Modeling and Simulation in Science, Engineering and Technology

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Preface

This book provides a unified treatment of continuum physics. A systematic approach to the balance equations is elaborated for wide-ranging classes of materials (classical continua, micropolar continua, mixtures, electromagnetic continua) following the lines of the encyclopedic handbook articles of Truesdell and Toupin and Truesdell and Noll. As is standard in Rational Thermodynamics, the constitutive properties are required to obey the objectivity principle and to be consistent with the second law of thermodynamics. Yet here a rather new approach is developed by viewing the entropy production as a constitutive function per se as is the case for the entropy and the entropy flux. While this does not determine any new result for simple materials, it proves conceptually and practically advantageous in the modelling of nonlinear phenomena such as those occurring in hysteretic continua, e.g. in plasticity, electromagnetism, and physics of shape-memory alloys.

The book is suitable for engineers, physicists, and mathematicians. The derivations of the sought results are fairly detailed through careful proofs. Though a wide variety of subjects are examined, the contents are developed so as to get a self-contained and consistent presentation of the various topics.

Part I reviews the kinematics of continuous bodies and illustrates the general setting of balance laws. Kinematics treats essential preliminaries to continuum physics such as reference and current configurations, transport relations, singular surfaces, objectivity and objective time derivatives. Next, a chapter on balance equations develops the balance laws of mass, linear momentum, angular momentum, energy, entropy (Clausius–Duhem inequality), and the balance laws in electromagnetism.

Part II is first devoted to the general requirements of constitutive models. In this sense, emphasis is given to the application of objectivity (the constitutive equations must be invariant under changes of frame) and consistency with the second law of thermodynamics (the constitutive equations must satisfy the restrictions placed by the entropy inequality). Next, a review is given of common models of simple materials, namely materials described by the first-order gradient of deformation, velocity, and temperature. In this framework, detailed descriptions are given of (thermoelastic, elastic, and dissipative) solids and (elastic, thermoelastic, viscous, and Newtonian) fluids.

A wide variety of constitutive models is investigated in Part III, consisting of separate chapters of non-simple materials. The chapter on rate-type models reviews the rheological devices and next shows some schemes within the Eulerian and the Lagrangian description with emphasis on objective time derivatives. The chapter on materials with memory begins with a general setting of (fading) memory and next shows memory models for thermoelasticity, heat conduction, viscoelasticity, electromagnetic solids, and modelling via fractional derivatives. Next, the modelling of aging (thermoelastic, rate-type, and thermo-viscoelastic) materials is exhibited in connection with the thermodynamic restrictions. Also, materials of higher-order grade are examined in the form of fluids and solids or models via interstitial working. The possible consistency of the hyperstress with the standard balance laws is shown to hold. A chapter is devoted to mixtures; in addition to balance equations for the constituents and the whole mixture, some aspects are investigated such as models of diffusion, Soret and Dufour effects, immiscible mixtures, and models for the whole mixture. Micropolar media are modelled as materials with a physical internal structure. Each point of the continuum is viewed as a body with a finite number of degrees of freedom. The balance laws are then reviewed so that the internal degrees of freedom show up in the balance of orientational momentum and energy. The model, so established, is then applied to the description of liquid crystals and nanofluids. Porous materials are described as mixtures (solid–liquid or solid–void). Also, porous materials with double porosity are established. The chapter on electromagnetism of continuous media describes a number of phenomena. The interaction of the electromagnetic field with deformation is examined within electroelasticity, magnetoelasticity, and plasma theory. This in turn allows us to examine magneto-, electro-, and mechanical-optical effects. Memory effects are modelled in quite a general setting. Nonlinearity effects are especially framed within micromagnetics and ferrofluids. Chiral media and ferrites are investigated in detail also to show the optical activity effects on linearly- and circularly-polarized waves. Finally, superconductivity and superfluidity are developed in a common framework of mixtures of reacting fluids; normal electrons and superconducting electrons in one case, normal fluid and superfluid in the other.

Hysteretic effects and phase transitions are developed in Part IV. Hysteresis is modelled in ferroelectrics, ferromagnetism, and plasticity. In all of these contexts, the modelling is performed by having recourse to the entropy production as a (non-negative) constitutive function. It follows that the free energy governs the anhysteretic behaviour, while the entropy production characterizes the hysteretic properties. Phase transitions are described in different ways. A transition may occur at a sharp interface between two different phases; the jump conditions across the interface govern the transition. Instead, a transition may occur in a diffuse region where the pertinent fields change continuously; in essence, the transition region is occupied by a mixture of constituents in different phases. A scheme for phase transitions and hysteretic effects in shape-memory alloys is also outlined.

