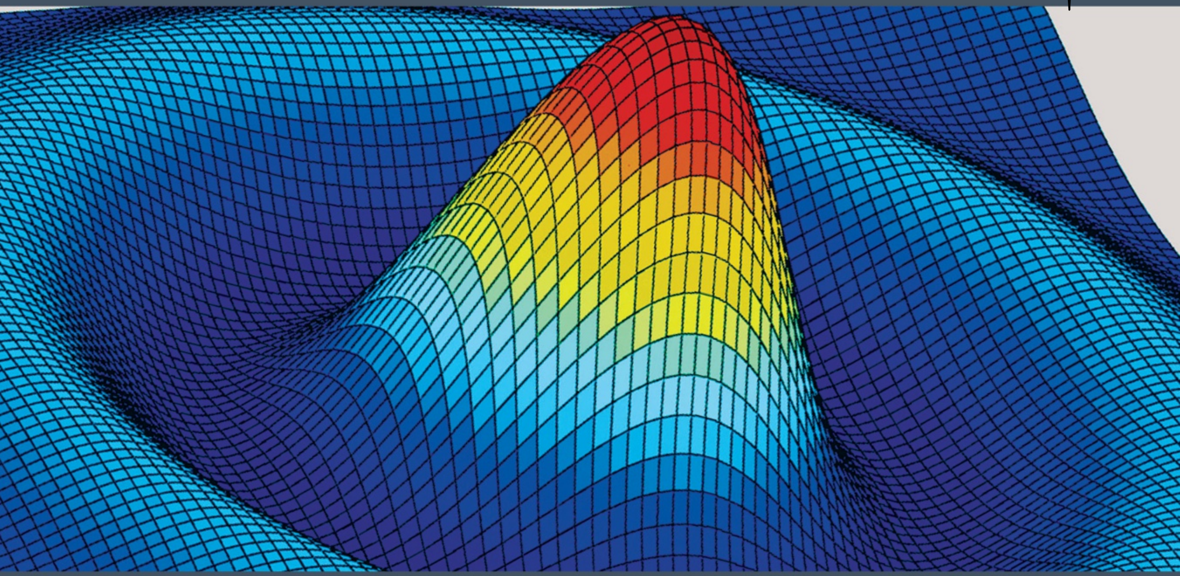
MATHEMATICS AND STATISTICS SERIES

ANALYSIS FOR PDEs SET



Volume 3

Distributions

Jacques Simon

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Distributions

To Laurent Schwartz,

For his Theory of Distributions, *obviously,*
without which this book could not have existed,
but also, and above all, for his kindness and courage.

The clarity of Schwartz's analysis classes at the École
Polytechnique in 1968 made the dunce that I was there happy.
Even if I arrived late, even if I had skipped a few sessions,
everything was clear, lively and easy to understand.
His soft voice, benevolent smile, mischievous eye —
especially when, with an air of nothing, he was
watching for reactions to one of his veiled jokes,
"a tore, from the Greek toro, the tyre"—
he made people love analysis.

When master's students at the university demanded
"a grade average for all" in 1969, most professors
either complied or slunk away. Not Schwartz.
When the results were posted—
I was there, to make up easily for the
calamitous grades I had earned at Polytechnique—
he came alone, frail, in front of a fairly excited horde.
He explained, in substance:

"An examination given to all, without any value, would
no longer allow one to rise in society through knowledge.
Removing selection on the basis of merit would leave
the field open to selection by money or social origin".

Premonitory, alas.

Analysis for PDEs Set

coordinated by
Jacques Blum

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First published 2022 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

Library of Congress Control Number: 2022936452

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-78630-525-1

Contents

Introduction

Objective. This book is the third of seven volumes dedicated to solving partial differential equations in physics:

Volume 1: *Banach, Frechet, Hilbert and Neumann Spaces*
Volume 2: *Continuous Functions*
Volume 3: *Distributions*
Volume 4: *Integration*
Volume 5: *Sobolev Spaces*
Volume 6: *Traces*
Volume 7: *Partial Differential equations*

This third volume aims to construct the space of distributions with real or vectorial values and to provide the main properties that are useful in studying partial differential equations.

Intended audience. We [1] have looked for simple methods that require a minimal level of knowledge to make this tool accessible to as wide an audience as possible — doctoral students, university students, engineers — without loosing generality and even generalizing certain results, which may be of interest to some researchers.

This has led us to choose an unconventional approach that prioritizes semi-norms and sequential properties, whether related to completeness, compactness or continuity.

1. **We?** *We*, it's just "me"! There's no intention of using the Royal We, dear reader, but this *modest (?) "we"* is commonly used in scientific texts when an author wishes to speak of themselves. *It is out of modesty that the writers of Port-Royal made this the trend so that they could avoid, they say, the vanity of "me"* [Louis-Nicolas BESCHERELLE, *Dictionnaire universel de la langue française*, 1845].

Utility of distributions. The main advantage of distributions is that they provide derivatives of all continuous or integrable functions, even those which are not differentiable, and thus broaden the scope of application of differential calculus. This is especially useful for solving partial differential equations.

To this end, a family of objects, the distributions, is defined, with the following properties.
— Any continuous function is a distribution.
— Any distribution has partial derivatives, which are distributions.
— For a differentiable function, we find the conventional derivatives.
— Any limit of distributions is a distribution.
— Any Cauchy sequence of distributions has a limit.

These properties may be roughly summarized by saying that the space $\mathcal{D}'$ of distributions is the *completion with respect to derivation* of the space $\mathcal{C}$ of continuous functions. This construction, due to Laurent SCHWARTZ, [69] and [72], is completed here for distributions on an open subset Ω of $\mathbb{R}^d$ with values in a Neumann space E, i.e. a sequentially complete separable semi-normed space. This includes values in a Banach or Fréchet space.

Originality. The quest for simple methods [2] giving general properties led us to proceed as follows.
— Directly consider *vectorial values*, i.e. constructing $\mathcal{D}'(\Omega; E)$ without any prior study of real distributions.
— Assume that E *is sequentially complete*, i.e. a Neumann space.
— Use *semi-norms* to construct the topologies of E, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega; E)$, *etc.*
— Equip $\mathcal{D}'(\Omega; E)$ with the *simple topology*.
— Introduce *weighting* to generalize the convolution to open domains.
— Explicitly construct the *primitives*.
— *Separate the variables* using a "basic" method.
— Only use *integration* for *continuous functions*.

Let us take a closer look at these points that lie off the beaten track.

Vector values. We consider distributions with values in a general Neumann space E even though the partial differential equations in physics generally have real values. This is useful in evolution equations to separate the time t from the variable of space x. A distribution over t, x with real values is then identified with a distribution over t with values in a space E of distributions on x, for example, with an element of

2. **Focus and simplicity.** This was one of Steve JOBS' favourite mantras: "Simple can be harder than complex: You have to work hard to get your thinking clean to make it simple. But it's worth it in the end because once you get there, you can move mountains." [*BusinessWeek*, 1998].

$\mathcal{D}'((0,T);E)$ where $E = \mathcal{D}'(\Omega)$, which is itself a Neumann space. This identification is made possible by the fundamental *kernel theorem*, p. 312.

A list of the most useful Neumann spaces is given on page 43.

