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Chao Wang
Ravi P. Agarwal

Combined Measure and Shift Invariance Theory of Time Scales and Applications

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Chao Wang • Ravi P. Agarwal

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Chao Wang
Department of Mathematics
Yunnan University
Kunming, Yunnan, China

Ravi P. Agarwal
Department of Mathematics
Texas A&M University-Kingsville
Kingsville, TX, USA

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We dedicate this book to our family members:

Chao Wang dedicates the book to his daughter Yuhan Wang and wishes her to grow up healthily and happily;

Ravi P. Agarwal dedicates the book to his wife Sadhna Agarwal.

Preface

Measure theory was initiated at the beginning of the twentieth century and "measure" is an important notion in analyzing the subsets of Euclidian spaces. In 1989, E. Borel first established a measure theory on subsets of the real numbers known as Borel sets, and Lebesgue measure was introduced by H. Lebesgue in 1902 and the related integral based on measure theory is more comprehensive than the Riemann integral (see [51, 117]). In fact, the notion of measure and its significance widely generalize the classical definitions of "length" and "area" in Euclidian spaces. In 1918, the concept of outer measures was introduced and studied by C. Carathéodory (see [160]). Since then, measure theory and the calculus theory based on it were developed rapidly in the field of pure and applied mathematics.

In 1988, a new analysis theory called time scale theory that can unify continuous and discrete analysis was proposed by S. Hilger (see [118]). Then it has been widely used to study various classes of dynamic equations and models in the real-world applications (see [61, 63, 107, 141, 202]). Time scale is an arbitrary closed subset of the real line, and the calculus on a time scale includes the classical Riemann integral and the discrete sum, and the different form of the calculus between the classical Riemann integral and the discrete sum such as q-calculus and its generalizations is a big advantage. Measure theory based on time scales is powerful and significant in studying functions on time scales or hybrid domains (see [86]). In 1999, Agarwal et al. investigated the basic calculus on time scales and some of its applications (see [4]), and Guseinov et al. analyzed Riemann delta and nabla integration on time scales (see [112–114]). To understand basic integral on time scales, Cabada et al. formally introduced the concept of Δ-measure on time scales (see [69]) and the relation of the Lebesgue Δ-integral and the usual Lebesgue integral is thoroughly investigated. Based on these fundamental work, Deniz et al. introduced the notion of Lebesgue-Stieltjes measure on time scales in both Δ and ∇ forms, and the relation between Lebesgue-Stieltjes integral and Lebesgue-Stieltjes Δ-integral was established (see [87]).

With the rapid development of time scale analysis, a new view of dynamic derivatives on time scales was introduced by Q. Sheng (see [164]), and a new analysis method which can then unify Δ and ∇-cases together on time scales. In

2006, Q. Sheng et al. proposed the concept of combined dynamic derivatives on time scales, and it was the first time that Δ and ∇-dynamic equations can transfer mutually, and some new dynamic behavior which are between Δ and ∇-cases and outside of difference and differential equations can also be investigated based on this theory (see [165]). This combined idea was used to analyze different dynamic equations and inequalities and was also applied to hybrid dynamic behavior that will not appear in the real line and discrete time scales [16, 17, 159, 166].

Now it is natural to ask a question see how to analyze the measure of a time scale in a combined form which not only can the Δ and ∇ measurability transfer mutually but also can induce a new hybrid measure. In this book, a new measure theory called combined measure theory on time scales is established, and $\Diamond_\alpha$-measurability is proposed and studied.

