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Domination in Graphs: Core Concepts

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Dedication

The authors dedicate this book to their coauthor and long-time good friend Pete Slater, who coauthored the predecessor of this volume, *Fundamentals of Domination in Graphs*, with Teresa and Steve.

Peter J. Slater (1946–2016)

Teresa pays a special tribute to Ulysses Grant (Lit) Haynes. I love you, Dad.

Steve pays a special tribute to Gerd H. Fricke (1946–2016), an exceptional mathematician and graph theorist, who made special contributions to domination in graphs, and in particular to the understanding of the concept of irredundance in graphs. Steve also pays a special tribute to Sandee, his wife of 43 years, with whom he has coauthored 78 papers.

Michael pays a special tribute to his brother Paul Henning, for the countless lives he has saved as an Emergency Medicine Physician over the years.

Preface

While concepts related to domination in graphs can be traced back to the Roman Empire in the fourth century AD and to the mid-1800s in connection with various chessboard problems, the mathematical concept of domination in graphs was first suggested by Kőnig in 1936, and then defined as a graph theoretical parameter by Berge in 1958. Domination in graphs experienced rapid growth from its introduction, resulting in over 1200 papers published on domination in graphs by the late 1990s.

Much of the interest in domination theory in graphs is due to its applications in many areas of study, such as genetics, chemistry, computer communication networks, facility location, social networking, surveying, transporting hazardous materials, monitoring electrical power networks, school bus routing, voting, and several areas of mathematics, to name a few.

Noting the need for a comprehensive survey of the literature on domination in graphs, in 1998 Haynes, Hedetniemi, and Slater published the first two books on domination, writing *Fundamentals of Domination in Graphs* (ISBN: 9780429157769) and editing *Domination in Graphs: Advanced Topics* (ISBN: 9781315141428). The explosive growth of this field has continued since 1998, and today more than 5000 papers have been published on domination in graphs, and the material in these two books is now more than 20 years old. Thus, we thought it was time for an update on the developments in domination theory since 1998. We also wanted to give a comprehensive treatment of only the major topics in domination. This coverage of domination, including the major results and updates, is in the form of three books: this book and its two companion books, *Topics in Domination in Graphs* (ISBN: 9783030511173) and *Structures of Domination in Graphs* (ISBN: 9783030588915), which we will call Books I, II, and III, respectively.

This book, *Domination in Graphs: Core Concepts*, is limited to, as the title suggests, the most core concepts of domination in graphs: domination, total domination, and independent domination. It contains major results on these three types of domination, including a wide variety of proofs, both short and long, of selected results that illustrate many of the proof techniques used in domination theory.

For the companion books, Books II and III, we invited leading researchers in domination theory to contribute chapters.

Book II focuses on the most-studied types of domination that are not covered in Book I. Although well over 70 types of domination have been defined, Book II focuses on those that have received the most attention in the literature, and contains chapters on paired domination, connected domination, restrained domination, multiple domination, distance domination, dominating functions, fractional domination, Roman domination, rainbow domination, locating-domination, eternal and secure domination, global domination, stratified domination, and power domination.

Book III is divided into three parts. The first part covers several domination-related concepts: broadcast domination, alliances, domatic numbers, dominator colorings, irredundance in graphs, private neighbor concepts, game domination, varieties of Roman domination, and domination in spectral graph theory. The second part contains chapters on domination in hypergraphs, chessboards, and digraphs and tournaments.

The third part focuses on the development of algorithms and complexity of signed, minus, and majority domination, power domination, and alliances in graphs. The third part also includes a chapter on self-stabilizing algorithms for domination.

This book (Book I) is intended as a reference resource for researchers and is written to reach the following audiences: first, established researchers in the field of domination who want an updated, comprehensive coverage of domination theory; second, researchers in graph theory who wish to become acquainted with newer topics in domination, along with major developments in the field and some of the proof techniques used; and third, graduate students with interests in graph theory, who might find the theory and many real-world applications of domination of interest for master's and doctoral theses topics. We also believe that this book provides a good basis for use in a seminar on either domination theory or domination algorithms and complexity, including the new algorithm paradigm of self-stabilizing domination algorithms.

This book is intended as an in-depth introduction to domination in graphs, limited to its most core concepts of domination, total domination, and independent domination. We have therefore intentionally focused more on depth than breadth in Book I, and supplied several in-depth proofs for the reader to acquaint themselves with a tool box of proof techniques and methods with which to attack open problems in the field. We have identified many unsolved problems and open conjectures, which can be used as a launching pad for future researchers in the field.