For stationary equations, the real distributions (that is, the case where $E = \mathbb{R}$) are sufficient. We will directly work on the case where E is a Neumann space in order to avoid repetitions, the generalization often consisting of replacing $\mathbb{R}$ with E and the absolute value $|\ |$ with a semi-norm of E in the statements and proofs, when using appropriate methods.

Particular features in the case of vector values. The main differences as compared to distributions with real values are as follows, for a general space E.
— The space $\mathcal{D}'(\Omega;E)$ is not reflexive and its topology of pointwise convergence on $\mathcal{D}(\Omega)$ does not coincide with its weak topology.
— The bounded subsets of $\mathcal{D}'(\Omega;E)$ are not relatively compact.
— The distributions over Ω are not of a finite order over its compact parts: they cannot always be expressed as finite order derivatives of continuous functions.
— Variables may be separated by constructing a bijection from $\mathcal{D}'(\Omega_1 \times \Omega_2;E)$ onto $\mathcal{D}'(\Omega_1;\mathcal{D}'(\Omega_2;E))$ (even for a real distribution, i.e. for $E = \mathbb{R}$, this brings in vector values, in this case in $\mathcal{D}'(\Omega_2)$).

Sequential completeness. We assume that E is a Neumann space, i.e. that all its Cauchy series converge, since this is an essential condition for continuous functions to be distributions. That is, for $\mathcal{C}(\Omega;E) \subset \mathcal{D}'(\Omega;E)$, *see* section 3.4, *The case where E is not a Neumann space*, p. 53.

This property is simpler than the completeness, i.e. the convergence of all the Cauchy filters, and is especially more general: for example, if H is a Hilbert space with infinite dimensions, H-weak is sequentially complete but is not complete [Vol. 1, Property (4.11), p. 63].

It is also simpler and more general than quasi-completeness, i.e. the completeness of bounded subsets, used by Laurent SCHWARTZ [72, p. 2, 50 and 52].

Semi-norms. We use families of semi-norms rather than locally convex topologies, which are equivalent, in order to be able to define $L^p(\Omega;E)$ in Volume 4. Indeed, it is possible to raise a semi-norm to a power p, but not a convex neighborhood!

The handling of semi-normed spaces is simple, although it is less familiar than that of topological spaces: it follows the handling of normed spaces, the main difference being that there are several semi-norms or norms instead of a single norm. For example, we bring in the topology of $\mathcal{D}(\Omega)$ through the family of semi-norms

$\|\varphi\|_{\mathcal{D}(\Omega);p} = \sup_{x\in\Omega,\, |\beta|\le p(x)} p(x)|\partial^\beta\varphi(x)|$ indexed by $p \in \mathcal{C}^+(\Omega)$, which is much simpler than its (equivalent) construction as the inductive limit of the $\mathcal{D}_K(\Omega)$.

Simply topology. We equip the space $\mathcal{D}'(\Omega; E)$ with the family of semi-norms $\|f\|_{\mathcal{D}'(\Omega;E);\varphi,\nu} = \|\langle f, \varphi\rangle\|_{E;\nu}$ indexed by $\varphi \in \mathcal{D}(\Omega)$ and $\nu \in \mathcal{N}_E$ (set indexing the semi-norms of E), i.e. with the topology of simple convergence on $\mathcal{D}(\Omega)$, as it is well-suited to our study ... and is simple. This simplicity is achieved without restricting ourselves to a *pseudo-topology* as is done in several texts.

In addition, this topology has the same convergent sequences and the same bounded sets as the topology of uniform convergence on the bounded subsets of $\mathcal{D}(\Omega)$ used by Laurent SCHWARTZ. The reasons for our choices are detailed on p. 45.

Open domain and weighting. We consider distributions defined on an open subset Ω of $\mathbb{R}^d$. As these do not necessarily have an extension to all of $\mathbb{R}^d$, we introduce an operation, we call it weighting, which plays a role for Ω that is similar to the role played by convolution for $\mathbb{R}^d$ and which we constantly use.

The weighted distribution $f \diamond \mu$ of a distribution f, defined on an open set Ω, by a weight μ, which is a real distribution on $\mathbb{R}^d$ with a compact support D, is a distribution defined on the open set $\Omega_D = \{x \in \mathbb{R}^d : x + D \subset \Omega\}$. When f and μ are functions, it is given by $(f \diamond \mu)(x) = \int_{\mathring{D}} f(x+y)\mu(y)\, \mathrm{d}y$. When $\Omega = \mathbb{R}^d$, the convolution is recovered up to a symmetry on μ, and all its properties are recovered up to a possible sign.

Primitives. We show that a field of distributions $q = (q_1, \ldots, q_d)$ has a primitive f, that is $\nabla f = q$, if and only if it satisfies $\langle q, \psi\rangle = 0_E$ for all the test fields $\psi = (\psi_1, \ldots, \psi_d)$ such that $\nabla \cdot \psi = 0$. It is the *orthogonality theorem*. We explicitly determine all the primitives and among these determine one which depends continuously on q.

We also demonstrate that when Ω is simply connected it is necessary and sufficient that $\partial_i q_j = \partial_j q_i$ for all i and j. It is the *Poincaré's generalized theorem*.

Separation of variables. We show that the separation of variables is bijective from $\mathcal{D}'(\Omega_1 \times \Omega_2; E)$ onto $\mathcal{D}'(\Omega_1; \mathcal{D}'(\Omega_2; E))$ by means of inequalities. These are certainly laborious to establish, but they avoid the difficult topological properties used by Laurent SCHWARTZ in his diabolical proof of this *kernel theorem*.

The advantage of this method is laid out in the commentary *Originality...*, p. 317.

Integration. The integration of continuous functions is essential to identify them with distributions through the equality $\langle f, \varphi\rangle = \int f\varphi$, for all test functions φ. Since the theory of integration was not developed for values in a Neumann space, in Volume 2 we established results relative to uniformly continuous functions that meet our requirements. We remind them before using them in this volume.

The general theory of integration with values in a Neumann space will be done in a later volume in the context of *integrable distributions*, which play the role of the usual *classes of almost equal integrable functions*. Indeed, it has seemed simpler to thus construct general integration.

Prerequisite. The proofs in the main body of the text only use the definitions and results already established in Volumes 1 and 2, recalled either in the Appendix or in the text, with references to their proofs.

This book has been written such that it can be read in an out-of-order fashion by a non-specialist: the proofs are detailed and include arguments that may be trivial for an expert and the numbers of the theorems being used are systematically recalled. These details are even more necessary [3] since the majority of the results are generalizations, that are new, to functions and distributions with values in a Neumann space of properties that are classic for values in a Banach space.

I request the reader to be lenient with how heavy this may make the text.

Comments. Unlike the main body of the text, the comments, appearing in smaller font, may refer to external results or those not yet established. The appendix 'Reminders' is also written in smaller font as it is assumed that the content is familiar.

Historical overview. Wherever possible, the origin of the concepts and results is specified in footnotes [4].

