].

3 The Global Version of the Relative Hodge–Tate Spectral Sequence

3.1. Let X be a smooth S-scheme (see Footnote 1). We denote by E the category of morphisms $V \to U$ above the canonical morphism $X_{\overline{\eta}} \to X$, that is, commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X_{\overline{\eta}} & \longrightarrow & X \end{array} \tag{8}$$

such that U is étale over X and the canonical morphism $V \to U_{\overline{\eta}}$ is *finite étale*. It is useful to consider the category E as fibered by the functor

$$\pi : E \to \mathbf{\acute{E}t}_{/X}, \quad (V \to U) \mapsto U, \tag{9}$$

over the étale site of X. The fiber of π above an object U of $\mathbf{\acute{E}t}_{/X}$ is canonically equivalent to the category $\mathbf{\acute{E}t}_{\mathrm{f}/U_{\overline{\eta}}}$ of finite étale morphisms over $U_{\overline{\eta}}$. We equip it with the étale topology and denote by $U_{\overline{\eta},\mathrm{f\acute{e}t}}$ the associated topos. If $U_{\overline{\eta}}$ is connected and if $\overline{y}$ is a geometric point of $U_{\overline{\eta}}$, then the topos $U_{\overline{\eta},\mathrm{f\acute{e}t}}$ is equivalent to the classifying topos of the profinite group $\pi_1(U_{\overline{\eta}}, \overline{y})$, *i.e.*, the category of discrete sets equipped with a continuous left action of $\pi_1(U_{\overline{\eta}}, \overline{y})$.

We equip E with the *covanishing* topology ([3] VI.1.10), that is the topology generated by coverings $\{(V_i \to U_i) \to (V \to U)\}_{i\in I}$ of the following two types :

(v) $U_i = U$ for all $i \in I$ and $(V_i \to V)_{i\in I}$ is a covering;
(c) $(U_i \to U)_{i\in I}$ is a covering and $V_i = V \times_U U_i$ for all $i \in I$.

The resulting site is called *Faltings site* of X. We denote by $\widetilde{E}$ and call *Faltings topos* of X the topos of sheaves of sets on E. It is an analogue of the covanishing topos $X_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}}$ ([3] VI.4).

To give a sheaf F on E amounts to give:

(i) for any object U of $\mathbf{\acute{E}t}_{/X}$, a sheaf F_U of $U_{\overline{\eta},\text{fét}}$, namely the restriction of F to the fiber of π above U;
(ii) for any morphism $f\colon U' \to U$ of $\mathbf{\acute{E}t}_{/X}$, a morphism $\gamma_f\colon F_U \to f_{\overline{\eta}*}(F_{U'})$.

These data should satisfy a cocycle condition for the composition of morphisms and a gluing condition for coverings of $\mathbf{\acute{E}t}_{/X}$ ([3] VI.5.10). Such a sheaf will be denoted by $\{U \mapsto F_U\}$.

There are three canonical morphisms of topos

$$\begin{array}{ccccc} & & X_{\overline{\eta},\text{ét}} & & \\ & & \Big\downarrow{\scriptstyle\psi} & & \\ X_{\text{ét}} & \xleftarrow{\ \sigma\ } & \widetilde{E} & \xrightarrow{\ \beta\ } & X_{\overline{\eta},\text{fét}} \end{array} \tag{10}$$

such that

$$\sigma^*(U) = (U_{\overline{\eta}} \to U)^a, \quad \forall\, U \in \mathrm{Ob}(\mathbf{\acute{E}t}_{/X}), \tag{11}$$

$$\beta^*(V) = (V \to X)^a, \quad \forall\, V \in \mathrm{Ob}(\mathbf{\acute{E}t}_{\mathrm{f}/X_{\overline{\eta}}}), \tag{12}$$

$$\psi^*(V \to U) = V, \quad \forall\, (V \to U) \in \mathrm{Ob}(E), \tag{13}$$

where the exponent ^{a} means the associated sheaf. The morphisms σ and β in the diagram (10) are the analogues of the first and second projections of the covanishing topos $X_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}}$. The morphism ψ is an analogue of the co-nearby cycles morphism ([3] VI.4.13)

Any specialization map $\overline{y} \rightsquigarrow \overline{x}$ from a geometric point $\overline{y}$ of $X_{\overline{\eta}}$ to a geometric point $\overline{x}$ of X, determines a point of $\widetilde{E}$ denoted by $\rho(\overline{y} \rightsquigarrow \overline{x})$ ([3] VI.10.18). The collection of these points is conservative ([3] VI.10.21).

Proposition 3.2 (Abbes and Gros [2] 4.4.2) *For any locally constant constructible torsion abelian sheaf F of $X_{\overline{\eta},\text{ét}}$, we have $\mathrm{R}^i\psi_*(F) = 0$ for any $i \geq 1$.*

This statement is a consequence of the fact that for any geometric point $\overline{x}$ of X over s, denoting by $\underline{X}$ the strict localization of X at $\overline{x}$, $\underline{X}_{\overline{\eta}}$ is a $K(\pi, 1)$ scheme ([3] VI.9.21), *i.e.*, if $\overline{y}$ is a geometric point of $\underline{X}_{\overline{\eta}}$, for any locally constant constructible torsion abelian sheaf F on $\underline{X}_{\overline{\eta}}$ and any $i \geq 0$, we have an isomorphism

$$\mathrm{H}^i(\underline{X}_{\overline{\eta}}, F) \xrightarrow{\sim} \mathrm{H}^i(\pi_1(\underline{X}_{\overline{\eta}}, \overline{y}), F_{\overline{y}}). \tag{14}$$

This property was proved by Faltings ([8] Lemma 2.3 page 281), generalizing results of Artin ([5] XI). It was further generalized by Achinger to the log-smooth case ([4] 9.5).

3.3. For any object $(V \to U)$ of E, we denote by $\overline{U}^V$ the integral closure of $\overline{U}$ in V and we set

$$\overline{\mathscr{B}}(V \to U) = \Gamma(\overline{U}^V, \mathscr{O}_{\overline{U}^V}). \tag{15}$$

The presheaf on E defined above is in fact a sheaf ([3] III.8.16). We write $\overline{\mathscr{B}} = \{U \mapsto \overline{\mathscr{B}}_U\}$ (cf. 3.1). For any étale X-scheme U which is affine, the stalk of the sheaf $\overline{\mathscr{B}}_U$ of $U_{\overline{\eta},\text{fét}}$ at a geometric point $\overline{y}$ of $U_{\overline{\eta}}$, is the representation $\overline{R}_U$ of $\pi_1(U_{\overline{\eta}}, \overline{y})$ defined in (5) for U.

