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Daniel Alpay, Jussi Behrndt,
Fabrizio Colombo, Irene Sabadini,
Daniele C. Struppa, Editors

Recent Developments in Operator Theory, Mathematical Physics and Complex Analysis

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Preface

This volume contains papers written by some of the plenary and semi-plenary speakers of the 32nd *International Workshop on Operator Theory and Its Applications* held at Chapman University, Orange, from 9 to 13 August 2022. The workshop, held in hybrid mode, has been the occasion for researchers to share their recent results.

All contributed papers represent recent achievements as well as "state-of-the-art" expositions.

The editors are grateful to the contributors to this volume and to the referees for their painstaking and careful work.

Special thanks are due to Chapman University for hosting the event and for the financial support.

Orange, CA, USA — Daniel Alpay
Graz, Austria — Jussi Behrndt
Milano, Italy — Fabrizio Colombo
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September 2022

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Infinite Order Differential Operators with a Glimpse to Applications to Superoscillations

Takashi Aoki, Yasunori Okada, Irene Sabadini, and Daniele C. Struppa

Abstract In this paper we will consider operators that can be formally written as

$$P\left(z, \frac{d}{dz}\right) := \sum_{n=0}^{\infty} a_n(z) \frac{d^n}{dz^n}$$

where the functions a_n are entire functions on the complex plane (possibly satisfying suitable growth conditions), and we will study their action on suitable spaces of entire functions as well. After an introductory section, whose content is well known, that describes infinite order differential operators from the point of view of the theory of hyperfunctions, we will describe as well convolutors arising from analytic functionals, and how they can be represented by infinite series of derivatives. The next section is dedicated to the way in which the study of longevity phenomena for superoscillations has led to a renewed interest for the theory of infinite order differential operators, and will present some recent results on the continuity of such

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operators. Finally, we will show how the idea of infinite order differential operators extends fruitfully to the hypercomplex setting.

Keywords Infinite order differential operators · Entire functions with growth conditions · Entire hyperholomorphic functions

1 Introduction: Infinite Order Differential Operators and Convolution Operators

The theory of infinite order differential operators has a long and distinguished history, even though the term itself has been used with different meanings in different settings. In this paper we will consider operators that can be written as

$$P\left(z, \frac{d}{dz}\right) := \sum_{n=0}^{\infty} a_n(z) \frac{d^n}{dz^n}$$

where the functions a_n are entire functions on the complex plane satisfying suitable growth conditions, and we will study their action on spaces of entire functions as well. As it is well known, the term *differential operator* only loosely applies to such operators, since unless some restrictions are imposed, they actually behave like convolution operators. The simplest example of such a situation is the operator

$$P\left(\frac{d}{dz}\right) := \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dz^n},$$

which is such that for any entire function $f(z)$ one has

$$P\left(\frac{d}{dz}\right) f(z) = f(z+1).$$

In order to clarify this situation we will begin with considering the space $\mathcal{H}_{\{0\}}$ of germs of one-variable holomorphic functions at the origin. This space can be endowed with the inductive limit topology that derives from defining

$$\mathcal{H}_{\{0\}} := \varinjlim \mathcal{H}(U)$$

over a decreasing family of open sets whose intersection is the origin. Its dual $\mathcal{H}'_{\{0\}}$ is isomorphic to the space of one-variable hyperfunctions supported by the origin, and as such is isomorphic to the quotient

$$\mathcal{B}_{\{0\}} = \frac{\mathcal{H}(\mathbb{C} \setminus \{0\})}{\mathcal{H}(\mathbb{C})}.$$

By using the Fourier-Borel transform, one can further see that this space is isomorphic to the space $\mathrm{Exp}_0(\mathbb{C})$ of entire functions of infraexponential type, also called of exponential type zero, namely the space of entire functions that satisfy the following growth condition

$$\text{for any } \epsilon > 0, \text{ there exists } A_\epsilon > 0 \text{ such that } |F(z)| \leq A_\epsilon e^{\epsilon |z|}.$$

Any differential operator

$$P\left(\frac{d}{dz}\right) := \sum_{n=0}^{\infty} a_n \frac{d^n}{dz^n}$$

such that

$$P(z) := \sum_{n=0}^{\infty} a_n z^n$$

is a function of infraexponential type will be said to be an infinite order differential operator and acts as a sheaf homomorphism on the sheaf of germs of holomorphic functions. We therefore see that we have several objects that, while looking different, are essentially the same: the dual of germs of holomorphic functions in the origin, the space of hyperfunctions with support in the origin, the space of entire functions of infraexponential type, and the space of infinite order differential operators with constant coefficients. There are several questions of interest that these few considerations inspire. To begin with, one would like to see what the growth condition on the function $P(z)$ means in terms of the coefficients a_n. This question is very well understood and we know that the infinite order differential operator

$$P\left(\frac{d}{dz}\right) := \sum_{n=0}^{\infty} a_n \frac{d^n}{dz^n}$$

acts continuously on the space of entire functions on $\mathbb{C}$ if

$$\lim_{k\to\infty} \sqrt[k]{k!|a_k|} = 0.$$

A second interesting questions is to inquire as to what happens when the coefficients a_n are not constant but, say, holomorphic functions $a_n(z)$. This question is also well understood, and we still have infinite order differential operators (that is objects that act on the sheaf of holomorphic functions), as long as the same kind of growth conditions are imposed on the variable coefficients. More precisely, the infinite order differential operator

$$P\left(z, \frac{d}{dz}\right) := \sum_{n=0}^{\infty} a_n(z) \frac{d^n}{dz^n}$$

acts continuously on the space of entire functions on $\mathbb{C}$ if for every compact set K in $\mathbb{C}$ (see e.g. [29, Lemma 1.8.1])

$$\lim_{k\to\infty} \sqrt[k]{\sup_{z\in K} k!|a_k(z)|} = 0.$$

The results hold more in general in n complex variables, but in this introduction we limit our discussion to the one variable case.