---

3. **Necessary details.** As Laurent SCHWARTZ explained in the preamble to one of his articles [71, p. 88]: "*Although many proofs are relatively easy, we find it useful to write them in extenso, because whenever topological vector spaces come into play there are so many 'traps' that great rigor is needed*".
Given that the great and rigorous Augustin CAUCHY has himself arrived at an erroneous result, we have not treated any detail too lightly. Let us recall that, in 1821, in his remarkable *Cours d'Analyse de l'École Royale Polytechnique*, he declared that he had 'easily' [18, p. 46] proven that if a real function with two real variables is continuous with respect to both the variables, it is continuous with respect to their couple. It was not till 1870 that Carl Johannes THOMAE [89, p. 15] demonstrated that this was inexact.

4. **Historical overview. Objective.** This is first of all to honour the mathematicians whose work has made this book possible and inspire it. Although some may be missing, either due to limited space or knowledge. The other objective is to show that the world of mathematics is an ancient human construction, not a "revealed truth", and that behind each theorem there are one or more humans, our contemporaries or distant ancestors who — including the Greeks — reasoned just as well as us, without internet, computers or even printing and paper.

The forgotten. The French are probably over-represented here, as they are in all french libraries and teaching and, often, in french hearts. Among the French, I am over-represented, because this book is the result of thirty years of work I have carried out to simplify and generalize distributions with vectorial values.

Navigating this book.
— The **table of contents**, at the beginning of this book, lists the topics discussed.
— The **index**, p. 371, provides another thematic access.
— The **table of notations**, p. xv, specifies the meaning of the symbols used.
— The hypotheses are all stated within the theorems themselves.
— The numbering is common to all the statements, so that they can be easily found in numerical order (for instance, Theorem 2.2 is found between statements 2.1 and 2.3, which are definitions).

Acknowledgements. I am particularly grateful to Enrique FERNÁNDEZ-CARA who has proofread countless versions of this text, indefatigable, and who has given me various friendly suggestions for equally countless improvements in each version.

Fulbert MIGNOT suggested (among other things) that each chapter should be preceded by a brief introduction. This was very helpful: in order to reveal the guiding principle, I had to highlight it and re-write several sections.

The meticulous and knowledgeable readings carried out by Olivier BESSON, Didier BRESCH and Pierre DREYFUSS contributed substantial improvements to the text.

Jérôme LEMOINE, my disciple — a stigmata and a cross he'll bear for life! —, had the task of proofreading the demonstrations: he is, thus, entirely responsible for any errors there may be ... except, perhaps, those I may have added since.

Jacques BLUM was able to convince me that it was time to publish this. Indeed.

Thank you, my friends, for all your help and warm support.

Jacques SIMON
Chapdes-Beaufort, March 2021

Laurent SCHWARTZ, my primary source of inspiration and admiration, is also perhaps over-represented, since, his treatises having no historical notes due to his great modesty, I have attributed to him the totality of their contents. On the other hand, the Russians and Eastern Europeans are probably particularly under-represented, due to the language barrier, aggravated by the mutual ignorance of the West and East during the Cold War period.

Novelties. At the risk of sounding immodest (well, *nobody is perfect*) I have marked out a large number of results that I believe are new, both to arouse the reader's vigilance — it's not impossible that this book may contain some careless mistake — and to draw their attention to the new tools available to them.

Appeal to the reader. A number of important results lack historical notes because I am not familiar with their origin. I beg the reader's indulgence for these lacunae and, above all, the injustices that may result. And I call upon the erudite among you to flag any improvements to me for future editions.

Notations

SPACES OF DISTRIBUTION

$\mathcal{D}(\Omega)$	space of test functions (infinitely differentiable with compact support)	21
$\mathcal{D}_K(\Omega)$	*id.* with support in the compact set $K \subset \Omega$ (another notation for $\mathcal{C}_K^\infty(\Omega)$)	24
$\mathcal{D}(\Omega; \mathbb{R}^d)$	space of test fields	271
$\mathcal{D}'(\Omega)$	space of real distributions	42
$\mathcal{D}'(\Omega; E)$	space of distributions with values in E	42
$\mathcal{D}'_K(\Omega; E)$	*id.* with support in the closed set $K \subset \Omega$	137
$\mathcal{D}'(\Omega; E^d)$	space of distribution fields	82
$\mathcal{D}'_\nabla(\Omega; E^d)$	space of gradients	253
$\mathcal{D}'(\Omega)$-weak	$\mathcal{D}'(\Omega)$ equipped with its weak topology	80
$\mathcal{D}'(\Omega; E\text{-weak})$	space of distributions with values in E-weak	78
$\mathcal{D}'(\Omega; E)$-unif	$\mathcal{D}'(\Omega; E)$ with topology of uniform convergence on bounded subsets of $\mathcal{D}(\Omega)$	46
$\mathcal{D}'(\Omega_1; \mathcal{D}'(\Omega_2; E))$	space of distributions of distributions	285

SET OF DISTRIBUTIONS

$\mathcal{D}'^+(\Omega)$	set of real positive distributions	111

SPACE OF FUNCTIONS

$\mathcal{B}(\Omega; E)$	space of uniformly continuous functions with bounded support	14
$\mathcal{C}(\Omega; E)$	space of continuous functions	12
$\mathcal{C}_b(\Omega; E)$	*id.* bounded	12
$\mathcal{C}_K(\Omega; E)$	*id.* with support in the compact set $K \subset \Omega$	18
$\mathcal{C}^+(\Omega)$	set of real positive continuous functions	19
$\mathcal{C}^m(\Omega; E)$	space of m times continuously differentiable functions; case of $m = \infty$	12, 13
$\mathcal{C}_b^m(\Omega; E)$	*id.* with bounded derivatives, case of $m = \infty$	12, 13
$\mathcal{C}_K^m(\Omega; E)$	*id.* with support in the compact set $K \subset \Omega$	18
$\mathbf{C}(\Omega; E)$	space of uniformly continuous functions	342
$\mathbf{C}_b(\Omega; E)$	*id.* bounded	13
$\mathbf{C}_D(\Omega; E)$	*id.* with support in the compact set $D \subset \mathbb{R}^d$	18
$\mathbf{C}_b^m(\Omega; E)$	space $\mathcal{C}^m$ with uniformly continuous and bounded derivatives	13
$\mathcal{K}(\Omega)$	space of real continuous functions with compact support	57
$\mathcal{K}^\infty(\Omega)$	*id.* infinitely differentiable	24
$\mathcal{K}(\Omega; E)$	space of continuous functions with compact support	186
$\mathcal{K}^\infty(\Omega; E)$	space of infinitely differentiable functions with compact support	178

SET OF FUNCTIONS

$\mathcal{C}^m(\Omega;\Lambda)$	set of functions of $\mathcal{C}^m(\Omega;\mathbb{R}^d)$ with values in the set Λ	100
$\mathcal{C}^1([a,b];\Omega)$	set of "differentiable" functions on $[a,b]$, with values in Ω	222

SPACE OF MEASURES

$\mathcal{M}(\Omega)$	space of real measures	58
$\mathcal{M}(\Omega;E)$	space of measures with values in E	58