For any specialization map $\overline{y} \rightsquigarrow \overline{x}$, we have

$$\overline{\mathscr{B}}_{\rho(\overline{y} \rightsquigarrow \overline{x})} = \varinjlim_{U \in \mathscr{V}_{\overline{x}}} \overline{R}_U, \tag{16}$$

where $\mathscr{V}_{\overline{x}}$ is the category of $\overline{x}$-pointed étale X-schemes U which are affine.

3.4. For any topos T, projective systems of objets of T indexed by the ordered set of natural numbers $\mathbb{N}$, form a topos that we denote by $T^{\mathbb{N}^\circ}$ ([3] III.7).

For any integer $n \geq 0$, we set $\overline{\mathscr{B}}_n = \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}$. To take into account p-adic topology, we consider the $\mathscr{O}_C$-algebra $\breve{\overline{\mathscr{B}}} = (\overline{\mathscr{B}}_n)_{n\geq 1}$ of the topos $\widetilde{E}^{\mathbb{N}^\circ}$. We work in the category $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ of $\breve{\overline{\mathscr{B}}}$-modules up to isogeny ([3] III.6.1), which is a global analogue of the category of $\widehat{\overline{R}}[\frac{1}{p}]$-representations of Δ considered in 2.1.

Theorem 3.5 (Abbes and Gros [2] 6.7.5) *Let $g\colon X' \to X$ be a smooth projective morphism. We denote by*

$$X'^{\mathbb{N}^\circ}_{\overline{\eta},\text{ét}} \xrightarrow{\breve{g}_{\overline{\eta}}} X^{\mathbb{N}^\circ}_{\overline{\eta},\text{ét}} \xrightarrow{\breve{\psi}} \widetilde{E}^{\mathbb{N}^\circ} \tag{17}$$

the morphisms induced by $g_{\overline{\eta}}$ and ψ (10), and by $\breve{\mathbb{Z}}_p$ the $\mathbb{Z}_p$-algebra $(\mathbb{Z}/p^n\mathbb{Z})_{n\geq 1}$ of $X'^{\mathbb{N}^\circ}_{\overline{\eta},\text{ét}}$. Then, we have a canonical spectral sequence of $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-modules

$$\mathrm{E}_2^{i,j} = \sigma^*(\mathrm{R}^i g_*(\Omega^j_{X'/X})) \otimes_{\sigma^*(\mathscr{O}_X)} \breve{\overline{\mathscr{B}}}_{\mathbb{Q}}(-j) \Rightarrow \breve{\psi}_*(\mathrm{R}^{i+j}\breve{g}_{\overline{\eta}*}(\breve{\mathbb{Z}}_p)) \otimes_{\mathbb{Z}_p} \breve{\overline{\mathscr{B}}}_{\mathbb{Q}}. \tag{18}$$

The projectivity condition on g is used in Theorem 4.4 below. It should be possible to replace it by the properness of g.

The spectral sequence (18) is called the *relative Hodge–Tate spectral sequence*. We can easily prove that it is G_K-equivariant for the natural G_K-equivariant structures on the topos and objects that appear. We deduce the following.

Proposition 3.6 (Abbes and Gros [2] 6.7.13) *Under the assumptions of Theorem 3.5, the relative Hodge–Tate spectral sequence* (18) *degenerates at* E_2.

Remark 3.7 Using a different approach, Caraiani and Scholze have constructed a relative Hodge–Tate filtration for proper smooth morphisms of adic spaces ([7] 2.2.4).

4 Faltings' Main *p*-adic Comparison Theorems

4.1. The assumptions and notation of Sect. 3 are in effect in this section. We denote by $\mathscr{O}_{\overline{K}^\flat}$ the limit of the projective system $(\mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}})_{\mathbb{N}}$ whose transition morphisms are the iterates of the absolute Frobenius endomorphism of $\mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}$,

$$\mathscr{O}_{\overline{K}^\flat} = \varprojlim_{\mathbb{N}} \mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}. \tag{19}$$

It is a perfect complete non-discrete valuation ring of height 1 and characteristic p. We fix a sequence $(p_n)_{n\geq 0}$ of elements of $\mathscr{O}_{\overline{K}}$ such that $p_0 = p$ and $p_{n+1}^p = p_n$ for any $n \geq 0$. We denote by ϖ the associated element of $\mathscr{O}_{\overline{K}^\flat}$ and we set $\xi = [\varpi] - p$ in the ring $\mathrm{W}(\mathscr{O}_{\overline{K}^\flat})$ of p-typical Witt vectors of $\mathscr{O}_{\overline{K}^\flat}$. We have a canonical isomorphism

$$\mathscr{O}_C(1) \xrightarrow{\sim} p^{\frac{1}{p-1}}\xi\mathscr{O}_C. \tag{20}$$

Theorem 4.2 (Faltings [9] and Abbes and Gros [2] 4.8.13) *Assume that X is proper over S. Let i, n be integers ≥ 0, F a locally constant constructible sheaf of $(\mathbb{Z}/p^n\mathbb{Z})$-modules of $X_{\overline{\eta},\text{ét}}$. Then, the kernel and cokernel of the canonical morphism*

$$\mathrm{H}^i(X_{\overline{\eta},\text{ét}}, F) \otimes_{\mathbb{Z}_p} \mathscr{O}_C \to \mathrm{H}^i(\widetilde{E}, \psi_*(F) \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}}) \tag{21}$$

are annihilated by $\mathfrak{m}_C$.

We say that morphism (21) is an *almost isomorphism*.

This is *Faltings' main p-adic comparaison theorem* from which he derived all comparaison theorems between p-adic étale cohomology and other p-adic cohomologies. It is also the main ingredient in the construction of the absolute Hodge–Tate spectral sequence (3).