Since time scale theory is a new and exciting type of mathematics and is more comprehensive and versatile than the traditional theories of differential and difference equations as it can mathematically and precisely depict the continuous-discrete hybrid processes and hence is the optimal way forward for accurate mathematical modelling in applied sciences such as physics, chemical technology, population dynamics, biotechnology and economics and social sciences. The one of the most important of applications is to study of almost periodic and almost automorphic functions and dynamic equations on hybrid time scales. Almost periodic phenomena are common and important in the real world, and the concept of almost periodic functions was first studied by H. Bohr and later generalized by V. Stepanov, H. Weyl, and A.S. Besicovitch, among others (see [54, 56, 57, 168, 211]). In 1955–1962, S. Bochner observed in various contexts that a certain property enjoyed by the almost periodic functions on the group G can be applied in obtaining more concise and logical proofs of certain theorems in terms of these functions, and S. Bochner called his property "almost automorphy" since it first aroused in work on differential geometry (see [58–60]). Based on well-known almost periodic and almost automorphic functions proposed by Bohr and Bochner, many new generalized concepts were introduced and studied by several researchers on the real line. Unfortunately, these theories do not work on non-translational and irregular time scales since the classical concepts of almost periodic and almost automorphic functions purely depend on the translation of functions, the results of the current study of almost periodic and almost automorphic problems are restricted to periodic time scales under translations. Nevertheless, for example, $\overline{q^{\mathbb{Z}}} := \overline{\{q^n : q > 1, n \in \mathbb{Z}\}}$ is not periodic under translations. This time scale leads to q-difference dynamic equations and plays an important role in different fields of engineering and biological science (see [43, 65]). However, it was impossible to study almost periodic problems for q-difference dynamic equations in the past because $\overline{q^{\mathbb{Z}}}$ is so irregular (the graininess function μ is unbounded) and there was no concept of almost periodic functions defined on it. The irregular distribution on the real line leads to many difficulties in studying functions on time scales, especially in investigating functions defined by the arbitrary shifts of arguments such as periodic functions, almost periodic functions and almost automorphic functions, etc. The classes of functions defined by the shifts of arguments are referred to as **Shift Functions**. In fact, the time scale without shift invariance will change the classical concept of relatively dense set on the real line, the convergence of function sequences, the completeness of function spaces, an almost periodicity of the variable limit integrals, etc. Therefore, it is very significant to make clear these basic properties before studying dynamic equations with shift operators on irregular time scales. In this book, the theory of matched spaces of time scales is established under which the closedness of time scales under non-translational shifts will be guaranteed so that $\overline{q^{\mathbb{Z}}}$ and some other types of irregular time scales can be regarded as the periodic time scales under shifts. Furthermore, the periods set of this new type of periodic time scale may be completely separated from the time scale $\mathbb{T}$. These fundamental results established in this book will smooth the path to investigate the periodic, almost periodic and almost automorphic problems in which the definition of the functions is determined by the irregular shift of time variables.

In this monograph, we establish a theory of combined measure and shift invariance of time scales and present the applications to realistic dynamical mathematical models on irregular hybrid time scales. The monograph is organized as 9 chapters:

In Chap. 1, the basic knowledge of calculus on the time scales is introduced including Riemann integration, stochastic calculus, combined derivatives and shift operators, etc., to make the book self-contained. Based on this knowledge, the combined Liouville formula and α-matrix exponential solutions to diamond-α dynamic equations are obtained and fundamental results of the quaternion combined impulsive matrix dynamic equation on time scales are established. The content of this chapter is the necessary theoretical knowledge to be used in the later chapters.

In Chap. 2, the combined measure theory on time scales is established, some important notions such as $\Diamond_\alpha$-measurability, $\Diamond_\alpha$-measurable functions, Lebesgue-Stieltjes combined $\Diamond_\alpha$-measure and the related integrals are introduced and studied. Based on the theory of combined measurability on time scales, the Riemann measure calculus is highly unified and deeply discussed on irregular hybrid time scales.

In Chap. 3, the concept of matched spaces of time scales is introduced and their basic properties are established. Based on it, the fundamental theory of the shift invariance of time scales is addressed. The whole chapter is devoted to the theory of the matched spaces of time scales and the intrinsic connections of some basic concepts introduced in this chapter are presented. Moreover, singularity theory of time scales under the action of the shift operators is established.