OPERATIONS ON A DISTRIBUTION (OR A FUNCTION) f

$\langle f,\varphi\rangle$	value of a distribution on a test function φ	42
$\widetilde{f}$	extension by 0_E	139
$\check{f}$	image under the symmetry $x\mapsto -x$ of the variable	184
$\breve{f}$	image under permutation of variables; case of a distribution of distributions	109, 318
$\overline{f}$	image under grouping of variables	317
$\underline{f}$	image under separation of variables	309
$f\vert_\omega$	restriction	117
$\tau_x f$	translation by $x\in\mathbb{R}^d$	107
$R_n f$	global regularization	176
Lf	composition with a linear mapping L over E	91
$f\circ T$	composition with a regular change of variable T	101
$f\diamond\mu$	weighting by a weight μ; case of a regular weight; case of functions	153, 142, 144
$f\diamond\rho_n$	local regularization	170
$f\star\mu$	convolution with μ	155
$f\otimes g$	tensor product (of functions)	297
$\operatorname{nihil} f$	annihilation domain	130
$\operatorname{supp} f$	support; case of a function	131, 17

DERIVATIVES OF A DISTRIBUTION (OR OF A FUNCTION) f

f' or $\mathrm{d}f/\mathrm{d}x$	derivative of a function of a real variable	9
$\partial_i f$	partial derivative: $\partial_i f=\partial f/\partial x_i$; case of a function	85, 10
$\partial^\beta f$	derivative of order β: $\partial^\beta f=\partial_1^{\beta_1}\ldots\partial_d^{\beta_d} f$; case of a function	85, 11
β	positive multi-integer: $\beta=(\beta_1,\ldots,\beta_d)$, $\beta_i\geq 0$	10
$\vert\beta\vert$	derivation order: $\vert\beta\vert=\vert\beta_1\vert+\ldots\vert\beta_d\vert$	10
$\partial^0 f$	derivative of order 0: $\partial^0 f=f$	10
∇f	gradient: $\nabla f=(\partial_1 f,\ldots,\partial_d f)$; case of a function	85, 9
Δf	Laplacian: $\Delta f=\partial_1^2 f+\ldots+\partial_d^2 f$	99
q	field: $q=(q_1,\ldots,q_d)$	81
$\nabla\cdot q$	divergence: $\nabla\cdot q=\partial_1 q_1+\ldots\partial_d q_d$	239
$\nabla^{-1}q$	primitive depending continuously on q	257

INTEGRALS OF FUNCTIONS AND PATHS

$\int_\omega f$	Cauchy integral	15
$\mathbb{S}_\omega^n f$	approximate integral	15
$\widehat{\int}_\omega f$	completed integral	54
$\int_{S_r} f\,\mathrm{d}s$	surface integral over a sphere	191
$\fint_\Gamma q\cdot\mathrm{d}\ell$	line integral of a vector field along a path	222
Γ	path	221
$[\Gamma]$	image of a path: $[\Gamma]=\{\Gamma(t):t_\mathrm{i}\leq t\leq t_\mathrm{e}\}$	221

$\overleftarrow{\Gamma}$	reverse path	226
$\overrightarrow{\cup}$	concatenation of paths	228
H	homotopy	231
$[H]$	image of a homotopy	231

SEPARATED SEMI-NORMED SPACES

E	separated semi-normed space	1
$\\| \\|_{E;\nu}$	semi-norm of E of index ν	1
$\mathcal{N}_E$	set indexing the semi-norms of E	1
$\underset{\leftrightarrows}{\equiv}$	equality of families of semi-norms	4
$\underset{\leftrightarrows}{=}$	topological equality	5
$\underset{\to}{\subset}$	topological inclusion	5
E-weak	space E equipped with pointwise convergence on E'	77
E'	dual of E	76
$\widehat{E}$	sequential completion of E	54
E^d	Euclidean product $E \times \ldots \times E$	8
$E_1 \times \cdots \times E_d$	product of spaces	114

SUBSETS AND MAPPINGS OF SEMI-NORMED SPACES

$\mathring{U}$	interior of a subset U of a semi-normed space	351
$\overline{U}$	closure of U	351
∂U	boundary of U	351
$\mathrm{Lin}(E; F)$	set of linear mappings	285
$\mathcal{L}(E; F)$	space of continuous linear mappings	356
$\mathcal{L}^d(E_1 \times \ldots \times E_d; F)$	space of continuous multilinear mappings	360

POINTS AND SUBSETS OF $\mathbb{R}^d$

$\mathbb{R}^d$	Euclidean space: $\mathbb{R}^d = \{x = (x_1, \ldots, x_d) : \forall i,\ x_i \in \mathbb{R}\}$	354
$\|x\|$	Euclidean norm: $\|x\| = (x_1^2 + \cdots + x_d^2)^{1/2}$	354
$x \cdot y$	Euclidean scalar product: $x \cdot y = x_1 y_1 + \ldots + x_d y_d$	354
$\mathbf{e}_i$	i-th basis vector in $\mathbb{R}^d$	10
Ω	domain of definition of functions or of distributions	8
Ω_D	domain of the weighted distribution: $\Omega_D = \{x : x + D \subset \Omega\}$; figure	142, 143
$\Omega_{1/n}$	Ω minus a neighborhood of its boundary: $\Omega_{1/n} = \{x : B(x, 1/n) \subset \Omega\}$	171
$\Omega_{1/n}^{(a)}$	connected component of $\Omega_{1/n}$ containing a; figures	235, 266, 268
Ω_r^n	potato-shaped set: $\Omega_r^n = \{x : \|x\| < n,\ B(x, r) \subset \Omega\}$	33
κ_n	crown-shaped set: $\kappa_n = \Omega_{1/(n+2)}^{n+2} \setminus \overline{\Omega_{1/n}^n}$	33
ω	subset of $\mathbb{R}^d$	8
$\|\omega\|$	measure of the open set ω	364
$B(x, r)$	closed ball $B(x, r) = \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$	353
$\mathring{B}(x, r)$	open ball $\mathring{B}(x, r) = \{y \in \mathbb{R}^d : \|y - x\| < r\}$	353
$C_{a,b}$	open crown $C_{a,b} = \{x \in \mathbb{R}^d : a < \|x\| < b\}$	364
υ_d	measure of the unit ball: $\upsilon_d = \|\mathring{B}(0, 1)\|$	365
$\Delta_{s,n}$	closed cube of edge length 2^{-n} centered on $2^{-n}s$	15
$\Subset$	compact inclusion in $\mathbb{R}^d$	176

OTHER SETS [1]

$\mathbb{N}$	set of natural numbers: $\mathbb{N} = \{0, 1, 2, \ldots\}$	349
$\mathbb{N}^*$	set of positive natural numbers: $\mathbb{N}^* = \{1, 2, \ldots\}$	349
$\mathbb{Z}$	set of integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	349
$\mathbb{Q}$	set of rational numbers	349
$\mathbb{R}$	space of real numbers	350
$[\![m, n]\!]$	integer interval: $[\![m, n]\!] = \{i \in \mathbb{N} : m \leq i \leq n\}$	10
$[\![m, \infty]\!]$	extended integer interval: $[\![m, \infty]\!] = \{i \in \mathbb{N} : i \geq m\} \cup \{\infty\}$	349
(a, b)	open interval: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$	350
$[a, b]$	closed interval: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$	350

AND ALSO

$\emptyset$	empty set
$\subset$	algebraic inclusion
$\setminus$	set difference: $U \setminus V = \{u \in U : u \notin V\}$
$\times$	product: $U \times V = \{(u, v) : u \in U,\ v \in V\}$

SPECIFIC FUNCTIONS AND DISTRIBUTIONS

det	determinant	362
δ_x	Dirac mass	59
$\alpha,\ \alpha_n$	partition of unity; localizing function	34, 175
ρ_n	regularizing function	169
φ or ϕ	test function	21
ψ	divergence-free test field	271
ξ	elementary potential: $\Delta\xi = -\delta_0$	197
γ	local elementary potential: $\gamma = \theta\xi$	208
η	corrector term: $\eta = 2\nabla\xi \cdot \nabla\theta + \xi\Delta\theta$	208
e	exponential number	[Vol. 1, p. 323]
log	logarithm	[Vol. 1, p. 321]

TYPOGRAPHY

▮	end of statement
□	end of proof or comment

FIGURES

1. **Notation of natural numbers.** The notation of $\mathbb{N}$ and $\mathbb{N}^*$ follows the ISO 80000-2 standard, see p. 349.