We revisit in [2] Faltings' proof of this important result providing more details. It is based on Artin-Schreier exact sequence for the "perfection" of the ring $\overline{\mathscr{B}}_1 = \overline{\mathscr{B}}/p\overline{\mathscr{B}}$. One of the main ingredients is a structural statement for almost étale φ-modules on $\mathscr{O}_{\overline{K}^\flat}$ verifying certain conditions, including an almost finiteness condition in the sense of Faltings. In our application to the cohomology of Faltings' topos ringed by the "perfection" of $\overline{\mathscr{B}}_1$, the proof of this last condition results from the combination of three ingredients:

(i) local calculations of Galois cohomology using Faltings' almost-purity theorem ([9], [3] II.8.17);
(ii) a fine study of almost finiteness conditions for quasi-coherent sheaves of modules on schemes;
(iii) Kiehl's result on the finiteness of cohomology of a proper morphism ([12] 2.9'a) (cf. [1] 1.4.7).

4.3. Next, we explain Faltings' construction of the absolute Hodge–Tate spectral sequence (3). Assume that X is proper over S. By Proposition 3.2, for any $i, n \geq 0$, we have a canonical isomorphism

$$\mathrm{H}^i(X_{\overline{\eta},\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^i(\widetilde{E}, \psi_*(\mathbb{Z}/p^n\mathbb{Z})). \tag{22}$$

It is not difficult to see that the canonical morphism $\mathbb{Z}/p^n\mathbb{Z} \to \psi_*(\mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism. Then, by Faltings' main p-adic comparison Theorem 4.2, we have a canonical morphism

$$\mathrm{H}^i(X_{\overline{\eta},\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} \mathscr{O}_C \to \mathrm{H}^i(\widetilde{E}, \overline{\mathscr{B}}_n), \tag{23}$$

which is an almost isomorphism. To compute $\mathrm{H}^i(\widetilde{E}, \overline{\mathscr{B}}_n)$, we use the Cartan-Leray spectral sequence for the morphism $\sigma\colon \widetilde{E} \to X_{\text{ét}}$ (10),

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(X_{\text{ét}}, \mathrm{R}^j\sigma_*(\overline{\mathscr{B}}_n)) \Rightarrow \mathrm{H}^{i+j}(\widetilde{E}, \overline{\mathscr{B}}_n). \tag{24}$$

We deduce the absolute Hodge–Tate spectral sequence (3) using the following global analogue of Faltings' computation of Galois cohomology.

Theorem 4.4 (Abbes and Gros [2] 6.3.8) *There exists a canonical homomorphism of graded $\mathscr{O}_{\overline{X}_n}$-algebras of $X_{s,\text{ét}}$*

$$\wedge(\xi^{-1}\Omega^1_{\overline{X}_n/\overline{S}_n}) \to \oplus_{i\geq 0}\mathrm{R}^i\sigma_*(\overline{\mathscr{B}}_n), \tag{25}$$

where ξ is the element of $\mathrm{W}(\mathscr{O}_{\overline{K}^\flat})$ defined in 4.1, whose kernel (resp. cokernel) is annihilated by $p^{\frac{2d}{p-1}}\mathfrak{m}_{\overline{K}}$ (resp. $p^{\frac{2d+1}{p-1}}\mathfrak{m}_{\overline{K}}$), where $d = \dim(X/S)$.

We prove this result using Kummer theory over the special fiber of Faltings ringed topos $(\widetilde{E}, \overline{\mathscr{B}})$.

4.5. Let $g\colon X' \to X$ be a smooth (see Footnote 1) morphism. We associate to X' objects similar to those associated to X in Sect. 3 and we equip them with a prime $'$. We have a commutative diagram

$$\begin{array}{ccccc} X'_{\overline{\eta},\text{ét}} & \xrightarrow{\psi'} & \widetilde{E}' & \xrightarrow{\sigma'} & X'_{\text{ét}} \\ \downarrow{\scriptstyle g_{\overline{\eta}}} & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle g} \\ X_{\overline{\eta},\text{ét}} & \xrightarrow{\psi} & \widetilde{E} & \xrightarrow{\sigma} & X_{\text{ét}} \end{array} \tag{26}$$

where Θ is defined, for any object $(V \to U)$ of E, by

$$\Theta^*(V \to U) = (V \times_X X' \to U \times_X X')^a, \tag{27}$$

where the exponent ^{a} means the associated sheaf. We have also a canonical ring homomorphism

$$\overline{\mathscr{B}} \to \Theta_*(\overline{\mathscr{B}}'). \tag{28}$$

Theorem 4.6 (Faltings [9] § 6 and Abbes and Gros [2] 5.7.4) *Assume that $g\colon X' \to X$ is projective. Let i, n be integers ≥ 0, F' a locally constant constructible sheaf of $(\mathbb{Z}/p^n\mathbb{Z})$-modules of $X'_{\overline{\eta},\text{ét}}$. Then, the canonical morphism*

$$\psi_*(\mathrm{R}^i g_{\overline{\eta}*}(F')) \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}} \to \mathrm{R}^i \Theta_*(\psi'_*(F') \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}}') \tag{29}$$

is an almost isomorphism.

Observe that the sheaves $\mathrm{R}^i g_{\overline{\eta}*}(F)$ $(i \geq 0)$ are locally constant constructible on $X_{\overline{\eta}}$ by smooth and proper base change theorems.

Faltings formulated this *relative version* of his main p-adic comparison theorem in [9] and he very roughly sketched a proof in the appendix. Some arguments have to be modified and the actual proof in [2] requires much more work. It is based on a fine study of the local structure of certain almost-étale φ-modules which is interesting in itself ([2] 5.5.20).

The projectivity condition on g is used to prove an almost finiteness result for almost coherent modules. We rely on the finiteness results of [6] instead of those of [12]. It should be possible to replace the projectivity condition on g just by the properness of g.

Remark 4.7 Scholze has generalized Theorem 4.2 to rigid varieties following the same strategy ([15] 1.3). He has also proved an analogue of Theorem 4.6 in his setting of adic spaces and pro-étale topos ([15] 5.12). He deduces it from the absolute case using a base change theorem of Huber.