Notation

We use mostly direct notation and the recourse to index notation is made only when, otherwise, the pertinent expression might become ambiguous. The symbols of continuum mechanics are used, e.g. in the books of Truesdell and Noll [426] and Gurtin et al [214]. Yet difficulties arise whenever any well-established symbol has a different meaning in continuum mechanics and in electromagnetism. For instance, **E** is the Green-St Venant strain tensor in continuum mechanics and the electric field in electromagnetism while **D** is the stretching tensor in continuum mechanics and the electric displacement in electromagnetism. To avoid the introduction of newly defined symbols, we have maintained the standard symbols whenever the pertinent chapter makes it clear which meaning has to be assigned to the symbol. There are places where the stretching tensor and the electric displacement occur simultaneously; in such cases, we use $\mathfrak{D}$ for the stretching tensor. Another difficulty arises in connection with the vector $\boldsymbol{\omega}$; it is currently used for angular velocity and vorticity. To avoid any ambiguities, we have maintained the symbol $\boldsymbol{\omega}$ for the angular velocity and have used $\boldsymbol{\varpi}$ for the vorticity (i.e. $\nabla \times \mathbf{v}$).

Throughout, lightface letters indicate scalars, while boldface letters indicate vectors or tensors. In the mechanical context, lowercase (uppercase) boldface letters indicate vectors (tensors). In electromagnetism, following the literature, the (electric and magnetic) vectors are denoted by uppercase letters, e.g. $\mathbf{E}, \mathbf{B}$. Moreover, Lin is the set of all tensors, Lin^+ is the set of all tensors with positive determinant, Sym is the set of all symmetric tensors, Skw is the set of all skew tensors, Psym is the set of all symmetric, positive definite tensors, Orth is the set of all orthogonal tensors, Orth^+ is the set of all rotations (all orthogonal tensors with positive determinant). For the benefit of the reader an Appendix "Notes on vectors and tensors" reviews the essential contents of algebra and analysis of vector and tensor functions.

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Contents

Part I
Basic Principles and Balance Equations

Chapter 1
Kinematics

This chapter deals with the kinematics of deformable bodies. Both deformation and motion of a body are developed by using the reference configuration; the position vector in the reference configuration is the operative label of the points of the body.

The basic relations so determined for deformation and motion are essential to the next chapters. Attention is addressed to the topics of objectivity and objective time derivatives, thus establishing a general framework that proves remarkable in the description of material properties in terms of time derivatives. This framework shows the connection between various known objective time derivatives (Jaumann, Green–Naghdi, Cotter–Rivlin, Oldroyd, Truesdell). Transport relations are obtained for convecting (or non-convecting) sets, thus establishing basic properties for the derivation of (local) balance equations and jump conditions for discontinuous fields. Hence, the kinematical and the geometric relations are derived for singular surfaces which provide a general setting for the investigation of discontinuity waves. Moreover, the transport theorems for surface integrals are established thus leading, in particular, to the convected time derivative.

1.1 Frames of Reference and Configurations

A frame of reference or observer $\mathscr{F}$ is an arbitrary set of rigidly fixed axes relative to which the position of points is determined. For simplicity, the chosen fixed axes are taken to be orthogonal and hence the corresponding unit vectors are an orthonormal basis; denote by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the chosen orthonormal basis. The time at which an event takes place can be specified relative to a particular event taken as a reference. An event is then a pair $\{P, t\}$ consisting of a point P, in the three-dimensional Euclidean space $\mathscr{E}$, and a time t. A chosen position O, origin, is taken as reference and hence we can identify each position P with the position vector $\mathbf{x}$ relative to the origin,

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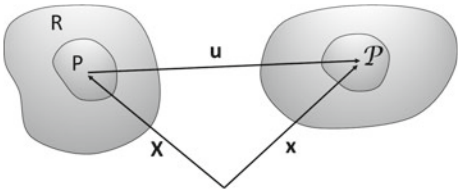

Fig. 1.1 A motion χ maps the point, labelled by the position vector $\mathbf{X} \in \mathrm{R}$, to the position vector $\mathbf{x} = \chi(\mathbf{X}, t)$

$\mathbf{x} = P - O$. Hence, a frame of reference $\mathscr{F}$ may be viewed as the Cartesian product of the set $\{\mathscr{E}, O, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\}$ and the real time axis $\mathbb{R}$.

A change of frame is a 1–1 mapping of space time onto itself such that distances, time intervals, and temporal order are preserved. An event $\{\mathbf{x}, t\}$ and its image $\{\mathbf{x}^*, t^*\}$ under a change of frame are related by rigid transformations and a time shift. Chosen possibly different origins O, O^*, for the two observers, $\mathscr{F}, \mathscr{F}^*$, we can express a change of frame by

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}, \qquad t^* = t - a,$$

where $a \in \mathbb{R}$, $\mathbf{c}(t) = O - O^*$, and $\mathbf{Q}(t)$ is a rotation tensor, $\det \mathbf{Q}(t) = 1$.