Chapter 1

Semi-Normed Spaces and Function Spaces

In this chapter, we provide definitions for the following essential notions.
— Semi-normed spaces and, in particular, Neumann, Fréchet and Banach spaces (§ 1.1).
— Topological equality and inclusion of semi-normed spaces (§ 1.2).
— Continuous mappings (§ 1.3) and differentiable functions (§ 1.4).
— Spaces of continuously differentiable functions and their semi-norms (§ 1.5).
— The integral of a uniformly continuous function with values in a Neumann space (§ 1.6).

We will make extensive use of their properties established in Volumes 1 and 2, and refer, as to make this book self-contained, to their precise statements in the course of the text or in the Appendix with references to their proofs.

1.1. Semi-normed spaces

Let us define separated semi-normed spaces [1] (the definitions of vector spaces and of semi-norms are recalled in the Appendix, § A.2).

Definition 1.1.– *A **semi-normed space** is a vector space E endowed with a family of semi-norms $\{\| \ \|_{E;\nu} : \nu \in \mathcal{N}_E\}$.*

*Any such space is said to be **separated** (or **Hausdorff**) if $u = 0_E$ is the only element such that $\|u\|_{E;\nu} = 0$ for all $\nu \in \mathcal{N}_E$.*

1. **History of the notion of semi-normed space.** John von NEUMANN introduced semi-normed spaces in 1935 [56] (with an unnecessary countability condition). He also showed [56, Theorem 26, p. 19] that they coincide with the locally convex topological vector spaces that Andrey KOLMOGOROV previously introduced in 1934 [46, p. 29].

History of the notion of Hausdorff space. Felix HAUSDORFF had included the separation condition in its original definition of a topological space in 1914 [40].

Distributions, First Edition. Jacques Simon.

*A **normed space** is a vector space E endowed with a norm $\| \ \|_E$.* ▮

Caution. Definition 1.1 of a separated semi-normed space is general but not universal. For Laurent SCHWARTZ [73, p. 240], a semi-normed space is a space endowed with a *filtering* family of semi-norms (Definition 1.8). This definition is equivalent, since every family is equivalent to a filtering family [Vol. 1, Theorem 3.15].

For Nicolas BOURBAKI [10, editions published after 1981, Chap. III, p. III.1] and Robert EDWARDS [30, p. 80], a semi-normed space is a space endowed with a *single* semi-norm, which drastically changes its meaning. □

Let us define bounded subsets [2] of a separated semi-normed space.

Definition 1.2.– *Let U be a subset of a separated semi-normed space E, whose family of semi-norms is denoted by $\{\| \ \|_{E;\nu} : \nu \in \mathcal{N}_E\}$. We say that U is **bounded** if, for every $\nu \in \mathcal{N}_E$,*

$$\sup_{u\in U} \|u\|_{E;\nu} < \infty. \quad ▮$$

Let us define convergent and Cauchy sequences [3] in a semi-normed space.

Definition 1.3.– *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in a separated semi-normed space E, whose family of semi-norms is denoted by $\{\| \ \|_{E;\nu} : \nu \in \mathcal{N}_E\}$.*

(a) *We say that $(u_n)_{n\in\mathbb{N}}$ **converges to a limit** $u \in E$, and we denote $u_n \to u$, if, for every $\nu \in \mathcal{N}_E$,*

$$\|u_n - u\|_{E;\nu} \to 0 \ \text{ when } n \to \infty.$$

(b) *We say that $(u_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** if, for every $\nu \in \mathcal{N}_E$,*

$$\sup_{m\geq n} \|u_m - u_n\|_{E;\nu} \to 0 \ \text{ when } n \to \infty. \quad ▮$$

CAUTION. We denote $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* \stackrel{\text{def}}{=} \{1, 2, \ldots\}$, conforming to the ISO 80000-2 standard for mathematical and physics notation (edited in 2009).

Any possible confusion will be of no consequence, apart from surprising the reader used to the opposite notation when seeing a term u_0 of a series indexed by $\mathbb{N}$, or the inverse $1/n$ of a number n in $\mathbb{N}^*$. □

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2. **History of the notion of bounded set.** Bounded sets in a semi-normed space were introduced in 1935 by John von NEUMANN [56]. Andrey KOLMOGOROV had introduced them in 1934 [46] for topological vector spaces.

3. **History of the notion of convergent sequence.** Baron Augustin CAUCHY gave Definition 1.3 for convergence in $\mathbb{R}$, in 1821 [18, p. 19]. Niels ABEL contributed to the emergence of this notion.

History of the notion of Cauchy sequence. Augustin CAUCHY introduced the convergence criterion of Definition 1.3 for real series, in 1821 [18, p. 115-116], admitting it (i.e. by implicitly considering $\mathbb{R}$ as the completion of $\mathbb{Q}$). Bernard BOLZANO previously stated this criterion in 1817 in [6], trying unsuccessfully to justify it due to the lack of a coherent definition for $\mathbb{R}$.

Let us define several types of sequentially complete spaces.

Definition 1.4.– *A separated semi-normed space is* ***sequentially complete*** *if all its Cauchy sequences converge.*

A ***Neumann space*** *is a sequentially complete separated semi-normed space.*

A ***Fréchet space*** *is a sequentially complete metrizable semi-normed space.*

A ***Banach space*** *is a sequentially complete normed space.* ▌

Neumann spaces. We named these spaces in Volume 1 in homage to John VON NEUMANN, who introduced sequentially complete separated semi-normed spaces in 1935 [56]. Thus, readers should recall the definition before using it elsewhere. Examples of such spaces are given in the commentary *Examples of Neumann spaces*, p. 43. □

Completeness and sequential completeness. A semi-normed space is **complete** if every Cauchy filter converges [SCHWARTZ, 73, Chap. XVIII, § 8, Definition 1, p. 251]. We shall not use this notion since sequential completeness is much simpler and more general (complete implies sequentially complete [SCHWARTZ, 73, p. 251]) and especially since certain useful spaces are sequentially complete but not complete. For example, it is the case of any reflexive Hilbert or Banach space of infinite dimension endowed with its weak topology [Vol. 1, Property (4.11), p. 63]. □

Completeness and Metrizability. Recall that, for a metrizable space, *completeness* is equivalent to *sequential completeness* [SCHWARTZ, 73, Theorem XVIII, 8; 1, p. 251]. This is why some authors speak of completeness for Banach or Fréchet spaces, while in reality they only use sequential completeness. □

Let us give a definition of metrizability.