4.8. We keep the assumptions of Theorem 4.6. Let n be an integer ≥ 0. Since the canonical morphism $\mathbb{Z}/p^n\mathbb{Z} \to \psi'_*(\mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism, in order to construct the relative Hodge–Tate spectral sequence (18), we are led by Theorem 4.6 to compute the cohomology sheaves $\mathrm{R}^q\Theta_*(\overline{\mathscr{B}}'_n)$ $(q \geq 0)$. Inspired by the absolute case (4.3), the problem is then to find a natural factorization of Θ, to which we can apply the Cartan-Leray spectral sequence. Consider the commutative diagram of morphisms of topos

$$\begin{array}{ccc} \widetilde{E}' & & \\ {\scriptstyle \tau}\downarrow & \searrow^{\sigma'} & \\ \widetilde{E} \times_{X_{\text{ét}}} X'_{\text{ét}} & \xrightarrow{\ \pi\ } & X'_{\text{ét}} \\ {\scriptstyle \Xi}\downarrow & & \downarrow{\scriptstyle g} \\ \widetilde{E} & \xrightarrow{\ \sigma\ } & X_{\text{ét}} \end{array} \qquad (30)$$

We prove that the fiber product of topos $\widetilde{E} \times_{X_{\text{ét}}} X'_{\text{ét}}$ is in fact a *relative Faltings topos*, whose definition was inspired by oriented products of topos, beyond the covanishing topos which inspired our definition of the usual Faltings topos.

5 Relative Faltings Topos

5.1. We keep the assumption and notation of 4.5. We denote by G the category of morphisms $(W \to U \leftarrow V)$ above the canonical morphisms $X' \to X \leftarrow X_{\overline{\eta}}$, that is, commutative diagrams

$$\begin{array}{ccccc} W & \longrightarrow & U & \longleftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & X & \longleftarrow & X_{\overline{\eta}} \end{array} \qquad (31)$$

such that W is étale over X', U is étale over X and the canonical morphism $V \to U_{\overline{\eta}}$ is *finite étale*. We equip it with the topology generated by coverings

$$\{(W_i \to U_i \leftarrow V_i) \to (W \to U \leftarrow V)\}_{i \in I} \qquad (32)$$

of the following three types :

(a) $U_i = U$, $V_i = V$ for all $i \in I$ and $(W_i \to W)_{i \in I}$ is a covering;
(b) $W_i = W$, $U_i = U$ for all $i \in I$ and $(V_i \to V)_{i \in I}$ is a covering;
(c) diagrams

$$\begin{array}{ccccc} W' & \longrightarrow & U' & \longleftarrow & V' \\ \| & & \downarrow & \square & \downarrow \\ W & \longrightarrow & U & \longleftarrow & V \end{array} \qquad (33)$$

where $U' \to U$ is any morphism and the right square is Cartesian.

The resulting site is called *relative Faltings site* of the morphism $g\colon X' \to X$. We denote by $\widetilde{G}$ and call *relative Faltings topos* of g the topos of sheaves of sets on G. It is an analogue of the oriented product of topos $X'_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}}$ ([3] VI.3).

There are two canonical morphisms

$$X'_{\text{ét}} \xleftarrow{\ \pi\ } \widetilde{G} \xrightarrow{\ \lambda\ } X_{\overline{\eta},\text{fét}}\ , \tag{34}$$

defined by

$$\pi^*(W) = (W \to X \leftarrow X_{\overline{\eta}})^a, \quad \forall\, W \in \mathrm{Ob}(\mathbf{\acute{E}t}_{/X'}), \tag{35}$$

$$\lambda^*(V) = (X' \to X \leftarrow V)^a, \quad \forall\, V \in \mathrm{Ob}(\mathbf{\acute{E}t}_{\mathrm{f}/X_{\overline{\eta}}}), \tag{36}$$

where the exponent ^{a} means the associated sheaf. They are the analogues of the first and second projections of the oriented product $X'_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}}$.

If $X' = X$, $\widetilde{G}$ is canonically equivalent to Faltings topos $\widetilde{E}$ (3.1). Hence, by functoriality of relative Faltings topos, we have a natural factorization of $\Theta\colon \widetilde{E}' \to \widetilde{E}$ into

$$\widetilde{E}' \xrightarrow{\ \tau\ } \widetilde{G} \xrightarrow{\ \mathfrak{g}\ } \widetilde{E}. \tag{37}$$

These morphisms fit into the following commutative diagram of morphisms of topos

$$\begin{array}{ccccc} & & \widetilde{E}' & \xrightarrow{\ \beta'\ } & X'_{\overline{\eta},\text{fét}} \\ & \swarrow^{\sigma'} & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle g_{\overline{\eta}}} \\ X'_{\text{ét}} & \xleftarrow{\ \pi\ } & \widetilde{G} & \xrightarrow{\ \lambda\ } & X_{\overline{\eta},\text{fét}} \\ {\scriptstyle g}\downarrow & \Box & \downarrow{\scriptstyle \mathfrak{g}} & \nearrow_{\beta} & \\ X_{\text{ét}} & \xleftarrow{\ \sigma\ } & \widetilde{E} & & \end{array} \tag{38}$$

We prove that *the lower left square is Cartesian* (4.8).

We have a canonical morphism

$$\varrho\colon X'_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}} \to \widetilde{G}. \tag{39}$$

To give a point of $X'_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}}$ amounts to give a geometric point $\overline{x}'$ of X', a geometric point $\overline{y}$ of $X_{\overline{\eta}}$ and a specialization map $\overline{y} \rightsquigarrow g(\overline{x}')$. We denote (abusively) such a point by $(\overline{y} \rightsquigarrow \overline{x}')$. We prove that the collection of points $\varrho(\overline{y} \rightsquigarrow \overline{x}')$ of $\widetilde{G}$ is conservative.

5.2. Let $\overline{x}'$ be a geometric point of X', $\underline{X}'$ be the strict localization of X' at $\overline{x}'$ and $\underline{X}$ the strict localization of X at $g(\overline{x}')$. We denote by $\underline{\widetilde{G}}$ the relative Faltings topos of the morphism $\underline{X}' \to \underline{X}$ induced by g, by $\underline{\lambda}\colon \underline{\widetilde{G}} \to \underline{X}_{\overline{\eta},\text{fét}}$ the canonical morphism (34) and by $\Phi\colon \underline{\widetilde{G}} \to \widetilde{G}$ the functoriality morphism. There is a canonical section θ of $\underline{\lambda}$,

$$\begin{array}{ccc} \underline{X}_{\overline{\eta},\text{fét}} & \xrightarrow{\theta} & \underline{\widetilde{G}} \\ & \searrow^{\text{id}} & \downarrow{\scriptstyle \underline{\lambda}} \\ & & \underline{X}_{\overline{\eta},\text{fét}} \end{array} \tag{40}$$

We prove that the base change morphism induced by this diagram

$$\underline{\lambda}_* \to \theta^* \tag{41}$$

is an isomorphism. We set

$$\phi_{\overline{x}'} = \theta^* \circ \Phi^*\colon \widetilde{G} \to \underline{X}_{\overline{\eta},\text{fét}}. \tag{42}$$