With the enormous literature that exists on domination in graphs and the dynamic nature of the subject, we were faced with the challenge of determining which topics to include and perhaps even more importantly which topics to exclude, even for the core concepts of domination, total domination, and independent domination. We have therefore been selective in the material included in this core domination book and wish to apologize in advance for omitting many important results and proofs due to space limitations.

We assume that the reader is acquainted with the basic concepts of graph theory and has had some exposure to graph theory at an introductory level. However, since graph theory terminology sometimes varies, we provide a glossary as a reference source for the reader regarding terminology and notation adopted in this book. Assuming that the reader has some familiarity with graph theory, this book is self-contained as we include the terminology and definitions involving domination in the glossary in Appendix A.

The material in this book has been organized into 18 chapters, an epilogue, and three appendices. It contains an extensive bibliography of more than 900 references, which we have cited throughout the book. A brief summary of the material covered in each chapter is presented below.

Chapter 1 *In the Beginning: Roots of Domination in Graphs* discusses the many origins, both historical and mathematical, of domination in graphs, dating as far back as the Roman Empire in the fourth century AD under Emperor Constantine.

Chapter 2 *Fundamentals of Domination* discusses how it is that the domination number, total domination number, and independent domination number can be defined in a variety of equivalent ways, each of which suggests natural generalizations of these three types of domination.

Chapter 3 *Complexity and Algorithms for Domination in Graphs* provides an overview of the core results on NP-completeness and algorithms for domination, total domination, and independent domination in graphs. It presents NP-completeness proofs for each type of domination, when restricted to several subclasses of graphs, and provides linear algorithms for computing each type of domination on trees.

Chapter 4 *General Bounds* presents some of the more basic lower and upper bounds on the domination, total domination, and independent domination numbers of graphs.

Chapter 5 *Domination in Trees* presents a wide variety of domination results for the class of trees, including lower and upper bounds, bounds in terms of the number of leaves in a tree, the Slater lower bound for trees, vertices in all or no minimum dominating sets in a tree, trees in which every vertex is a member of some minimum dominating set, trees having unique minimum dominating sets, trees in which the domination number equals the independent domination number, and trees in which the domination number equals the total domination number.

Chapter 6 *Upper Bounds in Terms of Minimum Degree* presents results which establish upper bounds on the core domination numbers in terms of the order of a graph and the minimum degree of a vertex in the graph, where for the domination number and total domination number the minimum degree ranges between one and six.

Chapter 7 *Probabilistic Bounds and Domination in Random Graphs* presents probabilistic bounds for the core domination numbers of a graph in terms of its order and minimum degree, and also considers bounds for the domination numbers of random graphs. It covers the basic questions of the probability that a randomly chosen set S of vertices in a graph G is a dominating set of one of the three basic types, if each vertex in the graph is chosen to be in the set S with a given probability.

Chapter 8 *Bounds in Terms of Size* discusses how the number of edges of a graph affects the values of the core domination numbers.

Chapter 9 *Efficient Domination in Graphs* considers classes of graphs that have dominating or total dominating sets S in which specified vertices are adjacent to exactly one vertex in S. Included in these classes of graphs are certain circulants, Cayley graphs, grid graphs, cylindrical graphs, toroidal graphs, prisms, Möbius ladders, and lexicographic graphs. Also included is a section on NP-completeness results for graphs having an efficient dominating set.

Chapter 10 *Domination and Forbidden Subgraphs* presents bounds on the three core domination numbers in classes of graphs which have certain subgraph restrictions, such as bipartite (no odd cycles), cubic (every vertex has degree three), and claw-free (no vertex has three neighbors, no two of which are adjacent).

Chapter 11 *Domination in Planar Graphs* covers domination and total domination in planar triangulations, outerplanar graphs, and in planar graphs having small diameter. Results on independent domination in planar graphs are also presented, including bipartite, cubic, and minimum diameter planar graphs.

Chapter 12 *Domination Partitions* covers vertex partitions $\pi = \{V_1, V_2, \ldots, V_k\}$ of a graph G such that each set V_i is a dominating set. Partitions into total and independent dominating sets are also considered. The nine possible ways of partitioning the vertices of a graph into two sets, say V_1 and V_2, such that V_1 is one of the three types of domination and V_2 is one of the three types of domination are also considered.