Bodies occupy regions of the Euclidean space $\mathscr{E}$. To describe the evolution of a body, we need to label the points.[1] This is accomplished by labelling the points of the body by the position in a reference configuration (or placement) $\mathrm{R} \subset \mathscr{E}$; sometimes the body is identified with the reference configuration [216]. Upon choosing an origin, the positions in R are associated with a position vector; to make it apparent that we are dealing with the reference configuration the vector is denoted by a uppercase boldface letter, most often $\mathbf{X}$. By definition, the reference configuration is independent of time. We denote by ∇ the gradient operator in $\mathscr{E}$ and by ∇_R the gradient operator in R (Fig. 1.1).

The motion of the body is described by saying the position of any point in $\mathscr{E}$ as a function of time. Granted a choice of origin also for the position in the current configuration, the current position vector, say $\mathbf{x}$, is then a function of the point (at) $\mathbf{X}$ and time t. Hence, a *motion* of the body is a (smooth) function χ that assigns to each position (vector) in R and time t a position

$$\mathbf{x} = \chi(\mathbf{X}, t)$$

in space; we say that $\mathbf{X}$ is a material vector while $\mathbf{x}$ is a spatial vector. Symbolically, $\chi(\mathrm{R}, t) \subset \mathscr{E}$ denotes the current configuration (at time t). The reference configuration R might be the configuration at a chosen initial time t_0 but need not be so. If $\mathbf{X}$ is fixed then $\chi(\mathbf{X}, t)$ describes the motion of the pertinent point in time. Hence

$$\dot{\chi}(\mathbf{X}, t) := \partial_t \chi(\mathbf{X}, t), \qquad \ddot{\chi}(\mathbf{X}, t) := \partial_t^2 \chi(\mathbf{X}, t),$$

[1] The term points is used, instead of particles, to make it clear that no internal structure is ascribed.

are the velocity and acceleration of the point $\mathbf{X}$ at time t. If, instead, t is fixed then χ is a mapping from R to the current configuration, at time t.

For brevity, throughout $\mathbf{X} \in \mathrm{R}$ means that $\mathbf{X}$ is the position vector of a point in R. Likewise, $\mathbf{x} \in \mathscr{D} \subset \mathscr{E}$ means that $\mathbf{x}$ is the position vector of a point in $\mathscr{D}$.

1.2 Deformation

To begin with, we restrict attention to a fixed time t and hence $\chi(\mathbf{X}, t)$, as a function of $\mathbf{X}$, describes the *deformation* at time t. Hence, the dependence on the parameter t is understood and we let

$$\mathbf{x} = \chi(\mathbf{X}).$$

It is assumed that $\chi(\mathbf{X})$ is a 1–1 mapping so that two material points, $\mathbf{X} \neq \mathbf{Y}$, cannot occupy the same position in the current configuration, $\mathbf{x} \neq \mathbf{y}$. Assume also that χ is differentiable. Hence, there is a tensor $\mathbf{F}(\mathbf{X})$ such that

$$\chi(\mathbf{Y}) - \chi(\mathbf{X}) = \mathbf{F}\,(\mathbf{Y} - \mathbf{X}) + o(|\mathbf{Y} - \mathbf{X}|),$$

$\mathbf{F}$ being a tensor function of the position $\mathbf{X}$ and possibly of t; $\mathbf{F}$ is said to be the *deformation gradient.* To within $o(|\mathbf{Y} - \mathbf{X}|)$, $\mathbf{F}$ maps material vectors, $\mathbf{Y} - \mathbf{X}$, to spatial vectors, $\chi(\mathbf{Y}) - \chi(\mathbf{X})$. If χ depends linearly on $\mathbf{X}$ then $\mathbf{F}$ is independent of $\mathbf{X}$,

$$\chi(\mathbf{Y}) - \chi(\mathbf{X}) = \mathbf{F}\,(\mathbf{Y} - \mathbf{X}),$$

and the deformation is said to be *homogeneous.* Hence, in general, a deformation approaches the corresponding homogeneous deformation, with $\mathbf{F} = \mathbf{F}(\mathbf{X})$; the closer $\mathbf{Y}$ is to $\mathbf{X}$ the more χ approaches a homogeneous deformation.

In components,[2] there is a matrix $F \in \mathbb{R}^{3\times 3}$ such that, for any two points $\mathbf{X}, \mathbf{Y} \in \mathrm{R}$,

$$\chi_h(\mathbf{Y}) - \chi_h(\mathbf{X}) = F_{hK}(\mathbf{X})(Y_K - X_K) + o(|\mathbf{Y} - \mathbf{X}|), \qquad h = 1, 2, 3.$$

The entries $\{F_{hK}\}$ are the components of the deformation gradient

$$\mathbf{F} = \nabla_R \chi, \qquad F_{hK} = \partial_{X_K} \chi_h.$$

To determine the effects of deformation on lengths, areas, and volumes, it is convenient to establish some properties of the deformation gradient. First we observe that the invertibility of χ implies

$$J := \det \mathbf{F} \neq 0.$$

[2] As in well-known textbooks [152, 428, 429], to make the notation more explicit, throughout in suffix notation, we use capital indices for quantities related to the reference configuration.