If $\overline{y}$ is a geometric point of $\underline{X}_{\overline{\eta}}$, we obtain naturally a point $(\overline{y} \rightsquigarrow \overline{x}')$ of $X'_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}}$. Then, for any sheaf F of $\widetilde{G}$, we have a canonical functorial isomorphism

$$F_{\varrho(\overline{y}\rightsquigarrow\overline{x}')} \xrightarrow{\sim} \phi_{\overline{x}'}(F)_{\overline{y}}. \tag{43}$$

Proposition 5.3 (Abbes and Gros [2] 3.4.34) *Under the assumptions of 5.2, for any abelian sheaf F of $\widetilde{G}$ and any $q \geq 0$, we have a canonical isomorphism*

$$\mathrm{R}^q\pi_*(F)_{\overline{x}'} \xrightarrow{\sim} \mathrm{H}^q(\underline{X}_{\overline{\eta},\text{fét}}, \phi_{\overline{x}'}(F)). \tag{44}$$

Corollary 5.4 (Abbes and Gros [2] 6.5.17) *Let $(\overline{y} \rightsquigarrow \overline{x}')$ be a point of $X'_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}}$, $\underline{X}'$ the strict localization of X' at $\overline{x}'$, $\underline{X}$ the strict localization of X at $g(\overline{x}')$, $\underline{g}\colon \underline{X}' \to \underline{X}$ the morphism induced by g,*

$$\varphi'_{\overline{x}'}\colon \widetilde{E}' \to \underline{X}'_{\overline{\eta},\text{fét}} \tag{45}$$

the canonical morphism analogue of (42). Then, for any abelian group F of $\widetilde{E}'$ and any $q \geq 0$, we have a canonical functorial isomorphism

$$(\mathrm{R}^q\tau_*(F))_{\varrho(\overline{y}\rightsquigarrow\overline{x}')} \xrightarrow{\sim} \mathrm{R}^q\underline{g}_{\overline{\eta},\text{fét}*}(\varphi'_{\overline{x}'}(F))_{\overline{y}}. \tag{46}$$

5.5. We consider the following ring of $\widetilde{G}$,

$$\overline{\mathscr{B}}^! = \tau_*(\overline{\mathscr{B}}'). \tag{47}$$

We have canonical homomorphisms $\overline{\mathscr{B}} \to \mathrm{g}_*(\overline{\mathscr{B}}^!)$ and $\hbar'_*(\mathscr{O}_{\overline{X}'}) \to \pi_*(\overline{\mathscr{B}}^!)$, where $\hbar'\colon \overline{X}' \to X'$ is the canonical projection. Hence, we may consider g and π as morphisms of ringed topos.

For any point $(\overline{y} \rightsquigarrow \overline{x}')$ of $X'_{\text{ét}} \overleftarrow{\times}_{X_{\text{ét}}} X_{\overline{\eta},\text{ét}}$, we prove that the ring $\overline{\mathscr{B}}^!_{\varrho(\overline{y}\rightsquigarrow\overline{x}')}$ is normal and strictly henselian. Moreover, the canonical homomorphism $\mathscr{O}_{\overline{X}',\overline{x}'} \to \overline{\mathscr{B}}^!_{\varrho(\overline{y}\rightsquigarrow\overline{x}')}$ is local and injective.

5.6. Assume that $X = \mathrm{Spec}(R)$ and $X' = \mathrm{Spec}(R')$ are affine. Let $\overline{y}'$ be a geometric point of $X'_{\overline{\eta}}$, $\Delta' = \pi_1(X'_{\overline{\eta}}, \overline{y}')$, $(W_j)_{j\in J}$ the universal cover of $X'_{\overline{\eta}}$ at $\overline{y}'$, $\overline{y} = g_{\overline{\eta}}(\overline{y}')$, $\Delta = \pi_1(X_{\overline{\eta}}, \overline{y})$ and $(V_i)_{i\in I}$ the universal cover of $X_{\overline{\eta}}$ at $\overline{y}$. For every $i \in I$, $(V_i \to X)$ is naturally an object of E and for every $j \in J$, $(W_j \to X')$ is naturally an object of E'. We set

$$\overline{R} = \varinjlim_{i\in I} \overline{\mathscr{B}}(V_i \to X), \tag{48}$$

$$\overline{R}' = \varinjlim_{j\in J} \overline{\mathscr{B}}'(W_j \to X'). \tag{49}$$

We recover the $\mathscr{O}_{\overline{K}}$-algebras defined in (5). We equip them with the natural actions of Δ and Δ'. For every $i \in I$, there exists a canonical X'-morphism $\overline{y}' \to X' \times_X V_i$. We denote by V'_i the irreducible component of $X' \times_X V_i$ containing $\overline{y}'$ and by Π_i the corresponding subgroup of Δ'. Then, $(V'_j \to X')$ is naturally an object of E'. We set

$$\overline{R}^! = \varinjlim_{i\in I} \overline{\mathscr{B}}'(V'_i \to X'), \tag{50}$$

$$\Pi = \bigcap_{i\in I} \Pi_i. \tag{51}$$

We have canonical homomorphisms $\overline{R} \to \overline{R}^! \to \overline{R}'$.

For any geometric point $\overline{x}'$ and any specialization map $\overline{y} \rightsquigarrow g(\overline{x}')$, we prove that we have a canonical isomorphism (determined by the choice of $\overline{y}'$)

$$\overline{\mathscr{B}}^!_{\varrho(\overline{y}\rightsquigarrow\overline{x}')} \xrightarrow{\sim} \varinjlim_{\overline{x}'\to U'\to U} \overline{R}^!_{U'\to U}, \tag{52}$$

where the inductive limit is taken over the category of morphisms $\overline{x}' \to U' \to U$ over $\overline{x}' \to X' \to X$, with U' affine étale over X' and U affine étale over X, and $\overline{R}^!_{U'\to U}$ is the corresponding ring (50).