Definition 1.5.– *A semi-normed space is* ***metrizable*** *if it is separated and if its family of semi-norms is countable or is equivalent to a countable family of semi-norms.* ▌

Definitions of the equivalence of families of semi-norms and of their countability are recalled on pages 4 and 350 (Definitions 1.6 and A.1).

Justification for the name "metrizable". We refer here to a *metrizable* space since every countable family or, what leads to the same thing, any sequence $(\| \, \|_k)_{k \in \mathbb{N}}$ of semi-norms can be associated with a **distance**, or *metric*, d which generates the same topology, for instance

$$d(u, v) = \sum_{k \in \mathbb{N}} 2^{-k} \frac{\|u - v\|_k}{1 + \|u - v\|_k}.$$

Definition 1.5 is, in fact, that of a **separated countably semi-normable space**. We shall abuse terminology and speak of *metrizable* spaces since this equivalent notion is more familiar. To be precise, a **metrizable** space is a space that is "*topologically equal to a metric space*". □

Superiority of a sequence of semi-norms over a distance. The semi-norms of a metrizable space E allow us to characterize its bounded subsets U by (Definition 1.2)

$$\sup_{u\in U} \|u\|_{E;k} < \infty, \text{ for all } k \in \mathbb{N}.$$

On the contrary, *if E is not normable* and if d is a distance that generates its topology, its bounded subsets **are not characterized by**

$$\sup_{u\in U} d(u, 0_E) < \infty.$$

What is worse, is that no "ball" $\{u \in E : d(u, z) < r\}$, for $r > 0$, is bounded. Indeed, the existence of a non-empty bounded open subset is equivalent to normability, due to **Kolmogorov's Theorem** [4]. □

1.2. Comparison of semi-normed spaces

First, let us compare families of semi-norms on the same vector space.

Definition 1.6.– *Let $\{\|\ \|_{1;\nu} : \nu \in \mathcal{N}_1\}$ and $\{\|\ \|_{2;\mu} : \mu \in \mathcal{N}_2\}$ be two families of semi-norms on the same vector space E.*

*The first family **dominates** the second if, for every $\mu \in \mathcal{N}_2$, there exist a finite subset N_1 of $\mathcal{N}_1$ and $c_1 \in \mathbb{R}$ such that for every $u \in E$,*

$$\|u\|_{2;\mu} \leq c_1 \sup_{\nu\in N_1} \|u\|_{1;\nu}.$$

*Both families are **equivalent** if each one dominates the other. We also say that they **generate the same topology**.* ▌

Terminology. The **topology** of E is the family of its open subsets. We can say that two families of semi-norms *generate the same topology* instead of saying that they *are equivalent*, since the equivalence of the families of semi-norms implies the equality of the families of open subsets [Vol. 1, Theorem 3.4], and reciprocally [Vol. 1, Theorem 7.14 (a) and 8.2 (a), with $L = T =$ Identity]. □

Let us see how we can compare two semi-normed spaces (the definition of a vector subspace is recalled in the Appendix, § A.2).

Definition 1.7.– *Let E and F be two semi-normed spaces, whose families of semi-norms are denoted by $\{\|\ \|_{E;\nu} : \nu \in \mathcal{N}_E\}$ and $\{\|\ \|_{F;\mu} : \mu \in \mathcal{N}_F\}$.*

(a) *We denote $E \underset{\Leftrightarrow}{\equiv} F$ if $E = F$ and if their additions, multiplications and families of semi-norms coincide. That is to say, if they have the same vector space structure and the same semi-norms.*

4. **History of Kolmogorov's Theorem.** Andrey KOLMOGOROV showed in 1934 [46, p. 33] that a topological vector space is normable if and only if there exists a bounded convex neighborhood of the origin, which is equivalent here to the existence of a bounded open set.

(b) *We say that E is **topologically equal** to F and we denote $E \underset{\leftrightarrow}{=} F$ if $E = F$, if their additions and multiplications coincide and if their families of semi-norms are equivalent.*

(c) *We say that E is **topologically included** in F and we denote $E \underset{\rightarrow}{\subseteq} F$ if E is a **vector subspace** of F and if the family of semi-norms of E dominates the family of restrictions to E of the semi-norms of F.*

That is to say if, for every semi-norm $\mu \in \mathcal{N}_F$, there exist a finite subset N of $\mathcal{N}_E$ and $c \in \mathbb{R}$ such that, for every semi-norm $u \in E$,

$$\|u\|_{F;\mu} \leq c \sup_{\nu \in N} \|u\|_{E;\nu}.$$

(d) *We say that E is a **topological subspace** of F if it is a vector subspace of F and if it is endowed with the restrictions to E of the semi-norms of F or, more generally, with a family equivalent to the family of these restrictions.* ▮

Caution. Suppose that $E = F$ and that, for every $\mu \in \mathcal{N}_F$, there exists $\nu \in \mathcal{N}_E$ such that, for every $u \in E$,

$$\|u\|_{F;\mu} = \|u\|_{E;\nu}.$$

This equality **does not imply topological equality** $E \underset{\leftrightarrow}{=} F$: it only implies $E \underset{\rightarrow}{\subseteq} F$. Indeed, it does not ensure the existence, for every $\nu \in \mathcal{N}_E$, of a μ satisfying this equality or of a finite family of μ such that $\|u\|_{E;\nu} \leq c \sup_{\mu \in M} \|u\|_{F;\mu}$, which is necessary for the converse inclusion $F \underset{\rightarrow}{\subseteq} E$.

Such an equality of semi-norms occurs for example in step 3, p. 28, of the proof of Theorem 2.12, where we thus prove a converse inequality to get topological equality. □

Let us finally define filtering families of semi-norms.

Definition 1.8.– *A family $\{\| \, \|_\nu : \nu \in \mathcal{N}\}$ of semi-norms on a vector space E is **filtering** if, for every finite subset N of $\mathcal{N}$, there exist $\mu \in \mathcal{N}$ such that, for every $u \in E$,*

$$\sup_{\nu \in N} \|u\|_\nu \leq \|u\|_\mu. \ ▮$$

Utility of filtering families. The use of filtering families simplifies some statements, by substituting a single semi-norm to the upper envelope of a finite number of semi-norms. This is for example the case with the characterization of continuous linear mappings from Theorem 1.12 where we consider both cases. Definition 1.9 of continuous mappings could similarly be simplified with a filtering family.