Before looking at more general growth (and therefore at operators that act as convolutors, rather than as infinite order differential operators), we should probably take a detour and mention another reason for the interest in infinite order differential operators. To do so, we briefly recall the celebrated Ehrenpreis-Palamodov Fundamental Principle, [27, 30]. In brief this theorem states that if $P_1, \ldots, P_r$ are polynomials in n complex variables, and if f is a generalized function in a suitable space (for example the space of distributions or infinitely differentiable functions on $\mathbb{R}^n$, or entire functions on $\mathbb{C}^n$) satisfying

$$P_1(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})f = \ldots = P_r(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})f = 0,$$

then f admits an integral representation of the form

$$f(x) = \sum_{j=0}^{t} \int_{V_j} \partial_j(e^{ix\cdot z}) d\nu_j(z),$$

where the V_j are subvarieties of the algebraic variety $V = \{z \in \mathbb{C}^n : P_1(z) = \ldots = P_r(z) = 0\}$, the ∂_j are differential operators with polynomial coefficients, and the $d\nu_j$ are measures supported on the varieties V_j. In the case in which $n = 1$ this theorem is obviously very well known and it is often referred to as the Euler theorem for ordinary differential equations with constant coefficients. There is, however, a way to remain in one variable and yet take advantage of the deep ideas underlying the Fundamental Principle. In his book [27], Ehrenpreis suggests the consideration of a sequence of polynomials in one or several variables P_j, and their correspondent linear constant coefficients operators D_j (i.e. the differential operators whose symbols were the polynomials P_j), and then to consider series of solutions f_j to the equations $D_j(f_j) = 0$. Ehrenpreis provided some general ideas on the geometric conditions that the varieties $V_j := \{z \in \mathbb{C}^n : P_j(z) = 0\}$ needed to satisfy, in order for the series $\sum_j f_j$ to converge and have suitable integral representation. The study of infinite order differential operators, however, offers a somewhat more direct and simple approach. Rather than considering a sequence of differential operators, one may consider a series of the differential operators $a_j \frac{d^j}{dx^j}$, and study the geometry of the variety associated to the holomorphic function $\sum_j a_j z^j$ to see whether the solution of the infinite order differential equation naturally associated can be given an integral representation. That this was possible, at least under certain conditions, and with the use of special summation methods, was shown first by Schwartz, [31], whose work eventually led to the work of Berenstein and Taylor [16] that showed that under appropriate conditions, and suitable summation procedures, the solutions of infinite order differential equations, and more generally the solutions of convolution equations (see below), could be given exponential representations in terms of the associated varieties (this time the varieties are not algebraic anymore, but rather analytic, something that creates a host of technical challenges). The reader interested in learning more is referred to [17, 32, 33].

One can also become interested in the more general case in which we allow symbols with faster growth, and to see what they imply in terms of the operators that are associated. A well known case of this situation occurs when one considers the space $\mathcal{H}(\mathbb{C})$ of entire functions, and its dual, the space of entire functionals. This space, that obviously contains the space of analytic functionals supported at the origin (or hyperfunctions supported at the origin), is itself isomorphic, by Fourier-Borel transform, to the space of entire functions of exponential type, namely entire functions F such that for some constants $A, B > 0$ their growth is controlled by

$$|F(z)| \leq Ae^{B|z|}. \tag{1.1}$$

The operators associated to such functions are not infinite order differential operators, though they can still be formally written as series of derivatives. A typical example is the function

$$e^z = \sum_{j=0}^{+\infty} \frac{z^j}{j!};$$

this function clearly satisfies (1.1) and the operator naturally associated to it is the operator that acts on a function f as

$$\sum_{j=0}^{+\infty} \frac{1}{j!} \frac{d^j f}{dz^j},$$

but it is easy to show that such function corresponds to $f(z + 1)$, so that the operator whose symbol is e^z is not an infinite order differential operator, but rather the simplest of convolution operators, the translation by 1.

Functions F satisfying (1.1) form a space denoted by $A_1(\mathbb{C})$, see Sect. 2, and the notion can be further generalized to consider functions satisfying the bound

$$|F(z)| \leq Ae^{B|z|^p}$$

and which will belong to the space $A_p(\mathbb{C})$. The (strong) dual of $A_p(\mathbb{C})$ is related, via Fourier-Borel transform, to another important type of space denoted by $A_{q,0}(\mathbb{C})$. The topology in these type of spaces is given via inductive or projective limits but to compute explicit estimates it is useful to have a more direct description, that we shall provide in Sect. 2.

While infinite order differential operators, in their various forms, are of great intrinsic interest, the reason for this article is to discuss in some more detail how their properties are of importance in the study of superoscillatory phenomena, see Sect. 3, as they provide a tool to establish the longevity of superoscillations. As we mentioned, most of the results hold in several complex variables, and we refer the interested reader to the very recent works on this topic [3, 4, 22, 25]. In this paper, however, we consider another higher dimensional case, the one given by the hypercomplex setting, in which we have successfully extended the basics of the theory of infinite order differential operators.

Here is a description of our paper. After this introductory section, whose content is well known, we will devote an entire section to the study of spaces of entire functions with growth conditions, their topologies, and their duals. In Sect. 3 we will use these results and ideas to address the fundamental issue of longevity for superoscillations, thus showing how infinite order differential operators, and more generally convolution operators, play a crucial role in this area. In Sect. 4 we shall present some new results on the continuity of such operators. Finally, we will show how the idea of infinite order differential operators extends fruitfully to the hypercomplex setting. This last section will include both some early results that the authors obtained a few decades ago, as well as some newer results who became available because of recent developments in the theory of hypercomplex functions.

2 Spaces of Entire Functions

As we shall see in the next sections, our results in the framework of superoscillations and in the hypercomplex setting make use of some suitable spaces of functions satisfying growth conditions depending on a plurisubharmonic function which is used as a weight, see [17, 27, 32, 34]. For our purposes we do not need this generality, indeed to define the spaces of functions that we need, it is enough to select weights of the form $w(z) = |z|^p$, $p > 0$, where for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set $|z| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$.

Definition 2.1 Let p be a positive real number, and let $\mathcal{H}(\mathbb{C}^n)$ be the space of entire functions in $\mathbb{C}^n$. The space $A_p(\mathbb{C}^n)$ is defined as

$$A_p(\mathbb{C}^n) := \{f \in \mathcal{H}(\mathbb{C}^n) \ : \ \exists\, A > 0, \quad B > 0 \ : \ |F(z)| \leq A \exp(B|z|^p)\}$$

and it is called the space of entire functions of order less or equal to p and of finite type.

Definition 2.2 Let p be a positive real number. The space $A_{p,0}(\mathbb{C}^n)$ is defined as

$$A_{p,0}(\mathbb{C}^n) := \{f \in \mathcal{H}(\mathbb{C}^n) \ : \ \forall \varepsilon > 0, \exists A_\varepsilon > 0 \ : \ |f(z)| \leq A_\varepsilon \exp(\varepsilon|z|^p)\},$$

and it is called the space of entire functions of order less or equal p and of minimal type.