Proposition 5.7 (Abbes and Gros [2] 5.2.29) *We keep the assumptions of 5.6 and we assume moreover that g fits into a commutative diagram (see Footnote 1)*

$$\begin{array}{ccc} X' & \xrightarrow{\iota'} & \mathbb{G}_{m,S}^{d'} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \gamma} \\ X & \xrightarrow{\iota} & \mathbb{G}_{m,S}^{d} \end{array} \tag{53}$$

where the morphisms ι and ι' are étale, d and d' are integers ≥ 0 and γ is a smooth homomorphism of tori over S. Let n be an integer ≥ 0. Then,

(i) There exists a canonical homomorphism of graded $\overline{R}^!$-algebras

$$\wedge\,(\Omega^1_{R'/R} \otimes_{R'} (\overline{R}^! / p^n \overline{R}^!)(-1)) \to \oplus_{i \geq 0} \mathrm{H}^i(\Pi, \overline{R}' / p^n \overline{R}'), \tag{54}$$

which is almost injective and its cokernel is killed by $p^{\frac{1}{p-1}} \mathfrak{m}_{\overline{K}}$.

(ii) The $\overline{R}^!$-module $\mathrm{H}^i(\Pi, \overline{R}'/p^n\overline{R}')$ is almost finitely presented for all $i \geq 0$, and it almost vanishes for all $i \geq r+1$, where $r = \dim(X'/X)$.

This is a relative version of Faltings' computation of Galois cohomology of $\overline{R}$, that relies on his almost purity theorem ([9] Theorem 4 page 192, [3] II.6.16). The statement can be globalized using Kummer theory on the special fiber of the ringed topos $(\widetilde{E}', \overline{\mathscr{B}}')$ into the following.

Theorem 5.8 (Abbes and Gros [2] 6.6.4) *For any integer $n \geq 1$, there exists a canonical homomorphism of graded $\overline{\mathscr{B}}^!$-algebras of $\widetilde{G}$*

$$\wedge\,(\pi^*(\xi^{-1}\Omega^1_{\overline{X}'_n/\overline{X}_n})) \to \oplus_{i \geq 0} \mathrm{R}^i \tau_*(\overline{\mathscr{B}}'_n), \tag{55}$$

where π^* denotes the pullback by the morphism of ringed topos $\pi\colon (\widetilde{G}, \overline{\mathscr{B}}^!) \to (X'_{\text{ét}}, \hbar'_*(\mathscr{O}_{\overline{X}'}))$, whose kernel (resp. cokernel) is annihilated by $p^{\frac{2r}{p-1}} \mathfrak{m}_{\overline{K}}$ (resp. $p^{\frac{2r+1}{p-1}} \mathfrak{m}_{\overline{K}}$), where $r = \dim(X'/X)$.

5.9. Next we consider the Cartan-Leray spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{R}^i \mathrm{g}_*(\mathrm{R}^j \tau_*(\overline{\mathscr{B}}'_n)) \Rightarrow \mathrm{R}^{i+j}\Theta_*(\overline{\mathscr{B}}'_n). \tag{56}$$

Taking into account Theorem 5.8, to obtain the relative Hodge–Tate spectral sequence (18), we need to prove a base change theorem relatively to the Cartesian diagram

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\pi} & X'_{\text{ét}} \\ {\scriptstyle \mathrm{g}}\downarrow & & \downarrow{\scriptstyle g} \\ \widetilde{E} & \xrightarrow{\sigma} & X_{\text{ét}} \end{array} \tag{57}$$

Theorem 5.10 (Abbes and Gros [2] 6.5.5) *Assume g proper. Then, for any torsion abelian sheaf F of $X'_{\text{ét}}$ and any $q \geq 0$, the base change morphism*

$$\sigma^*(\mathrm{R}^q g_*(F)) \to \mathrm{R}^q \mathrm{g}_*(\pi^*(F)) \tag{58}$$

is an isomorphism.

The proof is inspired by a base change theorem for oriented products due to Gabber. It reduces to proper base change theorem for étale topos.

Proposition 5.11 (Abbes and Gros [2] 6.5.29) *For any integer $n \geq 0$, the canonical homomorphism*

$$\overline{\mathscr{B}}_n \boxtimes_{\mathscr{O}_X} \mathscr{O}_{X'} \to \overline{\mathscr{B}}^!_n, \tag{59}$$

where the exterior tensor product of rings is relative to the Cartesian diagram (57), is an almost isomorphism.

Theorem 5.12 (Abbes and Gros [2] 6.5.31) *Assume that the morphism g is proper. Then, there exists an integer $N \geq 0$ such that for any integers $n \geq 1$ and $q \geq 0$, and any quasi-coherent $\mathscr{O}_{X'_n}$-module, the kernel and cokernel of the base change morphism*

$$\sigma^*(\mathrm{R}^q g_*(\mathscr{F})) \to \mathrm{R}^q \mathrm{g}_*(\pi^*(\mathscr{F})), \tag{60}$$

where σ^* and π^* denote the pullbacks in the sense of ringed topos, are annihilated by p^N.

Proposition 5.13 (Abbes and Gros [2] 6.5.32) *Let n, q be integers ≥ 0, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_{X'_n}$-module which is X_n-flat (1). Assume that the morphism g is proper and that for any integer $i \geq 0$, the $\mathscr{O}_{X_n}$-module $\mathrm{R}^i g_*(\mathscr{F})$ is locally free (of finite type). Then, the base change morphism*

$$\sigma^*(\mathrm{R}^q g_*(\mathscr{F})) \to \mathrm{R}^q \mathrm{g}_*(\pi^*(\mathscr{F})), \tag{61}$$

where σ^* and π^* denote the pullbacks in the sense of ringed topos, is an almost isomorphism.

5.14. Let n, q be integers ≥ 0. Since the canonical morphism $\mathbb{Z}/p^n\mathbb{Z} \to \psi'_*(\mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism, we deduce from Theorem 4.6, for any $q \geq 0$, a canonical morphism

$$\psi_*(\mathrm{R}^q g_{\overline{\eta}*}(\mathbb{Z}/p^n\mathbb{Z})) \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}} \to \mathrm{R}^q \Theta_*(\overline{\mathscr{B}}'_n), \tag{62}$$

which is an almost isomorphism. The relative Hodge–Tate spectral sequence (18) is then deduced from the Cartan-Leray spectral sequence (56) using Theorems 5.8 and 5.12.

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Semisimplicity of the Frobenius Action on π_1

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1 Introduction

Let K be a finite extension of $\mathbb{Q}_p$ with residue field of size q, and fix a prime $\ell \neq p$. Fix a choice of geometric Frobenius $\varphi_K \in G_K$ and an element $\sigma \in I_K$ of inertia which generates tame inertia. If X/K is a smooth proper variety with good reduction, then the Galois action on the étale cohomology $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified, and the eigenvalues of φ_K are all q-Weil numbers of weight i. The strong Tate Conjecture also predicts that the action of φ_K should be semisimple.