Chapter 4 is mainly devoted to establish a theory of almost periodic functions through the theory of matched spaces of time scales. Some basic notions and properties of periodic functions under the complete-closed time scales in shifts are presented. Moreover, a theory of δ-almost periodic functions under matched spaces is developed and a generalized notion of δ-almost periodic functions called n_0-order Δ-almost periodic functions is proposed and investigated.

In Chap. 5, based on the theory of matched space of time scales, we develop a theory of almost automorphic functions. A notion of weighted pseudo δ-almost automorphic functions and a new generalized type of almost automorphic functions called n_0-order weighted pseudo δ-almost automorphic functions under matched spaces are introduced and studied. Some of their basic properties are established. Moreover, some fundamental results of the discontinuous S-almost automorphic functions are obtained through introducing the **S**-equipotentially almost automorphic sequence under the complete-closed time scales under shift operators (short for S-CCTS).

In Chap. 6, some basic notions of C_0-semigroup and Stepanov-like almost automorphic functions in matched spaces of time scales are introduced and studied. The concept of C_0-semigroup on a quantum time scale is proposed and some basic properties are established and the notion of the Stepanov-like almost automorphic functions is introduced on a quantum time scale and their fundamental properties are investigated. Moreover, the weak automorphy of such functions in the quantum case is discussed. Finally, the theory of shift-semigroup and Stepanov-like almost automorphic functions under the matched space of time scales is established.

In Chap. 7, based the theory of δ-almost periodic functions and n_0-order Δ-almost periodic functions developed in Chap. 4, some fundamental results of the δ-almost periodic solutions and n_0-order Δ-almost periodic solutions of a general dynamic equations are established under matched spaces of time scales. Particularly, the basic results of almost periodic problems of the q-dynamic equations on a quantum time scale are included as the special case. In addition, by using the developed theory of matched spaces of time scales, the basic theory of dynamic equations under matched space of time scales is established and some effective methods are provided to study the almost periodic solutions of dynamic equations on hybrid time scales.

In Chap. 8, based on the theory established in Chap. 5, two types of almost automorphic solutions of dynamic equations under the matched spaces are discussed systematically. The weighted pseudo δ-almost automorphic solutions and the n_0-order weighted pseudo Δ-almost automorphic solutions of the general inhomogeneous dynamic equations are studied. Moreover, the almost automorphy of the solutions to dynamic equations with shift operators is analyzed, and some basic results of the discontinuous cases are established.

In Chap. 9, we will introduce some new types of neutral impulsive stochastic dynamical models on irregular time scales. By using almost periodic results of stochastic dynamic equations with shift operators, the mean-square almost periodic solutions for a new type of neutral impulsive stochastic Lasota-Wazewska timescale model are investigated. Moreover, the mean square almost periodic stochastic process with shift operators is applied to study the almost periodic oscillations for delay impulsive stochastic Nicholson's blowflies timescale model on hybrid time scales.

This is a monograph devoted to developing a theory of combined measure and shift invariance of time scales with the related applications to shift functions and dynamic equations. The study of shift closedness of time scales is significant to investigate the shift functions such as the periodic functions, the almost periodic functions, and the almost automorphic functions and their generalizations, which have many important applications in dynamic equations on arbitrary time scales.

The book is written at a graduate level and is intended for university libraries, graduate students, and researchers working in the field of general dynamic equations on time scales, and it will stimulate further research into the time scale theory. The book is also a good reference material for those undergraduates who are interested in dynamic equations on time scales and familiar with functional analysis, measure theory, and ordinary differential equations.