Remark 2.3 For the sake of simplicity, from now on we shall write $\mathcal{H}$, A_p, $A_{p,0}$ instead of $\mathcal{H}(\mathbb{C}^n)$, $A_p(\mathbb{C}^n)$, $A_{p,0}(\mathbb{C}^n)$.

Remark 2.4 Note that if $p = 1$, then the space A_1 is isomorphic, via Fourier-Borel transform, to the space $\mathcal{H}'$ of analytic functionals. Moreover, the space $A_{1,0}$ is the space of the so-called functions of *infraexponential type*, or of order one and type zero, and as we remarked in the introduction is isomorphic, via Fourier-Borel transform, to the space of analytic functionals carried by the origin (which in turn coincides with the space of hyperfunctions supported at the origin). This identification is a crucial fact in the theory of infinite order differential operators.

To describe the topology in these spaces, we consider $\tau > 0$ and we introduce the set $A_{p,\tau}$ of entire functions f such that

$$\|f\|_{p,\tau} := \sup_{z\in\mathbb{C}^n} |f(z)| \exp(-\tau|z|^p) < \infty.$$

We call $\|f\|_{p,\tau}$ the (p, τ)-norm of f. The space $A_{p,\tau}$ equipped with this norm is a Banach space. We also note that when $\tau < \tau'$ there is an inclusion map $A_{p,\tau} \hookrightarrow A_{p,\tau'}$ (which is also a compact operator). We have that A_p and $A_{p,0}$ are the inductive and projective limit, respectively, of these spaces. In symbols:

$$A_p = \varinjlim_{\tau>0} A_{p,\tau}, \qquad A_{p,0} = \varprojlim_{\tau>0} A_{p,\tau},$$

and they turn out to be a DFS (Dual Fréchet-Schwartz space) or an FS space (Fréchet-Schwartz space), respectively. It should be noted that the above topologies are independent of the choice of a sequence $\{\tau_n\}$ of positive numbers τ_n. Moreover, to say that a sequence $\{f_j\}$ is convergent to f in A_p means that there exists $\tau > 0$ such that $f_j, f \in A_{p,\tau}$, for all j, and $\|f_j - f\|_{p,\tau} \to 0$ for $j \to \infty$.

These function spaces are well known in the literature, however, for our scope, it is necessary to make precise the notion of continuity of linear operators, since in some proofs it is required to perform careful estimates, see [3, 12–14]. For this reason we provide the explicit proofs of the next two theorems.

Theorem 2.5 *Let $F : A_p \to A_p$ be a linear operator. Then F is continuous if and only if the following condition holds:*

For any $\tau > 0$, there exist $C > 0$ and $\tau' > 0$ for which $F(A_{p,\tau}) \subset A_{p,\tau'}$ and

$$\|Ff\|_{p,\tau'} \le C\|f\|_{p,\tau} \quad \text{for any } f \in A_{p,\tau}. \tag{2.1}$$

Proof For any $\tau > 0$, $A_{p,\tau}$ is a subspace of A_p and the natural inclusion mapping $A_{p,\tau} \to A_p$ is continuous. By the definition of the topology on A_p, a map $F : A_p \to A_p$ is continuous if and only if its restriction $F|_{A_{p,\tau}} : A_{p,\tau} \hookrightarrow A_p$ is continuous for every $\tau > 0$. Since F is a linear operator, applying [28] [Chap. 4, Part 1, 5, Corollary 1] to $F|_{A_{p,\tau}}$ for each $\tau > 0$, we know that there exists $\tau' > 0$ such that

$$F|_{A_{p,\tau}} : A_{p,\tau} \to A_{p,\tau'}$$

is a continuous operator. Hence it is a bounded operator and we have (2.1) for some $C > 0$. Conversely, if (2.1) holds, then $F : A_{p,\tau} \to A_{p,\tau'} \subset A_p$ is clearly a continuous operator.

□

Theorem 2.6 *Let $F : A_{p,0} \to A_{p,0}$ be a linear operator. Then F is continuous if and only if the following condition holds:*

For any $\tau > 0$, there exist $C > 0$ and $\tau' > 0$ for which

$$\|Ff\|_{p,\tau} \le C\|f\|_{p,\tau'} \quad \text{for any } f \in A_{p,0}. \tag{2.2}$$

Proof The space $A_{p,0}$ has a system of semi-norms $\|\cdot\|_{p,\tau}$ $(\tau > 0)$. Set

$$U_{\varepsilon,\tau} = \{f \in A_{p,0} \mid \|f\|_{p,\tau} < \varepsilon\}.$$

Then $\{U_{\varepsilon,\tau}\}$ $(\varepsilon > 0, \tau > 0)$ is a fundamental system of neighborhoods of the origin in $A_{p,0}$. Condition (2.2) implies that for any $\varepsilon > 0$, $\tau > 0$, there exist $\delta > 0$, $\tau' > 0$ for which

$$F(U_{\delta,\tau'}) \subset U_{\varepsilon,\tau}.$$

Indeed we can take $\delta = \varepsilon/C$. Hence F is a continuous mapping.

To show the converse, we assume that F is a continuous mapping and we take a countable fundamental system

$$\{U_{\frac{1}{k},\frac{1}{\ell}}\} \quad k, \ell \in \mathbb{N}$$

of neighborhoods of the origin in $A_{p,0}$. Hence the continuity of a linear operator $F : A_{p,0} \to A_{p,0}$ can be characterized by using the limits of sequences: F is continuous if and only if

$$F(f_k) \to 0 \quad \text{for any} \quad f_k \to 0 \ \text{ in } \ A_{p,0}. \tag{2.3}$$

Here $f_k \to 0$ means that for any $\varepsilon > 0$, $\tau > 0$, there exists $N \in \mathbb{N}$ such that $k > N$ implies $f_k \in U_{\varepsilon,\tau}$.