Chapter 13 *Domination Critical and Stable Graphs* presents the classes of graphs whose types of domination numbers change upon the removal of any vertex, the removal of any edge, or the addition of any edge. It also considers the classes of graphs whose domination numbers do not change, regardless of which vertex or edge is removed or which new edge is added to the graph.

Chapter 14 *Upper Domination Parameters* covers the upper domination number, the upper total domination number, and the independence number, that is, the maximum cardinalities of a minimal type of dominating set. Since the independence number, that is, the maximum cardinality of an independent set, is very well-studied in the literature, the focus of this chapter is mainly on the upper domination and upper total domination numbers, although several important results on the independence number are presented.

Chapter 15 *Relating the Core Parameters* presents relationships, inequalities, and bounds that exist between the three types of domination numbers, for example bounds on the ratio of the independent domination number to the domination number and the total domination number to the domination number. Also considered are the classes of graphs in which two of these domination numbers are always equal, for example the classes of graphs in which the independence number equals the upper domination number.

Chapter 16 *Nordhaus-Gaddum Bounds* discusses bounds on the sum and product of the domination numbers of a graph G and its complement $\overline{G}$. Bounds on the sum and product for total domination and independent domination numbers are also presented.

Chapter 17 *Domination in Grids and Hypercubes* presents results on domination, total domination, and independent domination in grids, which are chessboard-like graphs. There is also a brief discussion of cylinders (chessboards with column wrap-arounds) and tori (chessboards with both column and row wrap-arounds). The chapter concludes with domination in hypercubes.

Chapter 18 *Domination and Vizing's Conjecture* provides an overview of the most well-known conjecture in domination theory, that the domination number of the Cartesian product of two graphs G and H is greater than or equal to the product of the domination number of G and the domination number of H. Similar conjectures are also discussed for all six core domination numbers, including the lower and upper domination, total domination, and independent domination numbers.

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We have tried to eliminate errors, but surely several remain. We welcome any comments or corrections the reader may have. A list of typographical errors, corrections, and suggestions can be sent to any of our email addresses below.

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Contents

Chapter 1

In the Beginning: Roots of Domination in Graphs

1.1 Introduction

While domination in graphs was first formally defined by Berge in 1958, the roots of domination can be traced back to defense strategies used by the Roman Empire in the fourth century AD, to a precursor of the game of chess in India in the sixth century AD, and later in the mid-to-late 1800s, to a variety of chess problems. Other sources of domination can be found in a wide array of real-world areas such as radio broadcasting, computer communication networks, systems of distinct representatives, school bus routing, electrical power networks, influence in social networks, surveying, resource allocation, and even transporting hazardous materials.

In the 1900s, a variety of international researchers began to develop the mathematical foundations of domination in graphs, including the British mathematician, lawyer, and fellow at Trinity College Cambridge, W.W. Rouse Ball; the Hungarian mathematician who wrote the first book on graph theory, Dénes Kőnig; the English mathematician and statistician, Patrick Michael Grundy; the Hungarian-American mathematician, physicist, computer scientist, and engineer, John von Neumann; the German-American economist, Oskar Morgenstern; the French mathematician recognized as one of the founders of graph theory, Claude Berge; the Hungarian graph theorist Tibor Gallai; the Norwegian-American mathematician who worked in ring theory, Galois theory, and graph theory, Øystein Ore; the Soviet and Ukrainian graph theorist, Vadim Vizing; the Finnish mathematician, Juho Nieminen; and the Canadian graph theorists, Amram Meir, John Moon, and E.J. Cockayne. In this chapter, we discuss the many origins, both historical and mathematical, of domination in graphs and highlight some of the most significant contributions of these mathematicians to the theory of domination up to the year 1998 when the first two books on domination in graphs were produced by the American graph theorists, Teresa Haynes, Stephen Hedetniemi, and Peter Slater [416, 417].

T. W. Haynes et al., *Domination in Graphs: Core Concepts*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-031-09496-5_1

Before delving into the roots of domination in graphs, we give some basic definitions and notation in Section 1.2 that will be used throughout the book. To avoid repeating terminology in every chapter, we also provide a glossary in Appendix A including these basic terms and other definitions and refer the reader to it for terminology not defined on the spot.

1.2 Basic Terminology

A *graph* $G = (V, E)$ consists of a finite nonempty set $V(G)$ of objects called *vertices* together with a possibly empty set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. Throughout, unless otherwise stated, the graphs in this book are *simple graphs* with no loops or multiple edges and G is a graph with vertex set V and edge set E. The number of vertices $n = |V|$ is called the *order* of G and the number of edges $m = |E|$ is the *size* of G. An edge $\{u, v\}$ is denoted by uv. If $uv \in E$, then u and v are *adjacent vertices*. The vertex u (respectively, v) and edge uv are said to be *incident* to each other. Two distinct edges are *adjacent* if they are incident to a common vertex.