Indeed since the trivial deformation

$$\mathbf{x} = \mathbf{X}$$

gives $J = 1$ we may regard the deformation as a continuous process and hence we assume

$$J > 0.$$

Also, we let

$$j = \det \mathbf{F}^{-1}$$

and hence $J\,j = 1$.

As any invertible tensor, the deformation gradient $\mathbf{F}$ satisfies the *polar decomposition*

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR},$$

where $\mathbf{R} \in$ Orth while $\mathbf{U}, \mathbf{V} \in$ Psym.

To show this property, we first observe that $\mathbf{F}^T\mathbf{F} \in$ Psym in that, for any vector $\mathbf{w}$,

$$\mathbf{w} \cdot \mathbf{F}^T\mathbf{F}\mathbf{w} = (\mathbf{F}\mathbf{w}) \cdot (\mathbf{F}\mathbf{w}) \geq 0$$

and, since $\mathbf{F}$ is invertible,

$$\mathbf{F}\mathbf{w} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{w} = \mathbf{0}.$$

We can then define[3] $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$. Since $\mathbf{F}^T\mathbf{F} \in$ Psym then the eigenvalues, say $\{\mu_i\}$, are positive; let $\{\mathbf{N}_i\}$ be the eigenvectors and then $\mathbf{F}^T\mathbf{F} = \sum_{i=1}^3 \mu_i \mathbf{N}_i \otimes \mathbf{N}_i$. Hence $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ is defined as $\mathbf{U} = \sum_{i=1}^3 \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i$, $\lambda_i = \sqrt{\mu_i}$. Hence also $\mathbf{U}$ is invertible and $\mathbf{U}^{-1} = \sum_{i=1}^3 (1/\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i$. We can then write $\mathbf{F} = \mathbf{RU}$ and take it as the definition of $\mathbf{R}$; we find that

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}, \qquad \mathbf{R}^T\mathbf{R} = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{1}$$

and, if $J > 0$ then $\det \mathbf{R} = 1$. Accordingly, $\mathbf{R}$ is a rotation if $J > 0$ and is orthogonal if merely $J \neq 0$. Given $\mathbf{U} : \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ the tensor $\mathbf{R}$ is unique in that by

$$\mathbf{RU} = \hat{\mathbf{R}}\mathbf{U}$$

it follows $\mathbf{R} = \hat{\mathbf{R}}$.

Likewise, since $\mathbf{F}\mathbf{F}^T \in$ Psym we can define $\mathbf{V} \in$ Psym such that

$$\mathbf{F}\mathbf{F}^T = \mathbf{V}^2.$$

[3] The uniqueness of $\mathbf{U}$ is proved in [216], p. 32.

Let $\mathbf{F} = \mathbf{V}\tilde{\mathbf{R}}$. It follows $\det \tilde{\mathbf{R}} = 1$ if $J > 0$. Now by

$$\mathbf{F} = \mathbf{V}\tilde{\mathbf{R}} = \tilde{\mathbf{R}}\tilde{\mathbf{R}}^T\mathbf{V}\tilde{\mathbf{R}} = \tilde{\mathbf{R}}(\tilde{\mathbf{R}}^T\mathbf{V}\tilde{\mathbf{R}})$$

and the uniqueness of the decomposition **RU**, it follows

$$\tilde{\mathbf{R}} = \mathbf{R}, \qquad \mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T.$$

We can then write the following statement.

Theorem 1.1 (Polar decomposition) *If* $\mathbf{F}$ *is an invertible tensor with* $\det \mathbf{F} > 0$ *then there are unique, symmetric, positive-definite tensors* $\mathbf{U}$ *and* $\mathbf{V}$ *and a rotation* $\mathbf{R}$ *such that* $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$.

We say that $\mathbf{F} = \mathbf{R}\mathbf{U}$ and $\mathbf{F} = \mathbf{V}\mathbf{R}$ are the right and left polar decompositions of $\mathbf{F}$.