Suppose now that (2.2) does not hold. Then there exists $\tau > 0$ such that for any $C > 0$ and $\tau' > 0$, there is an $f \in A_{p,0}$ satisfying

$$\|Ff\|_{p,\tau} > C\|f\|_{p,\tau'}. \tag{2.4}$$

For every $k \in \mathbb{N}$, we can find $f_k \in A_{p,0}$ for which

$$\|Ff_k\|_{p,\tau} > k\|f_k\|_{p,1/k}.$$

If we set

$$g_k = \frac{f_k}{\sqrt{k}\,\|f_k\|_{p,1/k}},$$

then we have $\|g_k\|_{p,1/k} = 1/\sqrt{k}$. The norm $\|\cdot\|_{p,1/k}$ is increasing with respect to k. Hence, for any $\tau > 0$, there exists $N \in \mathbb{N}$ such that $k > N$ implies $\|f\|_{p,\tau} < \|f\|_{p,1/k}$ $(f \in A_{p,0})$. Therefore we have $g_k \to 0$. On the other hand, we have that

$$\|Fg_k\|_{p,\tau} = \frac{\|Ff_k\|_{p,\tau}}{\sqrt{k}\,\|f_k\|_{p,1/k}} > \sqrt{k} \to \infty,$$

and so F is not continuous. This is in contradiction with our hypothesis and this concludes the proof. □

In [13, Lemma 2.2] we proved the following result which contains a useful characterization of the Taylor coefficients of a function in A_p:

Proposition 2.7 *The function*

$$f(z) = \sum_{j=0}^{\infty} f_j z^j$$

belongs to A_p if and only if there exist $C_f > 0$ and $b > 0$ such that

$$|f_j| \leq C_f \frac{b^j}{\Gamma(\frac{j}{p}+1)}.$$

Remark 2.8 The proof in [13] is based on an estimate on the norms of $f \in A_p$. More precisely, one needs to estimate the derivatives

$$f^{(j)}(z) = \frac{j!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{j+1}}\, dw, \qquad j = 0, 1, \ldots$$

where the path of integration γ is the circle $|w - z| = s|z|$, s is a positive real number and $z \neq 0$. Since $f \in A_p$ is equivalent to $f \in A_{p,\tau}$ for some $\tau > 0$, we can set $C_f = \max_\gamma \|f\|_{p,\tau}$. This fact shows that $f \to 0$ in A_p if and only if $C_f \to 0$ which is crucial in various results later on.

The spaces we have just introduced satisfy an interesting duality property. Consider $p(z) = |z|^p$ and $q(z) = |z|^q$ where $p > 1, q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let us denote by A'_p the strong dual of A_p, namely the space of continuous linear functional on A_p endowed with the strong topology. Define the Fourier-Borel transform of $\mu \in A'_p$ as the entire function

$$\hat{\mu}(w) = \mu(\exp(-z \cdot w)), \qquad z \in \mathbb{C}^n.$$

The duality between $\mathcal{H}_{\{0\}}$ and $A_{1,0}$, as well as the duality between $\mathcal{H}$ and A_1 can be generalized to this new setting:

Theorem 2.9 *Let $p, q \in \mathbb{R}$, $p > 1$, $q > 1$ be such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The following isomorphisms

$$\widehat{A'_p} \cong A_{q,0}$$

and

$$\widehat{A'_{p,0}} \cong A_q,$$

are algebraic and topological as well.

3 Infinite Order Differential Operators and Superoscillations

In this section we show how infinite order differential operators can naturally arise in the study of superoscillations.

Superoscillatory functions were introduced in physics, see e.g. [1] and the more recent [7], and from a mathematical point of view, they can be described as a superposition of small Fourier components with a bounded Fourier spectrum, in modulus less than 1, that can nevertheless result in a shift by an arbitrarily large a.

They can also be thought of as an approximation of e^{iax}, $|a| > 1$, in terms of a sequence of the form

$$\{F_n(x,a)\}_{n=0}^{\infty}, \qquad F_n(x,a) = \sum_{j=0}^{n} C_j(n;a)e^{ik_j(n)x}$$

with $|k_j(n)| \leq 1$. The prototypical example is the sequence of functions:

$$F_n(x,a) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n = \left(\frac{1+a}{2}e^{ix/n} + \frac{1-a}{2}e^{-ix/n}\right)^n \tag{3.1}$$

where $a \in \mathbb{R}$, $a > 1$. By performing a binomial expansion, this sequence can be written as

$$\sum_{j=0}^{n} C_j(n;a)e^{i(1-2j/n)x}$$

for suitable coefficients $C_j(n;a)$. However, in the limit $F_n(x,a) \to e^{iax}$, that is, it displays a wavelength much larger than one.

An important question that was posed originally by both Aharonov and Berry is whether the superoscillating behavior persists after evolving a superoscillatory function according to some differential equations, for example the Schrödinger equation. To answer the question, one may use a method in two steps. First, one uses Fourier analysis, to solve the Cauchy problem associated to the Schrödinger equation. Second, one complexifies the setting, and demonstrates the permanence of the superoscillatory behavior as a consequence of a continuity theorem for suitable (convolution) operators with nonconstant coefficient of the form:

$$\mathcal{U}\left(t, \frac{\partial}{\partial z}\right) := \sum_{m=0}^{\infty} b_m(t,z)\frac{\partial^m}{\partial z^m}.$$

For brevity we will from now on write ∂_z instead of $\frac{\partial}{\partial z}$. In the sequel, we shall consider a subclass of this class of operators. In order to prove our main results we need some more notations and definitions:

Definition 3.1 Let $p \geq 1$. We denote by $\mathcal{D}_{p,0}$ the set of operators of the form

$$P(z,\partial_z) = \sum_{n=0}^{\infty} a_n(z)\partial_z^n$$

satisfying the properties:

(i) the functions $a_n(z)$ $(n = 0, 1, 2, \ldots)$ are entire;

(ii) there exists a constant $B > 0$ such that for every $\varepsilon > 0$ one can take a constant $C_\varepsilon > 0$ for which

$$|a_n(z)| \le C_\varepsilon \frac{\varepsilon^n}{(n!)^{1/q}} \exp(B|z|^p),$$

holds, where

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and $1/q = 0$ when $p = 1$.

Remark 3.2 Note that in [14], a wider class $\boldsymbol{D}_p$ of operators is introduced. It is proved that the class includes $\mathcal{D}_{p,0}$ as a proper subset and that any linear continuous operator acting on A_p is represented by an operator in $\boldsymbol{D}_p$.

We now repeat the statement of [13, Theorem 2.4] and its proof. In fact it is useful, for further reference, to make the constants appearing in the proof more explicit.