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Kunming, Yunnan, China — Chao Wang
Kingsville, TX, USA — Ravi P. Agarwal

Contents

Chapter 1
Riemann Integration, Stochastic Calculus, and Shift Operators on Time Scales

This chapter mainly introduces basic knowledge of calculus on time scales. In Sect. 1.1, concepts and fundamental properties of Riemann delta and nabla integration on time scales are introduced including some basic results of Riemann integral and fundamental theorems of calculus. In Sect. 1.2, stochastic calculus and some basic results of stochastic dynamic equations on time scales are provided. Section 1.3 is mainly devoted to introducing the concept of shift operators on time scales by which a new concept of periodicity is introduced; shift operator plays an important role in discussing shift invariance of time scales in later chapters. In Sect. 1.4, momentous hybrid derivatives called combined derivatives or diamond-α derivatives which can strictly include delta and nabla derivatives are introduced, and some basic properties of combined dynamic derivatives and integrations are established. In Sect. 1.5, the combined Liouville formula and α-matrix exponential solutions to diamond-α dynamic equations are obtained. In Sect. 1.6, fundamental results of the quaternion combined impulsive matrix dynamic equation on time scales are established. The content of this chapter is the necessary theoretical knowledge to be used in the later chapters.

1.1 Riemann Integration on Time Scales

A time scale $\mathbb{T}$ (which is a special case of a measure chain) is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Thus, it is a complete metric space with the metric $d(t, s) = |t - s|$.

For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

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while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, we put in addition $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$. Obviously, both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$ since that $\mathbb{T}$ is a closed subset of $\mathbb{R}$.

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \max \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \min \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense.

We introduce the sets $\mathbb{T}^{\kappa}$ and $\mathbb{T}_{\kappa}$ which are derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{t_1\}$; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_{\kappa} = \mathbb{T} - \{t_2\}$; otherwise $\mathbb{T}_{\kappa} = \mathbb{T}$.

For $a, b \in \mathbb{T}$ with $a \leq b$, we define the closed interval $[a, b]_{\mathbb{T}}$ in $\mathbb{T}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant b\}$. Open intervals, half-open intervals, etc. are defined accordingly.

If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^{\kappa}$, then the delta (or Hilger) derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U (in $\mathbb{T}$) of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

If $t \in \mathbb{T}_{\kappa}$, then we define the nabla derivative of f at the point t to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U (in $\mathbb{T}$) of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)[\rho(t) - s]| \leqslant \varepsilon|\rho(t) - s| \quad \text{for all } s \in U.$$

Note that the delta and nabla derivatives are particular cases of the alpha derivative introduced by Ahlbrandt et al. in [28], namely, with $\alpha = \sigma$ and $\alpha = \rho$, respectively. In fact, some important discrete dynamic equations such as Hamiltonian systems, Riccati equation, etc. (see [2, 24–27]) can be unified by time scales effectively (see [29–31]). The special and common feature of the continuous and discrete dynamic systems could be generally built by this powerful mathematical tool (see Aulbach et al. [20–23], Agarwal et al. [2, 3, 5–8, 10–13, 16, 17], Akhmet et al. [34, 35], Akın et al. [36–38], Atici et al. [44–47]).

If $\mathbb{T} = \mathbb{R}$, then f is delta differentiable (nabla differentiable) at t if f is differentiable in the ordinary sense at t. In this case we then have $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$, then f is delta differentiable (nabla differentiable) at t and we have

$$f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t), \quad f^{\nabla}(t) = f(t) - f(t-1) = \nabla f(t),$$

where Δ is the usual forward difference operator and ∇ is the usual backward difference operator defined by the last equations above.

A partition of $[a, b)_{\mathbb{T}}$ is any finite ordered subset $P = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_{\mathbb{T}}$, where $a = t_0 < t_1 < \ldots < t_n = b$. The number n depends on the particular partition, so we have $n = n(P)$. Let σ and ρ be the forward and backward jump operators in $\mathbb{T}$, respectively, and f be a real-valued bounded function on $[a, b)_{\mathbb{T}}$.