The graph consisting of a single vertex is called the *trivial graph*; a *nontrivial graph* has order $n \geq 2$. Given a graph $G = (V, E)$, the *complement* $\overline{G}$ of G is the graph $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$. The *complete graph* K_n is a graph of order n in which every two vertices are adjacent, while its complement $\overline{K_n}$ is an *empty graph*, that is, a graph on n vertices with no edges. Note that K_1 is the trivial graph.

The *open neighborhood* of a vertex $v \in V$ is the set $\mathrm{N}_G(v) = \{u : uv \in E\}$ of vertices adjacent to v, called the *neighbors* of v, and its *closed neighborhood* is the set $\mathrm{N}_G[v] = \mathrm{N}_G(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ of vertices is $\mathrm{N}_G(S) = \bigcup_{v \in S} \mathrm{N}_G(v)$, while the *closed neighborhood of a set* S is the set $\mathrm{N}_G[S] = \bigcup_{v \in S} \mathrm{N}_G[v]$. The *degree* of a vertex v is $\deg_G(v) = |\mathrm{N}_G(v)|$. If the graph G is clear from the context, then we omit the G subscript in the above expressions. A vertex $v \in V$ is called an *isolated vertex* if $\deg(v) = 0$, and is called a *leaf* if $\deg(v) = 1$. In a graph G of order n, a vertex v for which $\deg(v) = n - 1$ is called a *dominating vertex*. A graph G is called *k-regular* if every vertex $v \in V$ has $\deg(v) = k$. We say that a graph is *isolate-free* if it has no isolated vertices. The largest degree among the vertices of G is the *maximum degree* $\Delta(G)$ and the smallest degree is the *minimum degree* $\delta(G)$.

A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ and G' is a *spanning subgraph* of G if $V' = V$. For a nonempty subset $S \subseteq V$, the subgraph $G[S]$ of G *induced by* S has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. A *clique* is a complete subgraph.

A set S of vertices of a graph G is a *dominating set* if every vertex in $V \setminus S$ has a neighbor in S, that is, $\mathrm{N}[S] = V$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set of G and a dominating set with cardinality $\gamma(G)$ is called a *γ-set* of G.

A set S of vertices of an isolate-free graph G is a *total dominating set*, abbreviated TD-set, if every vertex in V is adjacent to at least one vertex in S. Thus, a subset $S \subseteq V$ is a TD-set of G if $\mathrm{N}(S) = V$. Note that since every vertex must have a neighbor in S, total domination is only defined for isolate-free graphs. The *total domination number* $\gamma_{\mathrm{t}}(G)$ equals the minimum cardinality of a TD-set of G and a TD-set with cardinality $\gamma_{\mathrm{t}}(G)$ is called a γ_{t}*-set* of G.

A *minimal dominating set* in a graph G is a dominating set that contains no dominating set of G as a proper subset, and a *minimal TD-set* of G is a TD-set that contains no TD-set of G as a proper subset. The *upper domination number* $\Gamma(G)$ equals the maximum cardinality of a minimal dominating set in G. Similarly, the *upper total domination number* $\Gamma_{\mathrm{t}}(G)$ equals the maximum cardinality of a minimal TD-set of G.

A set $S \subseteq V$ is *independent* if no two vertices in S are adjacent in G, and an independent set S is called *maximal* if no proper superset of S is independent. The *vertex independence number*, or just *independence number*, $\alpha(G)$ equals the maximum cardinality of an independent set of G. A set $S \subseteq V$ is an *independent dominating set*, abbreviated ID-set, if it is both independent and dominating. The *independent domination number* $i(G)$ equals the minimum cardinality of any ID-set of G and an ID-set with cardinality $i(G)$ is called an i*-set* of G. We note that $i(G)$ is the minimum cardinality of any maximal independent set of G. A set $M \subseteq E$ is *independent* if no two edges in M are adjacent in G, and a set of independent edges is called a *matching*. The *matching number* $\alpha'(G)$ equals the maximum number of edges in a matching of G.