Theorem 3.3 *Let $P(z, \partial_z) \in \mathcal{D}_{p,0}$ and let $f \in A_p$. Then $P(z, \partial_z) f \in A_p$ and $P(z, \partial_z)$ is continuous on A_p, that is $P(z, \partial_z) f \to 0$ as $f \to 0$.*

Proof The computations in [13] show that if we apply the operator $P(z, \partial_z)$ to $f \in A_p$ we obtain

$$P(z, \partial_z) f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n(z) f_{n+k} \frac{(k+n)!}{k!} z^k.$$

Then we have

$$|P(z, \partial_z) f(z)| \le \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_n(z)|\, |f_{n+k}| \frac{(k+n)!}{k!} |z|^k.$$

Since $f \in A_p$ means that $f \in A_{p,\tau}$ for some $\tau > 0$, then using the assumptions on a_n and f_j, we get

$$|P(z, \partial_z) f(z)| \le C_f C_\varepsilon \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\varepsilon^n}{(n!)^{1/q}} \exp(B|z|^p) \; \frac{b^{n+k}}{\Gamma\left(\frac{n+k}{p} + 1\right)} \frac{(k+n)!}{k!} |z|^k,$$

where C_f is as in Remark 2.8. Since

$$(n!)^{1/q} \ge \Gamma\left(\frac{n}{q} + 1\right) \quad \text{and} \quad (k+n)! \le 2^{k+n} k! n!$$

we get

$$
\begin{aligned}
&|P(z,\partial_z)f(z)| \leq \\
&\leq C_f C_\varepsilon \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\varepsilon^n}{\Gamma\left(\frac{n}{q}+1\right)} \frac{b^{n+k}}{\Gamma\left(\frac{n+k}{p}+1\right)} \frac{2^{k+n}k!n!}{k!}|z|^k \exp(B|z|^p) \\
&\leq C_f C_\varepsilon \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\varepsilon^n}{\Gamma\left(\frac{n}{q}+1\right)} \frac{b^{n+k}}{\Gamma\left(\frac{n+k}{p}+1\right)} 2^{k+n}n!|z|^k \exp(B|z|^p) \\
&\leq C_f C_\varepsilon \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (2b)^k(2\varepsilon b)^n \frac{1}{\Gamma\left(\frac{n}{q}+1\right)} \frac{n!}{\Gamma\left(\frac{n+k}{p}+1\right)}|z|^k \exp(B|z|^p).
\end{aligned}
\tag{3.2}
$$

Using

$$
\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \leq 1
$$

we deduce

$$
\Gamma\left(\frac{k+n}{p}+1\right) \geq \Gamma\left(\frac{k}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{p}+\frac{1}{2}\right).
$$

We can now rewrite (3.2) in the form

$$
\begin{aligned}
|P(z,\partial_z)f(z)| \leq C_f C_\varepsilon \sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\left(\frac{k}{p}+\frac{1}{2}\right)}|z|^k \exp(B|z|^p) \\
\times \sum_{n=0}^{\infty} (2\varepsilon b)^n \frac{n!}{\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{q}+1\right)}.
\end{aligned}
$$

Reasoning as in [13], we consider

$$
\sum_{n=0}^{\infty} (2\varepsilon b)^n \frac{n!}{\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{q}+1\right)}
$$

and observe that

$$
(2\varepsilon b)^n \frac{n!}{\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{q}+1\right)} \sim \frac{n^n(2\varepsilon b)^n}{\left(\frac{n}{p}\right)^{n/p}\left(\frac{n}{q}\right)^{n/q}} = (2\varepsilon b)^n[p^{1/p}q^{1/q}]^n.
$$

Since ε is arbitrary small the series converges, i.e.

$$\sum_{n=0}^{\infty}(2\varepsilon b)^n \frac{n!}{\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{q}+1\right)} = C'.$$

Using the properties of the Mittag-Leffler function, we have

$$\sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\left(\frac{k}{p}+\frac{1}{2}\right)}|z|^k \le C' \exp(B'|z|^p),$$

and from this with some more computations we conclude that there exists $B'' > 0$ such that

$$|P(z, \partial_z) f(z)| \le C' C_f C_\varepsilon \exp(B''|z|^p)$$

which means that $P(z, \partial_z) f(z) \in A_p$. As we observed in Remark 2.8, $C_f \to 0$ if and only if $f \to 0$ and so the same estimate proves the continuity, i.e. $|P(z, \partial_z) f(z)| \to 0$ when $f \to 0$. □

The importance of these results can be shown by going back to the fundamental question of longevity of superoscillations, and analyzing a bit more in detail how that question can be answered.

To begin with, Aharonov posed the following problem. Consider the solution of the Cauchy problem

$$i\frac{\partial \psi_n}{\partial t} = H(\psi_n)$$

$$\psi_n(x, 0) = F_n(x, a)$$

where H is the Hamiltonian of a given physical system, and $F_n(x, a)$ is the prototypical superoscillating sequence. Can we prove that the sequence of the solutions ψ_n is still superoscillatory? In a series of papers, we have shown that the answer to this question is positive, at least for a very large class of Hamiltonians. While we have not been able yet to give a general theorem that allows us to determine whether the answer is positive for a given Hamiltonian, current results (some obtained by the authors, others by colleagues and collaborators) include the case of the free particle [5], the quantum harmonic oscillator [19], the uniform magnetic field [20], the centrifugal potential [21], the step potential [8], where an interest phenomenon tied to tunneling emerges, the Klein Gordon field [9], where a two boundaries value problem has to be solved, the radial harmonic oscillator [11], the uniform electric field [6], the Dirac δ and δ' potentials [2, 15], the Dirac field [26], to mention a few. What is remarkable, however, is the fact that the strategy for each of these situations is essentially the same, though the technical challenges are very different from case to case. We will describe here the general approach in the simplest of cases, and that will clarify why the study of the spaces A_p is so relevant to our interests.

So, let us consider the case in which the Hamiltonian is simply $H = -\frac{\partial^2}{\partial x^2}$, namely the case of the free particle, which is historically the first case to which we applied our approach. Because the theory of infinite order differential operators is better suited to the holomorphic case, we will rewrite the problem replacing the real variable x with a complex variable z. Note that the initial value is a finite sum of exponentials, and thus it is a real analytic function that extends to an entire function of the variable z. In this case, it is then easy to verify (purely by substitution) that the solution for the Cauchy problem is given by

$$\psi_n(z,t) = \sum_{j=0}^{n} C_j(n;a) e^{izk_{j,n}} e^{-itk_{j,n}^2},$$

where we write $k_{j,n} = 1 - \frac{j}{2n}$ instead of $k_j(n)$. This step, immediate in the case of the free particle, ends up being very complicated in many of the cases we mentioned above. It usually entails finding the Green function associated to the Hamiltonian, and then working with it to obtain a manageable formula for the solution of the Cauchy system; the reader should not be misled by the apparent ease with which this result can be obtained in the free particle case. At this point, the next step consists in manipulating the solution of the Cauchy system in such a way as to rewrite it as the result of the action of an operator of infinite order (usually, in fact, a convolution operator) on the original superoscillatory function. In this particular case, this second step is also fairly simple, since one can use the fact that