Denote

$$M = \sup\{f(t) : t \in [a, b)_{\mathbb{T}}\}, \quad m = \inf\{f(t) : t \in [a, b)_{\mathbb{T}}\},$$

$$M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}, \quad m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}.$$

The upper Darboux Δ-sum $U(f, P)$ and the lower Darboux Δ-sum $L(f, P)$ of f with respect to P are defined by

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}), \quad L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

Then

$$m(b - a) \leqslant L(f, P) \leqslant U(f, P) \leqslant M(b - a). \tag{1.1}$$

The upper Darboux Δ-integral $U(f)$ of f from a to b is defined by

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b)_{\mathbb{T}}\}$$

and the lower Darboux Δ-integral $L(f)$ is

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b)_{\mathbb{T}}\}$$

In view of (1.1), $U(f)$ and $L(f)$ are finite real numbers. Obviously, $L(f) \leqslant U(f)$.

1.1.1 Riemann Delta and Nabla Integration on Time Scales

In this subsection, we will introduce the concepts and properties of the Riemann delta and nabla integration on time scales (see [112–114]).

Definition 1.1 (see [112, 113]) We say that f is Δ-integrable (delta-integrable) from a to b (or on $[a, b)_{\mathbb{T}}$) provided $L(f) = U(f)$. In this case, we write $\int_a^b f(t)\Delta t$ for this common value. We call this integral the Darboux Δ-integral.

If P and Q are two partitions of $[a, b)_{\mathbb{T}}$ such that every point of P belongs to Q, i.e., $P \subset Q$, then we say that Q is a refinement of, or is finer than, P. The following lemma can be proved by using the similar way as that in the case $\mathbb{T} = \mathbb{R}$.

Lemma 1.1 (see [112, 113]) *Let f be a bounded function on $[a, b)_{\mathbb{T}}$. If P and Q are partitions of $[a, b)_{\mathbb{T}}$ and Q is a refinement of P, then $L(f, P) \leqslant L(f, Q) \leqslant$*

$U(f, Q) \leqslant U(f, P)$, i.e., adding more points to a partition increases the lower sum and decreases the upper sum.

Lemma 1.2 (see [112, 113]) *If f is a bounded function on $[a, b)_{\mathbb{T}}$, and if P and Q are any two partitions of $[a, b)_{\mathbb{T}}$, then $L(f, P) \leqslant U(f, Q)$, i.e., every lower sum is less than or equal to every upper sum.*

Proof The set $P \cup Q$ is also a partition of $[a, b)_{\mathbb{T}}$. By applying Lemma 1.1 and $P \subset P \cup Q$ and $Q \subset P \cup Q$, it follows that $L(f, P) \leqslant L(f, P \cup Q) \leqslant U(f, P \cup Q) \leqslant U(f, Q)$. This completes the proof. □

Obviously, Lemma 1.2 yields the following result:

Theorem 1.1 (see [112, 113]) *If f is a bounded function on $[a, b)_{\mathbb{T}}$, then $L(f) \leqslant U(f)$.*

In fact, we can easily have $L(f, P) \leqslant L(f) \leqslant U(f) \leqslant U(f, Q)$ for all partitions P and Q of $[a, b)_{\mathbb{T}}$. In particular, for all partitions P of $[a, b)_{\mathbb{T}}$, it follows that $L(f, P) \leqslant L(f) \leqslant U(f) \leqslant U(f, P)$. Hence the following result is immediate:

Theorem 1.2 (see [112, 113]) *If $L(f, P) = U(f, P)$ for some partition P of $[a, b)_{\mathbb{T}}$, then the function f is Δ-integrable from a to b and $\int_a^b f(t)\Delta t = L(f, P) = U(f, P)$.*

In the following theorem, a "Cauchy criterion" for integrability and its proof can be given similar to the case $\mathbb{T} = \mathbb{R}$.