A set $S \subseteq V$ is a *packing* in G if for any two vertices $u, v \in S$, $\mathrm{N}[u] \cap \mathrm{N}[v] = \emptyset$. The *packing number* $\rho(G)$ equals the maximum cardinality of a packing of G. A *vertex cover* is a set S of vertices such that every edge in E is incident to at least one vertex in S. The *vertex covering number* $\beta(G)$, also denoted $\tau(G)$, equals the minimum cardinality of a vertex cover of G. An *edge cover* is a set F of edges such that every vertex in V is incident to at least one edge in F. The *edge covering number* $\beta'(G)$ equals the minimum cardinality of an edge cover of G. These concepts will be explored in more detail in Chapters 2 and 4.

A graph G is *bipartite* if its vertex set V can be partitioned into two sets X and Y such that every edge in G joins a vertex in X and a vertex in Y. The sets X and Y are called the *partite sets* of G. We note that the partite sets of a bipartite graph are independent sets. The *complete bipartite graph* $K_{r,s}$ is a bipartite graph with partite sets X and Y, where $|X| = r$, $|Y| = s$, and every vertex in X is adjacent to every vertex in Y. The *union* $G = G_1 \cup G_2$ of two graphs G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If G is a union of k copies of a graph F, we write $G = kF$.

For an integer $k \geq 1$, let $[k] = \{1, 2, \ldots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$. A *walk* in a graph G from a vertex u to a vertex v, called a (u, v)-walk, is a finite alternating sequence of vertices and edges, starting with the vertex u and ending with the vertex v, in which each edge of the sequence joins the vertex that precedes it in the sequence to the vertex that follows it in the sequence. A (u, v)*-trail* is a (u, v)-walk containing no repeated edges and a (u, v)*-path* is a (u, v)-walk containing no repeated vertices. A *cycle* is a closed (u, v)-trail. The *length* of a path (respectively, cycle) equals the number of edges in the path (respectively, cycle). A graph of order n which itself is a path is called the *path* P_n. Thus, the path P_n is the graph of order n whose vertices can be labeled $v_1, v_2, \ldots, v_n$ and whose edges are $v_i v_{i+1}$ for $i \in [n-1]$. For an integer $n \geq 3$, the *cycle* C_n is the graph of order n whose vertices can be labeled $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i \in [n-1]$. The cycle C_n is also referred to as an *n-cycle*. We write $P_n : v_1 v_2 \ldots v_n$ and $C_n : v_1 v_2 \ldots v_n v_1$ to denote the paths and cycles, respectively, with vertex sequence $(v_1, v_2, \ldots, v_n)$.

Two vertices u and v are *connected* if there is a (u, v)-path in G, and a graph G is said to be *connected* if every two of vertices in V are connected. The *distance* $d(u, v) = d_G(u, v)$ between two vertices u and v in a connected graph G is the minimum length of a (u, v)-path in G. The *eccentricity* $\mathrm{ecc}(v) = \mathrm{ecc}_G(v)$ of a vertex v in a connected graph G is the maximum of the distances from v to the other vertices of G; that is, $\mathrm{ecc}(v) = \max\{d(u, v) : u \in V\}$. The *diameter* $\mathrm{diam}(G)$ is the maximum eccentricity taken over all vertices of G and the *radius* $\mathrm{rad}(G)$ is the minimum eccentricity taken over all vertices of G. A vertex of G with eccentricity equal to $\mathrm{rad}(G)$ is called a *central vertex*. Abusing notation slightly, we refer to a central vertex as simply a *center* and say that a graph having exactly one central vertex x is *centered at* x.

A *tree* is an acyclic connected graph. A *star* is a tree with at most one vertex that is not a leaf, that is, a star is a tree with diameter at most 2. Thus, stars consist of complete bipartite graphs $K_{1,s}$ for $s \geq 1$ along with the trivial graph K_1. A double star $S(r, s)$, for $1 \leq r \leq s$, is a tree with exactly two (adjacent) vertices that are not leaves, with one of the vertices having r leaf neighbors and the other s leaf neighbors.

The *subdivision* of edge $uv \in E$ consists of deleting the edge uv from E, adding a new vertex w to V, and adding the new edges uw and wv to E. In this case, we say that the edge uv has been *subdivided*. In general, for an edge $uv \in E$ to be subdivided $k \geq 1$ times, we mean that edge uv is removed and replaced by a (u, v)-path of length $k + 1$. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing every edge of G exactly once.

1.3 Origins

In this section, we present the origins of domination in military tactics and chessboard problems.