$$e^{-itk_{j,n}^2} = \sum_{m=0}^{+\infty} \frac{(-itk_{j,n}^2)^m}{m!}$$

to rewrite the solution as

$$\begin{aligned}
\psi_n(z,t) &= \sum_{j=0}^{n} C_j(n;a) e^{izk_{j,n}} \sum_{m=0}^{+\infty} \frac{(-itk_{j,n}^2)^m}{m!} = \\
&= \sum_{m=0}^{+\infty} \frac{(-it)^m}{m!} \sum_{j=0}^{n} C_j(n;a) e^{izk_{j,n}} k_{j,n}^{2m} = \\
&= \sum_{m=0}^{+\infty} \frac{(it)^m}{m!} \sum_{j=0}^{n} C_j(n;a) \frac{d^{2m}}{dz^{2m}} e^{izk_{j,n}} = \\
&= \sum_{m=0}^{+\infty} \frac{(it)^m}{m!} \frac{d^{2m} F_n(z;a)}{dz^{2m}}.
\end{aligned}$$

We have therefore proved that the solution to the Cauchy problem for the free particle with superoscillatory initial data can be obtained by applying to the initial data $F_n(z; a)$ the variable coefficients convolution operator

$$U\left(t, \frac{d}{dz}\right) := \sum_{m=0}^{+\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dz^{2m}}.$$

Note that the symbol of this operator is the entire function

$$f(t, \zeta) = \sum_{m=0}^{+\infty} \frac{(it)^m \zeta^{2m}}{m!} = e^{it\zeta^2},$$

which is not of exponential type but rather belongs to the space A_2. For this reason, even though U formally looks like an infinite order differential operator, it actually acts as a convolution operator and we will use this terminology from now on.

Remark 3.4 We have therefore shown that if we denote by T_H the operator that associates to an entire function F the solution to the Cauchy problem with initial data F, and given Hamiltonian H, this operator is in fact a variable coefficients convolution operator U_H. How this operator can be constructed depends, of course, on the specific Hamiltonian being used, but our results show the existence of a fairly general correspondence between Hamiltonians and convolution operators (we do not have a general proof of this fact, but we have shown it in a very large number of cases as we indicated before). Equally important is to note that the symbol f_H of the operator U_H is an entire function that belongs to a space A_p where the weight p depends on the Hamiltonian itself. If, for example, the Hamiltonian were $H = -\frac{\partial^m}{\partial x^m}$, then the symbol of the associated operator would be an entire function in A_m. In more general cases, we actually have to look at weights $p(z)$ that are more general than the powers $|z|^p$, but we will not get into this detail at this point. Suffice to say that the step that transforms the solution, obtained via Green function, into a differential operator is in general highly more complex than the one indicated here.

We can now use this transformation to answer the original question. We have shown that the solution $\psi_n(z, t)$ to the original (complexified) Cauchy problem is given by

$$\psi_n(z, t) = U\left(t, \frac{d}{dz}\right)(F_n(z; a)),$$

for the convolution operator described above, Since this operator has symbol in A_2, it acts continuously on the space $A_{2,0}$ and therefore certainly as a continuous operator from A_1 to $A_{2,0}$, and so we can conclude that

$$\lim_{n\to+\infty} \psi_n(z,t) = \lim_{n\to+\infty} U\left(t, \frac{d}{dz}\right)(F_n(z;a)) =$$
$$= U\left(t, \frac{d}{dz}\right)\left(\lim_{n\to+\infty} F_n(z;a)\right) =$$
$$= U\left(t, \frac{d}{dz}\right)(e^{iaz}) = e^{iaz}e^{-ia^2t}.$$

We can now restrict the equality $\lim_{n\to+\infty} \psi_n(z,t) = e^{iaz}e^{-ia^2t}$ to the real axis, thus showing that the superoscillatory behavior is being preserved through the evolution via the Schrödinger equation.

4 Hypercomplex Case: Cauchy-Fueter and Monogenic Case

In this section, we move to infinite order differential operators acting on spaces of entire hyperholomorphic functions. More precisely, we shall consider two classes of such functions, both extending the class of holomorphic functions of one complex variable: the slice monogenic and the monogenic functions. In both cases, the functions have values in $\mathbb{R}_n$, the real Clifford algebra over n imaginary units $e_1, \ldots, e_n$ and are defined on open spaces of the $n+1$ Euclidean space $\mathbb{R}^{n+1}$. The element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ are identified with elements in the Clifford algebra, namely the so-called paravectors, via:

$$(x_0, x_1, \ldots, x_n) \mapsto x = x_0 + \underline{x} = x_0 + \sum_{\ell=1}^{n} x_\ell e_\ell.$$

If $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ can thus be interpreted as a function of the paravector x. The real part x_0 of x will also be denoted by $\mathrm{Re}(x)$. An element in $\mathbb{R}_n$ is called a Clifford number, and can be written as

$$a = a_0 + a_1e_1 + \ldots + a_ne_n + a_{12}e_1e_2 + \ldots + a_{123}e_1e_2e_3 + \ldots + a_{12\ldots n}e_1e_2\ldots e_n,$$

or $a = \sum_A a_A e_A$, where A belongs to the power set $P(1, \ldots, n)$, and if $A = \{i_1 \ldots i_r\}$ we set $e_A = e_{i_1} \ldots e_{i_r}$ and $e_\emptyset = 1$. Note that, using the relations $e_i^2 = -1$, $e_ie_j + e_je_i = 0$, $i, j \in \{1, \ldots, n\}$, $i \neq j$, the indices can be put in increasing order $i_1 < \ldots < i_r$. The norm of an element $y \in \mathbb{R}_n$ is defined as its Euclidean norm $|y|^2 = \sum_A |y_A|^2$, and in particular the norm of the paravector $x \in \mathbb{R}^{n+1}$ is $|x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2$. We define the conjugate of x as $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{\ell=1}^{n} x_\ell e_\ell$.