In general—X not necessarily smooth, proper, or of good reduction—the behavior of the Galois representations $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is less well-understood, and is the subject of several major conjectures.

Conjecture 1 (Weight–Monodromy) Let N be the nilpotent endomorphism of $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ given by $\frac{1}{e}\log(\sigma^e)$ where σ^e is any power of σ acting unipotently on $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. Let

$$0 \leq W_0\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \leq W_1\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \leq \cdots \leq W_{2i}\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$$

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be the weight filtration on $\mathrm{H}^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ constructed by Deligne [13, §6]. Then for all j, the representation $\mathrm{gr}^W_j \mathrm{H}^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is *pure* of weight j (see Definition 3 for a precise definition).[1]

Conjecture 2 (Frobenius-Semisimplicity) φ_K acts semisimply on $\mathrm{H}^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$.

When X is smooth and proper, the first of these conjectures is due to Deligne [12, 31]; it is a folklore theorem[2] that this implies the general case. The second of these conjectures follows from the strong Tate Conjecture and the Rapoport–Zink spectral sequence when X is smooth and proper with semistable reduction, and de Jong's theory of alterations allows one to extend this to the case of arbitrary reduction. We will shortly explain (Corollary 1) why one should believe Conjecture 2 in general. There are also analogous conjectures in the case $\ell = p$ regarding the action of the crystalline Frobenius on $\mathsf{D}_{\mathrm{pst}}(\mathrm{H}^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p))$ (see e.g. [29, Conjecture 3.27]).

The main aim of this paper is to prove analogues of the above conjectures when the cohomology $\mathrm{H}^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is replaced by the *$\mathbb{Q}_\ell$-pro-unipotent étale fundamental groupoid* $\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}})$ of $X_{\overline{K}}$. This object, whose formal definition we will recall in Sect. 3, consists of affine $\mathbb{Q}_\ell$-schemes $\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, y)$ for every $x, y \in X(K)$, endowed with an action of G_K. These come endowed with various structure maps; in particular each $\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, x)$ is a pro-unipotent group over $\mathbb{Q}_\ell$. The Lie algebras $\mathrm{Lie}(\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, x))$ and the affine rings $\mathcal{O}(\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, y))$ admit natural weight filtrations compatible with their induced Galois actions. We will prove the following two results, along with their analogues in the case $\ell = p$.

Theorem 1 (Weight–Monodromy for the Fundamental Groupoid) *Suppose that X/K is smooth and geometrically connected. Then:*

1. $\mathrm{gr}^W_{-n}\mathrm{Lie}(\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, x))$ *is pure of weight $-n$, for all n and all $x \in X(K)$; and*
2. $\mathrm{gr}^W_n \mathcal{O}(\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, y))$ *is pure of weight n, for all n and all $x, y \in X(K)$.*

Theorem 2 (Frobenius-Semisimplicity for the Fundamental Groupoid) *Suppose that X/K is smooth and geometrically connected. Then φ_K acts semisimply on* $\mathrm{Lie}(\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, x))$ *and* $\mathcal{O}(\pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, y))$ *for all $x, y \in X(K)$.*

The crystalline version of Theorem 1 already appears in work of Vologodsky [34, Theorem 26], to whom this work owes a great debt; we present two proofs, both different to Vologodsky's and more direct.

[1] There are two competing definitions of the word "pure" in the literature: one referring to representations, all of whose Frobenius eigenvalues are Weil numbers of a single weight, and one only requiring this condition on each graded piece of the monodromy filtration. In this paper, we will work exclusively with the latter.

[2] A proof of this was outlined to the first author by Tony Scholl; to the best of our knowledge there is no published proof of this fact.

Remark 1 In [11, Conjecture 3.10], the authors conjecture a number of independence of ℓ results for Weil–Deligne representations associated to unipotent fundamental groups. Our Theorem 2 above implies that the "weak" and "strong" versions of these conjectures are equivalent. In particular, Chiarellotto–Lazda prove the weak forms of their conjectures for smooth projective curves over mixed characteristic local fields, which, by Theorem 2, implies the strong form of their conjectures.

1.1 Canonical Splittings

The main technical result in this paper is a pure linear algebra lemma (Definition 4), which shows that weight–monodromy conditions can be used to overcome "extension problems" when studying Frobenius actions. If $(V, W_\bullet)$ is a filtered representation of G_K, then it is not in general true that semisimplicity of the Frobenius action on the graded pieces $\mathrm{gr}_n^W V$ implies semisimplicity of the action on V itself. However, if each $\mathrm{gr}_n^W V$ is pure of weight n, we will show that in fact there is a canonical φ_K-equivariant way to split the $W_\bullet$-filtration on V. In particular, in this case Frobenius-semisimplicity of each $\mathrm{gr}_n^W V$ does imply Frobenius-semisimplicity of V.

As well as proving our Frobenius-semisimplicity Theorem 2, this also proves the following relation between the weight–monodromy conjecture and the Frobenius-semisimplicity conjecture.

Corollary 1 *Suppose that the weight–monodromy Conjecture 1 holds for all varieties over K, and that the Frobenius-semisimplicity Conjecture 2 holds for all smooth proper varieties over K. Then the Frobenius-semisimplicity Conjecture 2 holds for all varieties over K.*

Proof If X/K is a variety, then each $\mathrm{gr}_j^W \mathrm{H}^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is a subquotient of the degree j étale cohomology of a smooth proper variety, and hence Frobenius-semisimple. The canonical φ_K-equivariant splitting provided by the weight–monodromy conjecture for X implies that $\mathrm{H}^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is Frobenius-semisimple.

The canonical splittings we construct also have further consequences for the fundamental groupoids of smooth geometrically connected varieties, in that they provide, for each $x, y \in X(K)$, a canonical φ_K-invariant path $p_{x,y} \in \pi_1^{\mathbb{Q}_\ell}(X_{\overline{K}}; x, y)(\mathbb{Q}_\ell)^{\varphi_K}$. The analogues of these paths in the case $\ell = p$ already appear in the work of Vologodsky, and in the good reduction case they play a central role in the theory of iterated Coleman integration [6]; in the case $\ell \neq p$ these paths were used in [4] to explicitly calculate the non-abelian Kummer map associated to a smooth hyperbolic curve X. Our construction of these canonical paths provides a unified explanation for these phenomena.