1.3.1 Defensive and Offensive Strategies of the Roman Empire

In the fourth century AD, the Roman Empire dominated large areas of three continents, Europe, Africa, and Asia Minor. But it had begun to lose its power and it became increasingly difficult to secure all of its conquered regions. During the reign of Emperor Constantine the Great, who ruled between 306 and 337 AD, the Roman Empire controlled Britain, Gaul, Iberia (Spain and Portugal), southern Central Europe (including Italy), Asia Minor (including Turkey and Constantinople, a city named after the Emperor), and North Africa (including Egypt).

Under Emperor Constantine, the Roman army was reorganized to consist of mobile field units and garrison soldiers, or local militia, capable of countering internal threats and barbarian invasions. A region was *secured* by armies being stationed there, and a region without an army was protected by sending mobile armies from neighboring regions. But Emperor Constantine decreed that a mobile field army could not be sent to defend a region if doing so left its original region unsecured.

This defense strategy suggests a type of domination in graphs in which there are three types of vertices: unsecured (no armies), secured with one army (usually composed of local militia, which are not mobile armies), and secured with two armies (one being a highly trained, mobile army). The condition to be met is that every unsecured vertex must be adjacent to at least one vertex at which two armies are stationed. In this way, the set of vertices having one or two armies is a dominating set of the set of vertices having no armies.

This defense strategy inspired the papers of Stewart [691] in 1999 and ReVelle and Rosing [658] in 2000, and then was formally defined as a type of domination in graphs for the first time in 2004 by Cockayne, Dreyer, Hedetniemi, and Hedetniemi [183].

1.3.2 Chaturanga

Chaturanga is a war-oriented board game generally considered to have been developed in India during the sixth century AD. The name is a Sanskrit word meaning "four arms," which stood for the four arms of the military, being the chariots, the cavalry, the elephants, and the infantry. Considered to be the precursor to the modern game of chess, chaturanga is a chesslike, two-player game played on a board of 8×8 squares, and with pieces very similar to those in chess:

1. Raja (king): moves one square in any direction.
2. Mantri (early form of queen): moves one square diagonally in any direction.
3. Ratha (rook): moves across any number of unoccupied squares either vertically or horizontally.
4. Gaja (elephant, early form of bishop): moves two squares diagonally but can jump over the first square.
5. Ashva (horse, knight): moves like the knight in chess, either two squares horizontally and then one square vertically, or two squares vertically and then one square horizontally, jumping over all intermediate squares.
6. Padáti (foot soldier, pawn): moves only one unoccupied square vertically, but can capture one square diagonally, as in chess.

A *capture* in chaturanga consists of a piece of any type moving to a square, according to the rules for that piece, on which an opponent's piece is found. The opponent's piece is captured and removed from the game, and the piece that was moved to that square and made the capture remains on that square. In this way, every piece is said to *dominate* all squares it can reach in one move. Thus, the set of squares *dominated* by the pieces of one of the two players consists of all the squares occupied by the pieces plus all the squares which can be reached in one move by all of the pieces. Although there were other board games that preceded chaturanga, they are generally called *race games* in which the objective is to reach some designated location before your opponent. Chaturanga is one of the first games to consider the concept of capturing an opponent's pieces, and hence the concept of domination first appears.

1.3.3 Eight Queens Problem

A German chess player, named Max Bezzel [75], posed the following problem in the September 1848 issue of the chess journal *Berliner Schachzeitung*:

> **Eight Queens Problem.** In how many ways can 8 queens be placed on the squares of the 8×8 chessboard so that no two queens can attack each other, that is, no two queens lie on the same row, or the same column, or the same diagonal?

A chess piece is said to *cover* (attack or dominate) any square on a chessboard that it can reach in a single move. For example, in one move a queen can move any number of unoccupied squares horizontally, vertically, or diagonally. Thus, a queen covers all of the squares in the same row, column, or diagonal as the square on which it is located, as illustrated in Figure 1.1. Figure 1.2 illustrates one placement of 8 pairwise non-attacking queens.

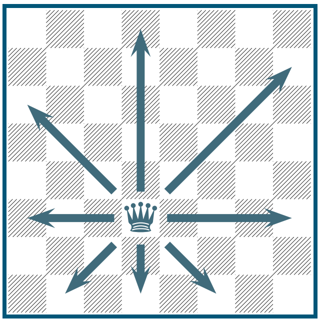

Figure 1.1 Moves of a queen on an 8×8 chessboard

The 8-Queens Problem quickly generalizes to the n-Queens Problem of placing n queens on an $n \times n$ board so that no two queens attack each other.