We denote by $\mathbb{S}$ the set

$$\mathbb{S} = \{\underline{x} = e_1x_1 + \ldots + e_nx_n \mid x_1^2 + \ldots + x_n^2 = 1\};$$

whose elements $\mathbf{j}$ are such that $\mathbf{j}^2 = -1$. Given a nonreal element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ we put $\mathbf{j}_x = \underline{x}/|\underline{x}|$ if $\underline{x} \neq 0$, and given an element $x \in \mathbb{R}^{n+1}$, the set

$$[x] := \{y \in \mathbb{R}^{n+1} \ : \ y = x_0 + \mathbf{j}|\underline{x}|, \ \mathbf{j} \in \mathbb{S}\}$$

is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R}+\mathbf{j}\mathbb{R}$, $\mathbf{j} \in \mathbb{S}$, is denoted by $\mathbb{C}_\mathbf{j}$ and an element belonging to $\mathbb{C}_\mathbf{j}$ will be indicated by $u + \mathbf{j}v$, for $u, v \in \mathbb{R}$. We say that $U \subseteq \mathbb{R}^{n+1}$ is axially symmetric if $[x] \subset U$ for any $x \in U$.

Definition 4.1 (Slice Hyperholomorphic Functions with Values in $\mathbb{R}_n$ (or Slice Monogenic Functions)) Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{R}_n$ is called a left slice function, if it is of the form

$$f(q) = f_0(u, v) + \mathbf{j} f_1(u, v) \qquad \text{for } q = u + \mathbf{j}v \in U$$

where the two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ satisfy the compatibility conditions

$$f_0(u, -v) = f_0(u, v), \qquad f_1(u, -v) = -f_1(u, v). \tag{4.1}$$

If in addition f_0 and f_1 satisfy the Cauchy-Riemann-equations

$$\frac{\partial}{\partial u} f_0(u, v) - \frac{\partial}{\partial v} f_1(u, v) = 0 \tag{4.2}$$

$$\frac{\partial}{\partial v} f_0(u, v) + \frac{\partial}{\partial u} f_1(u, v) = 0, \tag{4.3}$$

then f is called left slice hyperholomorphic (or left slice monogenic). Similar definitions can be given for $f(q) = f_0(u, v) + f_1(u, v)\mathbf{j}$ giving rise to the theory of right slice functions, or slice hyperholomorphic (slice monogenic) functions.

In both cases, when $U = \mathbb{R}^{n+1}$ we say that f is entire.

If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called intrinsic. We denote the sets of left or right slice hyperholomorphic functions on U by $\mathcal{SM}_L(U)$ and $\mathcal{SM}_R(U)$, respectively. When we do not distinguish between left of right we simply write $\mathcal{SM}(U)$.

Definition 4.2 Let $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$, let $x = u + \mathbf{j}v \in U$ and $f_\mathbf{j}$ be the restriction of f to the complex plane $\mathbb{C}_\mathbf{j}$. If x is not real, then we say that f admits left slice derivative in x if

$$\partial_S f(x) := \lim_{p \to x,\, p \in \mathbb{C}_\mathbf{j}} (p - x)^{-1}(f_\mathbf{j}(p) - f_\mathbf{j}(x)) \tag{4.4}$$

exists and is finite. If x is real, then we say that f admits left slice derivative in x if (4.4) exists for any $\mathbf{j} \in \mathbb{S}$. Similarly, we say that f admits right slice derivative in a nonreal point $x = u + \mathbf{j}v \in U$ if

$$\partial_S f(x) := \lim_{p \to x,\, p \in \mathbb{C}_{\mathbf{j}}} (f_{\mathbf{j}}(p) - f_{\mathbf{j}}(x))(p - x)^{-1} \tag{4.5}$$

exists and is finite, and we say that f admits right slice derivative in a real point $x \in U$ if (4.5) exists and is finite, for any $\mathbf{j} \in \mathbb{S}$.

We define

$$M_{f_{|\mathbb{C}_{\mathbf{j}}}}(r) = \max_{|z|=r, z \in \mathbb{C}_{\mathbf{j}}} |f(z)|, \quad \text{for } r \geq 0$$

and

$$M_f(r) = \max_{|x|=r} |f(x)|, \quad \text{for } r \geq 0.$$

Definition 4.3 Let f be an entire slice monogenic function. Then we say that f is of finite order if there exists $\kappa > 0$ such that

$$M_f(r) < e^{r^\kappa}$$

for sufficiently large r. The greatest lower bound ρ of such numbers κ is called order of f. Equivalently:

$$\rho = \limsup_{r \to \infty} \frac{\ln \ln M_f(r)}{\ln r}.$$

Let $f \in \mathcal{SM}(\mathbb{R}^{n+1})$ be of order ρ and let $A > 0$ be such that for sufficiently large values of r we have

$$M_f(r) < e^{Ar^\rho}.$$

We say that f of order ρ is of type σ if σ is the greatest lower bound of such numbers and

$$\sigma = \limsup_{r \to \infty} \frac{\ln M_f(r)}{r^\rho}.$$

Definition 4.4 Let $p \geq 1$. We denote by $\mathcal{SM}^p$ the space of entire slice monogenic functions with either order lower than p or order equal to p and finite type. It consists of those functions $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$, for which there exist constants

$B, C > 0$ such that

$$|f(x)| \leq Ce^{B|x|^p}, \forall x \in \mathbb{R}^{n+1}. \tag{4.6}$$

Let $(f_m)_{m\in\mathbb{N}}$, $f_0 \in \mathcal{SM}^p$. Then $f_m \to f_0$ in $\mathcal{SM}^p$ if there exists some $B > 0$ such that

$$\lim_{m\to\infty} \sup_{x\in\mathbb{R}^{n+1}} \left|(f_m(x) - f_0(x))e^{-B|x|^p}\right| = 0. \tag{4.7}$$

The following results, which extend the analogous results in the complex holomorphic case, are proved in [10]. We state them in the case of left slice monogenic functions but they are valid also in the right case, with obvious changes.

Proposition 4.5 *Let $p \geq 1$. A function*

$$f(x) = \sum_{k=0}^{\infty} x^k \alpha_k$$

belongs to $\mathcal{SM}_L^p$ if and only if there exist constants $C_f, b_f > 0$ such that

$$|\alpha_k| \leq C_f \frac{b_f^k}{\Gamma(\frac{k}{p}+1)}. \tag{4.8}$$

Furthermore, let f_m be a sequence in $\mathcal{SM}_L^p$; then $f_m \to 0$ for $m \to +\infty$ if and only if we can take constants C_{f_m} and b_m, $m \in \mathbb{N}$ such that the sequence $\{b_{f_m}\}_{m\in\mathbb{N}}$ is bounded and $C_{f_m} \to 0$ as $m \to +\infty$.