Any family of semi-norms is equivalent to a filtering family [Vol. 2, Theorem 3.15], but this one is not necessarily pleasant to use. □

Spaces endowed with filtering families. The "natural" family of some spaces is filtering. For example, $\mathcal{D}(\Omega)$ is endowed (Definition 2.5) with the family, which is filtering (Theorem 2.7), of the semi-norms, indexed by $p \in \mathcal{C}^+(\Omega)$,

$$\|\varphi\|_{\mathcal{D}(\Omega);p} = \sup_{x \in \Omega} \sup_{|\beta| \leq p(x)} p(x) |\partial^\beta \varphi(x)|.$$

A single norm also constitutes, on its own, a filtering family of semi-norms. □

Utility of non-filtering families. The "natural" family of some spaces is not filtering. For example, $\mathcal{D}'(\Omega)$ is endowed (Definition 3.1) with the non-filtering family of the semi-norms, indexed by $\varphi \in \mathcal{D}(\Omega)$,

$$\|f\|_{\mathcal{D}'(\Omega);\varphi} = |\langle f, \varphi\rangle|.$$

If the aim was to consider filtering families only, then $\mathcal{D}'(\Omega)$ should be endowed with the semi-norms

$$|||f|||_{\mathcal{D}'(\Omega);N} = \sup_{\varphi \in N} |\langle f, \varphi\rangle|$$

indexed by the finite subsets N of $\mathcal{D}(\Omega)$. The maximum over finite subsets would of course disappear from Definition 1.9 of continuous functions but it would reappear here. □

1.3. Continuous mappings

We now define various notions of continuity [5] of a mapping from a semi-normed space into another.

Definition 1.9.– *Let T be a mapping from a subset X of a separated semi-normed space E into another separated semi-normed space F, and let $\{\| \, \|_{E;\nu} : \nu \in \mathcal{N}_E\}$ and $\{\| \, \|_{F;\mu} : \mu \in \mathcal{N}_F\}$ be the families of semi-norms of E and F.*

(a) *We say that T is* ***continuous at the point*** *u of X if, for every $\nu \in \mathcal{N}_E$ and $\epsilon > 0$, there exist a finite subset M of $\mathcal{N}_F$ and $\eta > 0$ such that:*

$$v \in X, \ \sup_{\mu \in M} \|v - u\|_{F;\mu} \le \eta \quad \Rightarrow \quad \|T(v) - T(u)\|_{E;\nu} \le \epsilon.$$

We say that T is ***continuous*** *if it is so at every point of X.*

(b) *We say that T is* ***uniformly continuous*** *if, for every $\nu \in \mathcal{N}_E$ and $\epsilon > 0$, there exist a finite subset M of $\mathcal{N}_F$ and $\eta > 0$ such that:*

$$u \in X, \ v \in X, \ \sup_{\mu \in M} \|v - u\|_{F;\mu} \le \eta \quad \Rightarrow \quad \|T(v) - T(u)\|_{E;\nu} \le \epsilon.$$

(c) *We say that T is* ***sequentially continuous at the point*** *u of X if, for every sequence $(u_n)_{n \in \mathbb{N}}$ of X:*

$$u_n \to u \text{ in } E \quad \Rightarrow \quad T(u_n) \to T(u) \text{ in } F.$$

We say that T is ***sequentially continuous*** *if it is so at every point of X.*

---

5. **History of the notions of continuity.** Augustin CAUCHY defined sequential continuity for a real function on a line segment in 1821, in [18]. Bernard Placidus Johann Nepomuk BOLZANO also contributed to the emergence of this notion.
Eduard HEINE defined the uniform continuity of a function defined on a part of $\mathbb{R}^d$ in 1870, in [42]. It had already been used implicitly by Augustin CAUCHY in 1823 to define the integral of a real function [19, p. 122-126], and then explicitly by Peter DIRICHLET.

(d) *We say that T is* ***bounded*** *if its image $T(X) = \{T(u) : u \in X\}$ is bounded in F. That is to say if, for every $\mu \in \mathcal{N}_F$,*

$$\sup_{u \in X} \|T(u)\|_{F;\mu} < \infty. \blacksquare$$

Recall that continuity always implies sequential continuity [Vol. 1, Theorem 7.2] [6].

Theorem 1.10.– *Any continuous mapping from a subset of a separated semi-normed space into a separated semi-normed space is sequentially continuous.* ▮

The converse is true if the initial space is metrizable [Vol. 1, Theorem 9.1].

Theorem 1.11.– *A mapping from a subset of a metrizable separated semi-normed space into a separated semi-normed space is continuous if and only if it is sequentially continuous.* ▮

For linear mappings, Definition 1.9 gives [Vol. 1, Theorem 7.14]:

Theorem 1.12.– *Let L be a linear mapping from a separated semi-normed space E into a separated semi-normed space F, and let $\{\| \, \|_{E;\nu} : \nu \in \mathcal{N}_E\}$ and $\{\| \, \|_{F;\mu} : \mu \in \mathcal{N}_F\}$ be the families of semi-norms of E and F. Then:*

(a) *L is continuous if and only if, for every $\mu \in \mathcal{N}_F$, there exist a finite subset N of $\mathcal{N}_E$ and $c \geq 0$ such that: for every $u \in E$,*

$$\|Lu\|_{F;\mu} \leq c \sup_{\nu \in N} \|u\|_{E;\nu}.$$

This is also equivalent to the statement: L is uniformly continuous.

(b) *If the family of semi-norms of E is filtering, then L is continuous if and only if, for every $\mu \in \mathcal{N}_F$, there exist $\nu \in \mathcal{N}_E$ and $c \geq 0$ such that: for every $u \in E$,*

$$\|Lu\|_{F;\mu} \leq c \, \|u\|_{E;\nu}. \blacksquare$$

Observe that topological inclusion is equivalent to the continuity of identity.

6. **Numbering of statements.** The numbering is common to all the statements — **Definition 1.1**, ..., **Definition 1.9**, **Theorem 1.10**, **Theorem 1.11**, etc. —, to make it easier to find a given result by following the order of the numbers. It is not worthwhile therefore to look for Theorems 1.1 to 1.9, as these numbers have been assigned to definitions. There is also no need to look for Definitions 1.10, 1.11, etc.

Theorem 1.13.– *The topological inclusion $E \underset{\rightarrow}{\subsetneq} F$ of a separated semi-normed space into another is equivalent to the continuity of the identity mapping from E into F.* ▌

Proof. Definition 1.7 (c) of topological inclusion coincides with the characterization of continuity from Theorem 1.12 (a) applied to the identity, i.e. for $Lu = u$. □

For semi-norms, Definition 1.9 gives [Vol. 1, Theorem 7.11]:

Theorem 1.14.– *A semi-norm p on a separated semi-normed space E with a filtering family $\{\| \|_{E;\nu} : \nu \in \mathcal{N}_E\}$ of semi-norms is continuous if and only if there exist $\nu \in \mathcal{N}_E$ and $c \geq 0$ such that: for every $u \in E$,*

$$p(u) \leq c\|u\|_{E;\nu}.$$ ▌

Addition and multiplication by a real number t for mappings with values in a vector space are defined by

$$(T + S)(u) \stackrel{\text{def}}{=} T(u) + S(u), \quad (tT)(u) \stackrel{\text{def}}{=} tT(u). \tag{1.1}$$

1.4. Differentiable functions

We reserve the term **function** for a mapping defined on a subset of $\mathbb{R}^d$, which in general is denoted by Ω. Throughout the book, the dimension d is an integer ≥ 1.

Recall that a function is continuous if and only if it is sequentially continuous (Theorem 1.11, since $\mathbb{R}^d$ is normed).

Gradient being defined in $E^d \stackrel{\text{def}}{=} \{(u_1, \ldots, u_d) : u_i \in E, \forall i\}$, recall that this product space is endowed with the semi-norms, indexed by $\nu \in \mathcal{N}_E$,

$$\|u\|_{E^d;\nu} \stackrel{\text{def}}{=} (\|u_1\|^2_{E;\nu} + \cdots + \|u_d\|^2_{E;\nu})^{1/2}, \tag{1.2}$$

which makes it a separated semi-normed space [Vol. 1, Theorem 6.11].