A positive-definite symmetric tensor represents a state of pure stretches along three mutually orthogonal axes. Therefore, the polar decomposition means that any (homogeneous) deformation may be viewed as the result of a pure stretch $\mathbf{U}$ and a rotation $\mathbf{R}$ or the same rotation $\mathbf{R}$ followed by the stretch $\mathbf{V}$. Accordingly, $\mathbf{U}$ is called the *right stretch tensor* and $\mathbf{V}$ the *left stretch tensor*. In calculations, it proves more convenient to use the *right and left Cauchy–Green tensors*

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \qquad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T.$$

When $\mathbf{F}$ is a rotation, $\mathbf{F} = \mathbf{R}$, $\mathbf{F}^T\mathbf{F} = \mathbf{1}$. Also $\mathbf{F} = \mathbf{1}$ if the deformation is the identity transformation $\mathbf{x} = \mathbf{X}$. It is then useful to use the *Green-St. Venant strain tensor*

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{1}).$$

The vector

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \chi(\mathbf{X}) - \mathbf{X}$$

is called the *displacement* of the point (at $\mathbf{X}$). The *displacement gradient* is the tensor[4]

$$\mathbf{H} = \nabla_R \mathbf{u} = \mathbf{F} - \mathbf{1}.$$

Hence

$$\mathbf{C} = (\mathbf{H} + \mathbf{1})^T(\mathbf{H} + \mathbf{1}) = \mathbf{1} + \mathbf{H}^T + \mathbf{H} + \mathbf{H}^T\mathbf{H}$$

and

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T) + \tfrac{1}{2}\mathbf{H}^T\mathbf{H}.$$

[4] To follow the standard notation in continuum mechanics, the symbols **B**, **H**, **E**, and **D** are used. When **B**, **H**, **E**, and **D** are used within electromagnetism they denote the magnetic induction, the magnetic field, the electric field, and the electric displacement.

It is useful sometimes to consider the linear part $\mathcal{E}$ of $\mathbf{E}$, i.e.

$$\mathcal{E} := \frac{1}{2}(\mathbf{H}^T + \mathbf{H});$$

we refer to $\mathcal{E}$ as the *strain tensor.* Since

$$\mathbf{E} = \mathcal{E} + \mathbf{H}^T\mathbf{H}$$

then for small deformations, that is when $\|\mathbf{H}\| \ll 1$, we can use the approximation $\mathbf{E} \simeq \mathcal{E}$.

Further

$$\mathcal{E} = \mathrm{sym}\mathbf{H} = \mathrm{sym}[\nabla\mathbf{u}(\mathbf{1} + \mathbf{H})].$$

The *infinitesimal strain tensor* is defined by

$$\varepsilon := \mathrm{sym}\nabla\mathbf{u}.$$

Hence, for small deformations $\mathbf{E} \simeq \mathcal{E} \simeq \varepsilon$.

It is worth observing that $\mathbf{C}$ and $\mathbf{B}$ admit the same principal invariants. By direct calculations, it follows that

$$\mathrm{tr}\,\mathbf{C} = \mathrm{tr}\,\mathbf{F}^T\mathbf{F} = \mathbf{F}\cdot\mathbf{F}, \qquad \mathrm{tr}\,\mathbf{B} = \mathrm{tr}\,\mathbf{F}\mathbf{F}^T = \mathbf{F}\cdot\mathbf{F},$$

$$\mathrm{tr}\,\mathbf{C}^2 = \mathrm{tr}\,(\mathbf{F}^T\mathbf{F}\mathbf{F}^T\mathbf{F}) = (\mathbf{F}^T\mathbf{F})\cdot(\mathbf{F}^T\mathbf{F}), \quad \mathrm{tr}\,\mathbf{B}^2 = \mathrm{tr}\,(\mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{F}^T) = (\mathbf{F}^T\mathbf{F})\cdot(\mathbf{F}^T\mathbf{F}),$$

$$\det\mathbf{C} = \det\mathbf{F}^T\mathbf{F} = (\det\mathbf{F})^2, \qquad \det\mathbf{B} = \det\mathbf{F}\mathbf{F}^T = (\det\mathbf{F})^2.$$

Hence, we find the common values of the principal invariants

$$I_1 = \mathrm{tr}\,\mathbf{C} = \mathrm{tr}\,\mathbf{B},$$

$$I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}\,\mathbf{C}^2] = \frac{1}{2}[(\mathrm{tr}\,\mathbf{B})^2 - \mathrm{tr}\,\mathbf{B}^2],$$

$$I_3 = \det\mathbf{C} = \det\mathbf{B}.$$

By the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, it follows that, associated with a deformation gradient $\mathbf{F}$, there are various rotation-independent tensors related to $\mathbf{U}$ and $\mathbf{V}$. The right Cauchy–Green deformation tensor (or Green's deformation tensor) is defined by $\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$. The inverse $\mathbf{C}^{-1}$ is denoted by $\mathcal{F}$,

$$\mathcal{F} = \mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-T}.$$

As to $\mathbf{V}$, the left Cauchy–Green deformation tensor is denoted by $\mathbf{B}$ and defined by $\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$. The inverse $\mathbf{B}^{-1}$ is denoted by $\mathcal{C}$,

$$\mathcal{C} = \mathbf{B}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1}.$$

In the literature, $\mathcal{F}$ is called the Finger tensor (and denoted by $\mathbf{f}$). The inverse $\mathcal{C}$ of $\mathbf{B}$ was introduced by Cauchy and is denoted by $\mathbf{c}$; as with tensors, we prefer to use a capital letter ($\mathcal{C}$ instead of $\mathbf{c}$). In addition, $\mathcal{C}$ is called Piola tensor, and Finger tensor in the fluid dynamics literature. We avoid the use of Piola tensor for $\mathcal{C}$ because the Piola, or Piola-Kirchhoff, tensor is a well-known stress tensor of continuum mechanics.