Following [10], we now introduce a class of infinite order differential operators acting on spaces of entire slice monogenic functions. These operators are designed to preserve the slice monogenicity and in fact they involve the so-called $\star$-product, a special notion of multiplication of functions designed to preserve the slice monogenicity, see [23, 24]. Since $\mathbb{R}_n$ is non-commutative for $n \geq 2$ (while $\mathbb{R}_1 = \mathbb{C}$) we need a product preserving slice monogenicity on the left, denoted by $\star_L$ and one on the right, denoted by $\star_R$. We refer the reader to [23] for details. These operators are a generalization to the hypercomplex setting of the class of operators denoted by $\mathbf{D}_p$ in [14].

Definition 4.6 Let $(u_m)_{m\in\mathbb{N}_0} : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be entire functions in $\mathcal{SM}_L$ (resp. $\mathcal{SM}_R$) and $p \geq 1$. Assume that for every $\varepsilon > 0$ there exist $B_\varepsilon > 0$, $C_\varepsilon > 0$ for which

$$|u_m(x)| \leq C_\varepsilon \frac{\varepsilon^m}{(m!)^{1/q}} \exp(B_\varepsilon |x|^p), \quad for\ all \quad m \in \mathbb{N}_0, \tag{4.9}$$

where $1/p + 1/q = 1$ and $1/q = 0$ when $p = 1$. For $(u_m)_{m\in\mathbb{N}_0}$ in $\mathcal{SM}_L$ entire functions as above, $\mathbf{D}^L_{p,0}$ denotes the set of formal operators defined by

$$U_L(x, \partial_{x_0}) f(x) := \sum_{m=0}^{\infty} u_m(x) \star_L \partial^m_{x_0} f(x),$$

and acting on entire functions in $\mathcal{SM}_L$.

One of the main results in [10] is the following:

Theorem 4.7 *Let $p \geq 1$ and let $\mathbf{D}^L_{p,0}$ be the sets of formal operators as in Definition 4.6. Let $U_L(x, \partial_{x_0}) \in \mathbf{D}^L_{p,0}$ and let $f \in \mathcal{SM}^p_L$, then $U_L(x, \partial_{x_0}) f \in \mathcal{SM}^p_L$ and the operator $U_L(x, \partial_{x_0})$ acts continuously on $\mathcal{SM}^p_L$, i.e., if $(f_m) \subset \mathcal{SM}^p_L$ and $f_m \to 0$ in $\mathcal{SM}^p_L$ then $U_L(x, \partial_{x_0}) f_m \to 0$ in $\mathcal{SM}^p_L$.*

A similar characterization can be given in the case of right slice monogenic functions. The proof of these results mimics that one in the complex case, adapted with the $\star$-multiplication. In particular, it makes use of the the function

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)},$$

called Mittag-Leffler function, is an entire slice monogenic function of order $1/\alpha$ (and of type 1) for $\alpha > 0$ and $\mathrm{Re}(\beta) > 0$.

Another class of functions generalizing holomorphic functions to the hypercomplex case is that of monogenic functions.

Definition 4.8 Let $U \subseteq \mathbb{R}^{n+1}$ be an open subset. A function $f : U \to \mathbb{R}_n$, of class C^1, is called left monogenic if

$$\mathcal{D} f = \frac{\partial}{\partial x_0} f + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} f = 0.$$

A function $g : U \to \mathbb{R}_n$, of class C^1, is called right monogenic if

$$f \mathcal{D} = \frac{\partial}{\partial x_0} f + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f e_i = 0.$$

The set of left monogenic functions (resp. right monogenic functions) will be denoted by $\mathcal{M}_L(U)$ (resp. $\mathcal{M}_R(U)$); if $U = \mathbb{R}^{n+1}$ we simply denote it by $\mathcal{M}_L$ (resp. $\mathcal{M}_R$) and the functions are called entire monogenic.

The class of monogenic functions has been widely studied in the literature for a longtime. These functions are, in particular, harmonic in $n + 1$ variables, while slice monogenic functions are not (they are harmonic but in two variables). On the other hand, monomials x^k in the paravector variable x are not monogenic but it is possible to construct some special homogeneous polynomials, called Fueter polynomials, which are monogenic. We recall them below and we refer the reader to [18] for more information.

Let $k = (k_1, \dots, k_n)$ where k_i are integers, be a multi–index. Let $|k| = \sum_{i=1}^n k_i$ and $k! = \prod_{i=1}^n k_i!$. We define the homogeneous polynomials $P_k(x)$ as follows: for $0 = (0, \dots, 0)$ we set

$$P_0(x) := 1.$$

For a multi-index k with $|k| > 0$ and the integers k_j nonnegative, we define $P_k(x)$ as follows: for each k consider the sequence of indices $j_1, j_2, \dots, j_{|k|}$ be given such that 1 appears in the sequence k_1 times, 2 appears k_2 times, and so on, and let n appears k_n times. We define $z_i = x_i - x_0 e_i$ for any $i = 1, \dots, n$ and $z = (z_1, \dots, z_n)$. We set

$$z^k := z_{j_1} z_{j_2} \dots z_{j_{|k|}}$$

and

$$|z|^k = |z_1|^{k_1} \cdots |z_n|^{k_n}$$

these products contains z_1 exactly k_1-times, z_2 exactly k_2-times etc. We define

$$P_k(x) = \frac{1}{|k|!} \sum_{\sigma \in perm(k)} \sigma(z^k) := \frac{1}{|k|!} \sum_{\sigma \in perm(k)} z_{j_{\sigma(1)}} z_{j_{\sigma(2)}} \dots z_{j_{\sigma(|k|)}},$$

where $perm(k)$ is the permutation group with $|k|$ elements.

We note that the Fueter polynomials $P_k(x)$ are both left and right monogenic.

We then define the polynomials:

$$V_k(x) := k! P_k(x) = \frac{k!}{|k|!} \sum_{\sigma \in perm(k)} z_{j_{\sigma(1)}} z_{j_{\sigma(2)}} \dots z_{j_{\sigma(|k|)}}.$$

Remark 4.9 The following estimates holds

$$|P_k(x)| \le |x|^{|k|}.$$

Since the monomials x^k are not monogenic also the transcendental functions $\exp(x)$, $\sin(x)$, $\cos(x)$, etc. are not monogenic. However, it makes sense to consider exponential bounds and give the following:

Definition 4.10 Let f be an entire left monogenic function. Then we say that f is of finite order if there exists $\kappa > 0$ such that