In graph theory terminology the n-Queens Problem is easily stated as that of finding a maximum independent set S of n vertices in the *queens graph* Q_n. The queens graph Q_n has a vertex set V consisting of the n^2 squares of an $n \times n$ chessboard, and two vertices are adjacent if and only if the corresponding squares lie on a common row, a common column, or a common diagonal. The vertex independence number $\alpha(Q_n)$ of the queens graph Q_n, therefore, equals the maximum number of queens which can be placed on the $n \times n$ chessboard so that no two queens attack

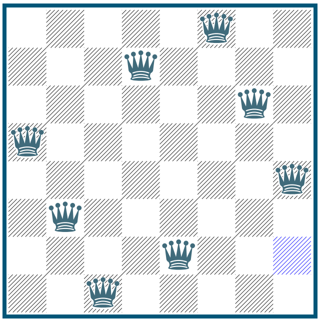

Figure 1.2 Maximum independent set of 8 queens

each other. It is obvious that $\alpha(Q_n) \leq n$, since any set of more than n queens would have to contain two queens that lie on a common row, column, or diagonal. It remains to be shown that for any n, $\alpha(Q_n) = n$. In 1910 Ahrens [9] was the first person to prove that for every positive integer $n \geq 4$, $\alpha(Q_n) = n$, that is, one can always place n queens on an $n \times n$ chessboard so that no two queens attack each other.

The 8-Queens Problem, posed by Max Bezzel, was reported to have attracted the attention of the famous mathematician Gauss, but it was Dr. Franz Nauck [608, 609] who in 1850 pointed out, apparently without proof, that there were 92 different ways to place 8 non-attacking queens on the standard chessboard. These solutions fell into 12 classes, 11 of which yield 8 solutions by rotations and reflections, and the 12th solution generates another 4 solutions. In 1874 Pauls [630] was the first to prove that 92 is indeed the total number of solutions to the 8-Queens Problem. In 1892, although no proofs were given, W.W. Rouse Ball [51] correctly reported that for boards of sizes 4, 5, 6, 7, 8, 9, and 10, there are altogether 2, 10, 4, 40, 92, 342, and 724 solutions, respectively, to the n-Queens Problem.

1.3.4 Five Queens Problem

Five Queens Problem. Show that 5 queens can be placed on the squares of the 8×8 chessboard so that every square is either occupied by a queen or is attacked by a queen. In how many ways can this be done?

It was known from the earliest times that five queens were sufficient to *cover* or *dominate* every square of the 8×8 chessboard; see for example Figure 1.3, in which the five queens mutually cover one another, and Figure 1.4, in which the five queens form an independent set.

But this was quickly generalized to the following.

Queens Domination Problem. What is the minimum number of queens which can be placed on an $n \times n$ chessboard so that every square is either occupied by a queen or is attacked by a queen?

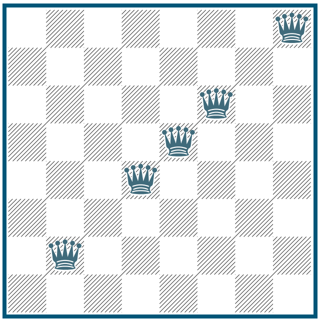

Figure 1.3 Five queens covering an 8×8 chessboard

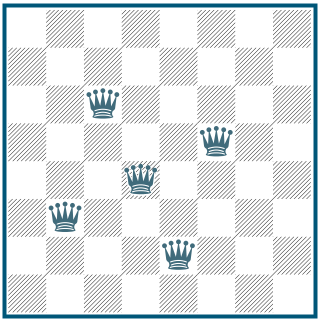

Figure 1.4 Five independent queens covering an 8×8 chessboard

According to Gibbons and Webb [335], this problem was first stated by Abbe Durand in 1861, but was also given in 1862 by C.F. de Jaenisch [218], a Finnish and Russian chess player (1813–1872) and theorist, who in the 1840s was among the top chess players in the world.

In graph theory terminology the Queens Domination Problem is to determine the *queens domination number* $\gamma(Q_n)$, that is, the minimum number of queens necessary to cover, or dominate, every square of an $n \times n$ chessboard. Although it proved to be relatively easy to determine the *queens independence number* $\alpha(Q_n) = n$, after all these years since 1861, the determination of the value of $\gamma(Q_n)$ for all $n \geq 1$, remains an unsolved, and quite difficult, problem.