We denote by $|z| \stackrel{\text{def}}{=} (z_1^2 + \cdots + z_d^2)^{1/2}$ the Euclidean norm of $z \in \mathbb{R}^d$ and, for $u \in E^d$,

$$z \cdot u \stackrel{\text{def}}{=} z_1 u_1 + \cdots + z_d u_d.$$

Observe that

$$\|z \cdot u\|_{E;\nu} \leq |z| \|u\|_{E^d;\nu}. \tag{1.3}$$

Indeed, from the Cauchy-Schwarz inequality in $\mathbb{R}^d$, see (A.1), p. 354,

$$\|z \cdot u\|_{E;\nu} \leq \sum_{i=1}^{d} |z_i| \|u_i\|_{E;\nu} \leq \Big(\sum_{i=1}^{d} z_i^2\Big)^{1/2} \Big(\sum_{i=1}^{d} \|u_i\|_{E;\nu}^2\Big)^{1/2} = |z| \|u\|_{E^d;\nu}.$$

Let us define various levels of differentiability for a function with values in a separated semi-normed space.

Definition 1.15.– *Let f be a function from an open subset Ω of $\mathbb{R}^d$ into a separated semi-normed space E, whose family of semi-norms is denoted by $\{\| \ \|_{E;\nu} : \nu \in \mathcal{N}_E\}$.*

*We say that f is **differentiable at the point** x of Ω, if there exists an element of E^d, denoted by $\nabla f(x)$ and called the **gradient** of f at the point x, such that, for every $\nu \in \mathcal{N}_E$ and $\epsilon > 0$, there exists $\eta > 0$ such that, if $z \in \mathbb{R}^d$, $|z| \leq \eta$ and $x + z \in \Omega$, then*

$$\|f(x+z) - f(x) - z \cdot \nabla f(x)\|_{E;\nu} \leq \epsilon |z|.$$

*We say that f is **differentiable** if it is differentiable at every point of Ω, and that it is **continuous differentiable** if, moreover, ∇f is continuous from Ω into E^d.*

*We say that f is m **times differentiable**, where $m \in \mathbb{N}^*$, if it has successive gradients ∇f, $\nabla^2 f, \ldots, \nabla^m f$ (these are elements of E^d, $E^{d^2}, \ldots, E^{d^m}$ respectively).*

*We say that f is m **times continuously differentiable** if, moreover, its successive gradients are continuous. We extend this notion and the previous one to $m = 0$ by denoting*

$$\nabla^0 f \stackrel{\text{def}}{=} f.$$

*We say that f is **infinitely differentiable** if it is m times differentiable for every $m \in \mathbb{N}^*$.* ▮

When $d = 1$, the differentiability of f at the point x reduces to the existence of an element of E, denoted by $f'(x)$ and called the **derivative**[7] at the point x, such

7. **History of the notion of the derivative of a real function.** EUCLIDE, in his *Elements* [31, Book III, p. 16] was already looking for the tangent to a curve. Pierre de FERMAT, in 1636, found the tangent for the curve of equation $y = x^m$ with a calculation prefiguring that of the derivative. Isaac NEWTON introduced in 1671 the *fluxion* of a function $y = f(x)$ which he denoted by $\dot{y}$ [57, p. 76]. Gottfried von LEIBNIZ developed *infinitesimal calculus* in 1675 [51]. The notion of derivative was made rigorous in 1821 by Augustin CAUCHY [18, p. 22].
History of the notation. The notation $\mathrm{d}y/\mathrm{d}x$ was introduced by Gottfried von LEIBNIZ in 1675 [51]. This was what Joseph Louis LAGRANGE denoted by $f'x$ in 1772 [49].
The symbol ∇ was introduced by Sir William Rowan HAMILTON in 1847, by inverting the Greek letter Δ, which had already been used in an analogous context (to designate the *Laplacian*); the name *nabla* was given to him by Peter Guthrie TAIT on the advice of William Robertson SMITH, in 1870, in analogy to the form of a Greek harp which in Antiquity bore this name ($\nu\acute{\alpha}\beta\lambda\alpha$).

that, for every $\nu \in \mathcal{N}_E$ and $\epsilon > 0$, there exists $\eta > 0$ such that, if $t \in \mathbb{R}$, $|t| \leq \eta$ and $x + t \in \Omega$, then

$$\|f(x+t) - f(x) - tf'(x)\|_{E;\nu} \leq \epsilon\,|t|. \tag{1.4}$$

The gradient here has only one component: $\nabla f(x) = f'(x)$. The derivative is often denoted as $\mathrm{d}f/\mathrm{d}x$ instead of f', in particular when we wish to specify the variable with respect to which we are differentiating.

Utility of assuming that Ω is open. This hypothesis guarantees the uniqueness of the gradient at every point where it exists [Vol. 2, Theorem 2.2]. If not, for example if Ω was just a point, then every function would be differentiable and would admit every element of E as its gradient. However, the notion of differentiability can be extended to the closure of an open set while preserving the uniqueness of the gradient [Vol. 2, Definition 2.26]. □

Let us define partial derivatives [8], denoting by $\mathbf{e}_i$ the i-th basis vector of $\mathbb{R}^d$, i.e.

$$(\mathbf{e}_i)_i = 1, \quad (\mathbf{e}_i)_j = 0 \text{ if } j \neq i,$$

and denoting by $[\![m, n]\!]$ the interval of integers $\{i \in \mathbb{N} : m \leq i \leq n\}$.

Definition 1.16.– *Let f be a function from an open subset Ω of $\mathbb{R}^d$ into a separated semi-normed space E, whose family of semi-norms is denoted $\{\|\ \|_{E;\nu} : \nu \in \mathcal{N}_E\}$.*

*We say that f has a **partial derivative** $\partial_i f(x) \in E$ at the point x of Ω, where $i \in [\![1, d]\!]$, if the function $x_i \mapsto f(x)$ is differentiable at the point x_i with derivative $\partial_i f(x)$.*

That is to say, if, for every $\nu \in \mathcal{N}_E$ and all $\epsilon > 0$, there exists $\eta > 0$ such that, if $t \in \mathbb{R}$, $|t| \leq \eta$ and $x + t\mathbf{e}_i \in \Omega$, then

$$\|f(x+t\mathbf{e}_i) - f(x) - t\partial_i f(x)\|_{E;\nu} \leq \epsilon\,|t|. \tag{1.5}$$

▮

Clarification. More precisely, f has a partial derivative $\partial_i f(x)$ at the point x if the function

$$s \mapsto f(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_d),$$

which is defined on the open subset $\{s \in \mathbb{R} : (x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_d) \in \Omega\}$ of $\mathbb{R}$, has the derivative $\partial_i f(x)$ at the point x_i. □

8. **History of partial derivatives.** Partial derivatives appeared in 1747 with Alexis Claude CLAIRAUT and Jean le Rond D'ALEMBERT [24], and in 1755 with Leonhard EULER [32].
The symbol ∂ was introduced by Nicolas de Caritat, Marquis of CONDORCET, in 1773 [22].