Theorem 1.3 (see [112, 113]) *A bounded function f on $[a, b)_{\mathbb{T}}$ is Δ-integrable if and only if for each $\varepsilon > 0$ there exists a partition P of $[a, b)_{\mathbb{T}}$ such that $U(f, P) - L(f, P) < \varepsilon$.*

Lemma 1.3 (see [112, 113]) *For every $\delta > 0$ there exists at least one partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b)_{\mathbb{T}}$ such that for each $i \in \{1, 2, \ldots, n\}$ either $t_i - t_{i-1} \leqslant \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$, where ρ denotes the backward jump operator in $\mathbb{T}$.*

Definition 1.2 (see [112, 113]) For given $\delta > 0$, the set of all partitions $P : a = t_0 < t_1 < \cdots < t_n = b$ is denoted by $\mathfrak{G}_\delta([a, b)_{\mathbb{T}})$ or simply by $\mathfrak{G}_\delta$ which possess the property indicated in Lemma 1.3.

The following criterion for integrability can easily follow from the case $\mathbb{T} = \mathbb{R}$.

Theorem 1.4 (see [112, 113]) *A bounded function f on $[a, b)_{\mathbb{T}}$ is Δ-integrable if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$P \in \mathfrak{G}_\delta \quad \textit{implies} \quad U(f, P) - L(f, P) < \varepsilon \tag{1.2}$$

for all partitions P of $[a, b)_{\mathbb{T}}$.

The Riemann definition of integrability is given as follows:

Definition 1.3 (see [112, 113]) Let f be a bounded function on $[a, b)_{\mathbb{T}}$, and let $P : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b)_{\mathbb{T}}$. In each interval $[t_{i-1}, t_i)_{\mathbb{T}}$, where $1 \leqslant i \leqslant n$, choose an arbitrary point ξ_i and form the sum

$$S = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}). \tag{1.3}$$

S is called a Riemann Δ-sum of f corresponding to the partition P. f is said to be Riemann Δ-integrable from a to b (or on $[a, b)_{\mathbb{T}}$) if there exists a number I with the property that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann Δ-sum S of f corresponding to a partition $P \in \mathfrak{G}_\delta$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$, $i = 1, 2, \ldots, n$. It is easily seen that such a number I is unique. The number I is the Riemann Δ-integral of f from a to b.

Theorem 1.5 (see [112, 113]) *A bounded function f on $[a, b)_{\mathbb{T}}$ is Riemann Δ-integrable if and only if it is (Darboux) Δ-integrable, in which case the values of the integrals are equal.*

Proof Assume that f is (Darboux) Δ-integrable from a to b in the sense of Definition 1.1. Let $\varepsilon > 0$ and $\delta > 0$ be chosen such that (1.2) of Theorem 1.4 holds. We claim that

$$\left| S - \int_a^b f(t)\Delta t \right| < \varepsilon \tag{1.4}$$

for every Riemann Δ-sum (1.3) associated with a partition $P \in \mathfrak{G}_\delta$. In fact, we have $L(f, P) \le S \le U(f, P)$ and so (1.4) follows from the inequalities

$$U(f, P) < L(f, P) + \varepsilon \leqslant L(f) + \varepsilon = \int_a^b f(t)\Delta t + \varepsilon,$$

$$L(f, P) > U(f, P) - \varepsilon \geqslant U(f) - \varepsilon = \int_a^b f(t)\Delta t - \varepsilon.$$

i.e., (1.4) holds; hence f is Riemann Δ-integrable and $I = \int_a^b f(t)\Delta t$.

Now assume that f is Riemann Δ-integrable in the sense of Definition 1.3, and consider $\varepsilon > 0$. Let $\delta > 0$ and let I be as given in Definition 1.3. Select any partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b)$ such that $P \in \mathfrak{G}_\delta$, and for each $i = 1, 2, \ldots, n$, choose ξ_i in $[t_{i-1}, t_i)$ such that $f(\xi_i) < m_i + \varepsilon$, where $m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)\}$. The Riemann Δ-sum S for this choice of ξ_i satisfies $S < L(f, P) + \varepsilon(b - a)$ as well as $|S - I| < \varepsilon$. It follows that $L(f) \geqslant L(f, P) > S - \varepsilon(b - a) > I - \varepsilon - \varepsilon(b - a)$. Since ε is arbitrary, we conclude that $L(f) \geqslant I$. A similar argument shows that $U(f) \leqslant I$. Since $L(f) \leqslant U(f)$, it follows that $L(f) = U(f) = I$. This shows that f is (Darboux) Δ-integrable and that $\int_a^b f(t)\Delta t = I$. The proof is completed. □