In 1862 De Jaenisch [218] determined the queens domination number $\gamma(Q_n)$, for $n \in [8]$, to be $1, 1, 1, 2, 3, 3, 4, 5$. In particular, he showed that $\gamma(Q_8) = 5$; see Figure 1.4. The values $\gamma(Q_9) = \gamma(Q_{10}) = \gamma(Q_{11}) = 5$ were correctly reported by Ahrens [9] in 1910; see for example Figure 1.5. These values have since been verified by computer programs.

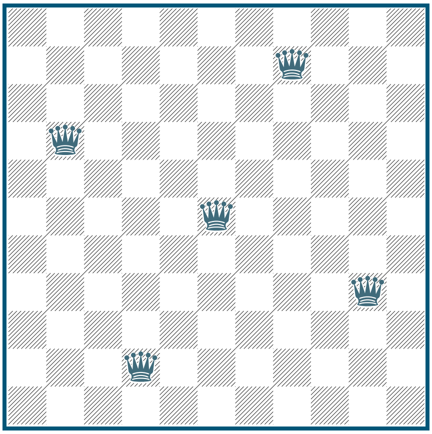

Figure 1.5 Five queens covering an 11×11 chessboard

1.3.5 Queens Independent Domination Problem

The independent domination number $i(Q_n)$ of the queens graph Q_n was identified as an interesting problem by De Jaenisch [218], who in 1862 correctly gave the first eight values of $i(Q_n)$, which are $1, 1, 1, 3, 3, 4, 4, 5$; see Figure 1.4 for $n = 8$. These have been verified by computer. It is interesting to note that $\gamma(Q_5) = 3 < i(Q_5) = 4$ and $\gamma(Q_6) = 3 < i(Q_6) = 4$, while $\gamma(Q_7) = 4 = i(Q_7)$ and $\gamma(Q_8) = 5 = i(Q_8)$.

Determining the domination numbers and independent domination numbers of the queens graph seem to be extremely difficult problems. As noted in [446] and [626], only relatively few exact values of these two domination numbers of the queens graph are known. The value of $\gamma(Q_n)$ is either known, or known to be one of two consecutive values, for all $n \leq 120$ (see [626]). The known values of $\gamma(Q_n)$ and $i(Q_n)$, for $4 \leq n \leq 20$, are summarized in Table 1.1; the values $\gamma(Q_{20}) = 11$ and $i(Q_{19}) = i(Q_{20}) = 11$ were discovered in the 2017 PhD thesis [77] of Bird at the University of Victoria; Bird [77] also found five other new values: $\gamma(Q_{22}) = 12$, $\gamma(Q_{24}) = 13$, $i(Q_{22}) = 12$, $i(Q_{23}) = 13$, and $i(Q_{24}) = 13$. An independent covering of five queens for an 11×11 chessboard is illustrated in Figure 1.5.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\gamma(Q_n)$	2	3	3	4	5	5	5	5	6	7	8	9	9	9	9	10	11
$i(Q_n)$	3	3	4	4	5	5	5	5	7	7	8	9	9	9	10	11	11

Table 1.1 First 20 values of $\gamma(Q_n)$ and $i(Q_n)$

1.3.6 Queens Total Domination Problem

Still another interesting variant of the above three types of problems was formally introduced in 1892 by W.W. Rouse Ball [51].

> **Queens Total Domination Problem.** What is the minimum number of queens which can be placed on an $n \times n$ chessboard so that every square is attacked by a queen, including the squares occupied by a queen?

This is, of course, the total domination number $\gamma_t(Q_n)$. Notice for example that Figure 1.3 shows five queens dominating the standard 8×8 chessboard, all of which lie on a common diagonal, and thus this set of five queens induces a connected subgraph, and thus this set is both a total dominating set and a connected dominating set. Hence, for $n = 8$, $\gamma(Q_8) = \gamma_t(Q_8) = 5$.

At this point we have seen examples of dominating sets of queens of several different types, for example, dominating sets, maximum and minimum independent dominating sets, and total dominating sets. We next discuss these types of domination for different chess pieces.

1.3.7 Generalizations to Other Chess Pieces

Figure 1.6 W.W. Rouse Ball

In 1939 W.W. Rouse Ball [52] listed these three basic types of problems that were being studied on chessboards at the time. A photograph of Rouse Ball is given in Figure 1.6.

- *Covering*: Determine the minimum number of chess pieces of a given type that are required to cover every square of an $n \times n$ chessboard (domination number).
- *Independent Covering*: Determine the minimum number of mutually non-attacking chess pieces of a given type that are required to cover every square of an $n \times n$ chessboard (independent domination number).