We now examine the spectral representation of $\mathbf{F}$. The common eigenvalues of $\mathbf{C} = \mathbf{U}^2$ and $\mathbf{B} = \mathbf{V}^2$ are λ_i^2, $i = 1, 2, 3$. Hence

$$\mathbf{C} = \textstyle\sum_{i=1}^{3} \lambda_i^2 \mathbf{N}_i \otimes \mathbf{N}_i, \qquad \mathbf{B} = \textstyle\sum_{i=1}^{3} \lambda_i^2 \mathbf{n}_i \otimes \mathbf{n}_i$$

and

$$\mathbf{U} = \textstyle\sum_{i=1}^{3} \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i, \qquad \mathbf{V} = \textstyle\sum_{i=1}^{3} \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i = \textstyle\sum_{i=1}^{3} \lambda_i \mathbf{R}\mathbf{N}_i \otimes \mathbf{R}\mathbf{N}_i.$$

Since $\mathbf{n}_i = \mathbf{R}\mathbf{N}_i$, we can write

$$\mathbf{R} = \textstyle\sum_{j=1}^{3} \mathbf{n}_j \otimes \mathbf{N}_j;$$

$\mathbf{R}$ is the active rotation $\mathcal{R}$ as described in Sect. 1.4.1.

Hence, $\mathbf{F}$ is represented in the form

$$\mathbf{F} = \mathbf{R}\mathbf{U} = (\textstyle\sum_{j=1}^{3} \mathbf{n}_j \otimes \mathbf{N}_j)(\textstyle\sum_{i=1}^{3} \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i) = \textstyle\sum_{j=1}^{3} \lambda_j \mathbf{n}_j \otimes \mathbf{N}_j.$$

From the spectral representation, we can derive the matrix representation. If the deformation is a rotation, then $\mathbf{U} = \mathbf{1}$. Let $\mathbf{R}$ be given by a rotation of the angle θ around $\mathbf{n}_3 = \mathbf{N}_3$. Since $\mathbf{R} = \sum_{j=1}^{3} \mathbf{n}_j \otimes \mathbf{N}_j$, we find

$$R_{hk} = \mathbf{N}_h \cdot \mathbf{R}\mathbf{N}_k = \mathbf{N}_h \cdot \mathbf{n}_j \; \mathbf{N}_j \cdot \mathbf{N}_k = \mathbf{N}_h \cdot \mathbf{n}_k.$$

It follows that

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Green-St Venant strain tensor $\mathbf{E}$ describes the deviation of the present configuration from the reference one, where $\mathbf{F} = \mathbf{1}$ and then also $\mathbf{C} = \mathbf{1}$. Hence, a better description of deformation might involve $\mathbf{E}$. Now

$$\mathbf{F}^{-T}(\mathbf{C} - \mathbf{1})\mathbf{F}^{-1} = \mathbf{F}^{-T}(\mathbf{F}^T\mathbf{F} - \mathbf{1})\mathbf{F}^{-1} = \mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1} = \mathbf{1} - \mathbf{B}^{-1}.$$

Hence, letting

$$\mathbf{E}_A := \tfrac{1}{2}(\mathbf{1} - \mathbf{B}^{-1})$$

we have

$$\mathbf{E}_A = \mathbf{F}^{-T}\mathbf{E}\mathbf{F}^{-1}.$$

The tensor $\mathbf{E}_A$ is called the Eulerian Almansi strain tensor.

1.2.1 Effects on Lengths, Areas, and Volumes

Distances and Lengths

Look at two position vectors $\mathbf{X}, \mathbf{Y} \in \mathrm{R}$ and the corresponding spatial position vectors $\mathbf{x} = \chi(\mathbf{X}), \mathbf{y} = \chi(\mathbf{Y})$ in the deformed region $\chi(\mathrm{R}) \subset \mathscr{E}$. It is

$$\mathbf{y} - \mathbf{x} = \mathbf{F}(\mathbf{Y} - \mathbf{X}) + o(|\mathbf{Y} - \mathbf{X}|),$$