We remark that in the archimedean setting, i.e. in the category of mixed Hodge structures, the existence of similar canonical splittings is well-known. The weight filtration on a mixed Hodge structure is canonically split over $\mathbb{C}$ by the Deligne splitting, and there is even a variant of this splitting which is defined over $\mathbb{R}$ [10, Proposition 2.20]. Thus there are also canonical choices of $\mathbb{R}$-pro-unipotent Betti paths between any two points in a smooth connected variety $X/\mathbb{C}$. The study of these paths will be taken up by the second author in forthcoming work.

Applications

As a further application of our main theorems, we apply these results in Sect. 4 to study the geometry of local Bloch–Kato Selmer varieties arising in the Chabauty-Kim program. These are three sub-presheaves $\mathrm{H}^1_e(G_K, U) \subseteq \mathrm{H}^1_f(G_K, U) \subseteq \mathrm{H}^1_g(G_K, U)$ of the continuous Galois cohomology presheaf $\mathrm{H}^1(G_K, U)$ associated to the $\mathbb{Q}_p$-pro-unipotent étale fundamental group $U = \pi_1^{\mathbb{Q}_p}(X_{\overline{K}}, \bar{x})$, and are all representable by affine schemes over $\mathbb{Q}_p$ [23, Proposition 3][24, Lemma 5]. The relevance of the local Selmer varieties to Diophantine geometry and the Chabauty–Kim method comes via a certain *non-abelian Kummer* or *higher Albanese* map

$$X(K) \to \mathrm{H}^1(G_K, U)(\mathbb{Q}_p)\,.$$

When X is a smooth projective curve with good reduction, then the image of this map is contained in $\mathrm{H}^1_e(G_K, U) = \mathrm{H}^1_f(G_K, U)$, which is known to be an affine space over $\mathbb{Q}_p$ [25, Proposition 1.4]. In the absence of properness or good reduction assumptions, the image is contained in $\mathrm{H}^1_g(G_K, U)$, but not in general in H^1_f.

Our contribution in this paper is to describe the geometry of $\mathrm{H}^1_g(G_K, U)$ (for X not necessarily proper, of arbitrary dimension, and with no restrictions on the reduction type). In fact, we find that its geometry is as good as could be hoped: it is *canonically* isomorphic to the product of $\mathrm{H}^1_e(G_K, U)$ and an explicit (pro-finite-dimensional) vector space $\mathcal{V}(\mathsf{D}_{\mathrm{pst}}(U))^{\varphi=1}$, and therefore also an affine space. As an illustration, we are able to write down explicit dimension formulae in the case that X is a curve.

2 Weil–Deligne Representations

In this section, we fix a finite extension K of $\mathbb{Q}_p$ with residue field k, along with an algebraic closure $\overline{K}$. We write W_K for the Weil group of K, i.e. the subgroup of the absolute Galois group G_K consisting of elements acting on the residue field $\overline{k}$ via an integer power of the absolute Frobenius $\sigma\colon x \mapsto x^p$, and we write $v\colon W_K \to \mathbb{Z}$ for the unique homomorphism such that $w \in W_K$ acts on $\overline{k}$ via $\sigma^{v(w)}$. We fix a geometric Frobenius $\varphi_K \in W_K$, i.e. an element such that $v(\varphi_K) = -f(K/\mathbb{Q}_p)$. The following definition is standard.

Definition 1 Let E be a field of characteristic 0. A *Weil representation* with coefficients in E is a representation $\rho\colon W_K \to \mathrm{Aut}(V)$ of W_K on a finite dimensional E-vector space V on which the inertia group I_K acts through a finite quotient. A *Weil–Deligne representation* with coefficients in E consists of a Weil representation V endowed with an E-linear endomorphism $N \in \mathrm{End}(V)$, called the *monodromy operator*, such that

$$\rho(w) \circ N \circ \rho(w)^{-1} = p^{v(w)} \cdot N$$

for all $w \in W_K$. It follows from this condition that N is necessarily nilpotent.

We denote the category of Weil–Deligne representations by $\mathbf{Rep}_E({}'W_K)$. The category $\mathbf{Rep}_E({}'W_K)$ has a canonical tensor product making it into a neutral Tannakian category, where the tensor $V_1 \otimes V_2$ is endowed with the tensor product W_K-action, and with the endomorphism $N_{V_1} \otimes 1 + 1 \otimes N_{V_2}$.

Example 1 The Weil–Deligne representation $E(1)$ has underlying vector space E, trivial monodromy operator N, and the Weil group acts via $w\colon x \mapsto p^{v(w)}x$.

As explained in [17], Weil–Deligne representations arise naturally from Galois representations, both ℓ-adic and p-adic.

Example 2 Let ℓ be a prime distinct from p, and choose a generator $t \in \mathbb{Q}_\ell(1)$. Let $t_\ell\colon I_K \to \mathbb{Q}_\ell(1)$ denote the ℓ-adic tame character $w \mapsto \left(\frac{w(p^{1/\ell^n})}{p^{1/\ell^n}}\right)_{n\in\mathbb{N}}$. Then there is a fully faithful exact $\otimes$-functor

$$\mathbf{Rep}_{\mathbb{Q}_\ell,\mathrm{cts}}(G_K) \to \mathbf{Rep}_{\mathbb{Q}_\ell}({}'W_K)$$

from the category of continuous $\mathbb{Q}_\ell$-linear[3] representations of G_K to the category of Weil–Deligne representations. This functor is defined as follows. If (V, ρ_0) is a continuous $\mathbb{Q}_\ell$-linear representation of G_K, there is an open subgroup $I_L \leq I_K$ acting unipotently on V by Grothendieck's ℓ-adic Monodromy Theorem. We let N denote the endomorphism of V such that

$$\rho_0(g) = \exp\left(t^{-1}t_\ell(g)N\right)$$

for all $g \in I_L$. We define an action ρ of W_K on V by

$$\rho\left(\varphi_K^n g\right) = \rho_0(\varphi_K^n g)\exp\left(-t^{-1}t_\ell(g)N\right)$$

[3] One can equally well work with continuous E-linear representations for any algebraic extension $E/\mathbb{Q}_\ell$. However, in our context only $\mathbb{Q}_\ell$-linear representations will appear, so we restrict to this case.