$\mathbf{F}$ being evaluated at $\mathbf{X}$. To within $o(|\mathbf{Y} - \mathbf{X}|)$, we have

$$\mathbf{y} - \mathbf{x} = \mathbf{F}(\mathbf{Y} - \mathbf{X}) \tag{1.1}$$

and hence

$$(\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) = [\mathbf{F}(\mathbf{Y} - \mathbf{X})] \cdot [\mathbf{F}(\mathbf{Y} - \mathbf{X})] = (\mathbf{Y} - \mathbf{X}) \cdot \mathbf{F}^T\mathbf{F}(\mathbf{Y} - \mathbf{X}) = (\mathbf{Y} - \mathbf{X}) \cdot \mathbf{C}(\mathbf{Y} - \mathbf{X}).$$

Then the distance $l = |\mathbf{y} - \mathbf{x}|$ between the points in the deformed region is given by

$$l^2 = (\mathbf{Y} - \mathbf{X}) \cdot \mathbf{C}(\mathbf{Y} - \mathbf{X}); \tag{1.2}$$

the length l depends on the material vector $\mathbf{Y} - \mathbf{X}$ and not merely on the length $|\mathbf{Y} - \mathbf{X}|$. That is why the right Cauchy–Green tensor $\mathbf{C}$ is viewed as a metric tensor.

If we are interested also in the direction of $\mathbf{y} - \mathbf{x}$, then we consider (1.1) and observe that the unit vector $\mathbf{e} = (\mathbf{Y} - \mathbf{X})/|\mathbf{Y} - \mathbf{X}|$ is mapped to

$$\frac{\mathbf{y} - \mathbf{x}}{|\mathbf{Y} - \mathbf{X}|} = \mathbf{F}\mathbf{e}.$$

The ratio of the lengths

$$\frac{|\mathbf{y} - \mathbf{x}|}{|\mathbf{Y} - \mathbf{X}|} = |\mathbf{F}\mathbf{e}|,$$

is consistent with the action of deformation via the right Cauchy–Green tensor $\mathbf{C}$,

$$|\mathbf{y} - \mathbf{x}|^2 = |\mathbf{Y} - \mathbf{X}|^2\, \mathbf{e} \cdot \mathbf{C}\mathbf{e} = |\mathbf{Y} - \mathbf{X}|^2\, \mathbf{e} \cdot \mathbf{F}^T\mathbf{F}\mathbf{e} = |\mathbf{Y} - \mathbf{X}|^2\, |\mathbf{F}\mathbf{e}|^2.$$

The deviation of the deformation from the identity deformation is well described by $\mathbf{E}$. To get a detailed effect of $\mathbf{E}$, we let $\mathbf{y} - \mathbf{x} = \mathbf{F}(\mathbf{Y} - \mathbf{X})$ and find

$$|\mathbf{y} - \mathbf{x}|^2 - |\mathbf{Y} - \mathbf{X}|^2 = (\mathbf{Y} - \mathbf{X}) \cdot (\mathbf{C} - \mathbf{1})(\mathbf{Y} - \mathbf{X}).$$

Since $\mathbf{Y} - \mathbf{X} = \mathbf{F}^{-T}(\mathbf{y} - \mathbf{x})$ we find

$$|\mathbf{y} - \mathbf{x}|^2 - |\mathbf{Y} - \mathbf{X}|^2 = \mathbf{F}^{-1}(\mathbf{y} - \mathbf{x}) \cdot (\mathbf{C} - \mathbf{1})\mathbf{F}^{-1}(\mathbf{y} - \mathbf{x}) = (\mathbf{y} - \mathbf{x}) \cdot \mathbf{F}^{-T}(\mathbf{C} - \mathbf{1})\mathbf{F}^{-1}(\mathbf{y} - \mathbf{x}).$$

Hence

$$|\mathbf{y} - \mathbf{x}|^2 - |\mathbf{Y} - \mathbf{X}|^2 = (\mathbf{Y} - \mathbf{X}) \cdot 2\mathbf{E}(\mathbf{Y} - \mathbf{X}) = (\mathbf{y} - \mathbf{x}) \cdot 2\mathbf{E}_A(\mathbf{y} - \mathbf{x}).$$

The Green-St. Venant tensor $\mathbf{E}$ yields the difference $|\mathbf{y} - \mathbf{x}|^2 - |\mathbf{Y} - \mathbf{X}|^2$ in terms of the vector $\mathbf{Y} - \mathbf{X}$ while the Eulerian Almansi tensor, $\mathbf{E}_A$ or $\mathbf{e}$, yields the difference in terms of $\mathbf{y} - \mathbf{x}$.

We now examine the length of a curve. Let C be a curve in R represented parametrically as

$$\mathbf{Y} = \hat{\mathbf{Y}}(\lambda), \qquad \lambda \in [a, b].$$

By deformation, C is mapped to $\mathcal{C}$ given by

$$\mathbf{Y}(\lambda) \;\rightarrow\; \mathbf{y}(\lambda) = \chi(\mathbf{Y}(\lambda)).$$

The line integral of a continuous function f defined on $\mathcal{C}$ reads

$$\int_a^b f(\mathbf{y}(\lambda))|\mathbf{y}'(\lambda)|d\lambda = \int_a^b f(\mathbf{y}(\chi(\lambda)))|\chi'(\mathbf{Y}(\lambda))|d\lambda$$

where $'$ means derivative with respect to the parameter λ. Since $\chi'(\mathbf{Y}(\lambda)) = \mathbf{F}(\mathbf{Y}(\lambda))\mathbf{Y}'(\lambda)$ then

$$\int_a^b f(\mathbf{y}(\lambda))|\mathbf{y}'(\lambda)|d\lambda = \int_a^b f(\chi(\mathbf{Y}(\lambda)))|\mathbf{F}(\mathbf{Y}(\lambda))\mathbf{Y}'(\lambda)|d\lambda.$$

Let $f = 1$ and observe that

$$\mathcal{L} = \int_a^b |\mathbf{y}'(\lambda)|d\lambda$$

is the length of $\mathcal{C}$ while $\int_a^b |\mathbf{Y}'(\lambda)|d\lambda$ is the length of C. Hence

$$\mathcal{L} = \int_a^b |\mathbf{F}(\mathbf{Y}(\lambda))\mathbf{Y}'(\lambda)|d\lambda.$$

By means of the polar decomposition, we can write

$$|\mathbf{F}\,\mathbf{Y}'| = (\mathbf{R}\,\mathbf{U}\,\mathbf{Y}' \cdot \mathbf{R}\,\mathbf{U}\,\mathbf{Y}')^{1/2}(\mathbf{R}^T\,\mathbf{R}\,\mathbf{U}\,\mathbf{Y}' \cdot \mathbf{U}\,\mathbf{Y}')^{1/2} = |\mathbf{U}\,\mathbf{Y}'|.$$

As a consequence

$$\mathcal{L} = \int_a^b |\mathbf{U}(\mathbf{Y}(\lambda))\mathbf{Y}'(\lambda)|d\lambda,$$

so that the length is unchanged if $\mathbf{F}$ is (locally) a pure rotation, $\mathbf{U} = \mathbf{1}$. This shows that $\mathbf{U} \neq \mathbf{1}$ does not preserve lengths.

A deformation that preserves the distance between points is said to be rigid; χ is *rigid* if

$$|\chi(\mathbf{Y}) - \chi(\mathbf{X})| = |\mathbf{Y} - \mathbf{X}|, \qquad \forall\, \mathbf{Y}, \mathbf{X} \in \mathcal{R}. \tag{1.3}$$

By (1.3), it follows

$$[\chi(\mathbf{Y}) - \chi(\mathbf{X})] \cdot [\chi(\mathbf{Y}) - \chi(\mathbf{X})] = (\mathbf{Y} - \mathbf{X}) \cdot (\mathbf{Y} - \mathbf{X}).$$

Differentiation with respect to $\mathbf{X}$ and $\mathbf{Y}$ yields

$$\mathbf{F}^T(\mathbf{Y})\mathbf{F}(\mathbf{X}) = \mathbf{1}.$$

Letting $\mathbf{X} = \mathbf{Y}$ we find

$$\mathbf{F}^T(\mathbf{X})\mathbf{F}(\mathbf{X}) = \mathbf{1}$$

and hence $\mathbf{F}$ is a rotation, say $\mathbf{R}$, at any point of the body. Consequently

$$\mathbf{F}(\mathbf{Y})\mathbf{F}^T(\mathbf{Y})\mathbf{F}(\mathbf{X}) = \mathbf{F}(\mathbf{Y})$$

implies $\mathbf{F}(\mathbf{X}) = \mathbf{F}(\mathbf{Y})$. Hence, in a rigid deformation $\mathbf{F}$ is a rotation $\mathbf{R}$, independent of the position. Moreover, by integration of

$$\nabla_R \chi = \mathbf{R}$$

we have

$$\chi(\mathbf{Y}) = \chi(\mathbf{X}) + \mathbf{R}(\mathbf{Y} - \mathbf{X}). \tag{1.4}$$

It is of interest to examine the (approximate) description of small rigid deformations. By (1.4), since $\mathbf{R} = \mathbf{F}$ then letting $\chi(\mathbf{Y}) = \mathbf{Y} + \mathbf{u}(\mathbf{Y})$, we have

$$\mathbf{u}(\mathbf{Y}) - \mathbf{u}(\mathbf{X}) = (-\mathbf{1} + \mathbf{F})(\mathbf{Y} - \mathbf{X}).$$

By $\mathbf{F}^T\mathbf{F} = \mathbf{C} = \mathbf{U}^2 = \mathbf{1}$, it follows

$$(\mathbf{1} + \mathbf{H}^T)(\mathbf{1} + \mathbf{H}) = \mathbf{1}, \qquad \mathbf{H}^T + \mathbf{H} = \mathbf{0}$$