In the definition of $\int_a^b f(t)\Delta t$, we assumed that $a < b$. The following definitions remove this restriction:

$$\int_a^a f(t)\Delta t = 0, \quad \int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t, \quad a > b. \tag{1.5}$$

Theorem 1.6 (see [112, 113]) *Assume that a and b are arbitrary points in* $\mathbb{T}$*. Every constant function* $f(t) = c(t \in \mathbb{T})$ *is* Δ*-integrable from a to b and*

$$\int_a^b c\Delta t = c(b-a). \tag{1.6}$$

Proof Let $a < b$. Consider a partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b)_{\mathbb{T}}$. Obviously we have $U(f, P) = L(f, P) = c(b-a)$ and therefore Theorem 1.2 shows that f is Δ-integrable and (1.6) holds. Formula (1.6) for $a = b$ and $a > b$ follows by the definition (1.5). Note that every Riemann Δ-sum of f associated with P is also equal to $c(b-a)$. The proof is completed. □

Theorem 1.7 (see [112, 113]) *Let t be an arbitrary point in* $\mathbb{T}$*. Every function f defined on* $\mathbb{T}$ *is* Δ*-integrable from t to* $\sigma(t)$ *and*

$$\int_t^{\sigma(t)} f(s)\Delta s = [\sigma(t) - t]f(t). \tag{1.7}$$

Proof If $\sigma(t) = t$, then (1.7) is obvious, because both sides of (1.7) are equal to zero in this case. Let now $\sigma(t) > t$. Then a single partition of $[t, \sigma(t))_{\mathbb{T}}$ is $P : t = s_0 < s_1 = \sigma(t)$, and since $[s_0, s_1)_{\mathbb{T}} = [t, \sigma(t))_{\mathbb{T}} = \{t\}$, we have $U(f, P) = f(t)[\sigma(t) - t] = L(f, P)$. Therefore, it follows from Theorem 1.2 that f is Δ-integrable from t to $\sigma(t)$ and (1.7) holds. Note that the Riemann Δ-sum of f associated with P is also equal to $f(t)[\sigma(t) - t]$. This completes proof. □

Theorem 1.8 (see [112, 113]) *Assume* $a, b \in \mathbb{T}$ *and* $a < b$*. Then we have the following:*

(i) *If* $\mathbb{T} = \mathbb{R}$*, then a bounded function f on* $[a, b)_{\mathbb{T}}$ *is* Δ*-integrable from a to b if and only if f is Riemann-integrable on* $[a, b)_{\mathbb{T}}$ *in the classical sense; in this case*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t,$$

where the integral on the right is the usual Riemann integral.

(ii) *If* $\mathbb{T} = \mathbb{Z}$, *then every function* f *defined on* $\mathbb{Z}$ *is* Δ*-integrable from* a *to* b *and*

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k). \tag{1.8}$$

Proof Clearly, the above given Definitions 1.1 and 1.3 of the Δ-integral coincide in case $\mathbb{T} = \mathbb{R}$ with the usual Darboux and Riemann definitions of the integral, respectively. Notice that the classical definitions of Darboux's and Riemann's integrals do not depend on whether the subintervals of the partition are taken closed, half-closed, or open. Moreover, if $\mathbb{T} = \mathbb{R}$, then $\mathfrak{G}_\delta([a, b)_\mathbb{T})$ consists of all partitions of $[a, b)_\mathbb{T}$ the norm (mesh) of which is less than or equal to δ. So part (i) of the theorem is valid.

To prove part (ii), let $a < b$. Then $b = a + p$ for some positive integer p. Consider the partition P^* of $[a, b)_\mathbb{Z}$ defined by $P^* : a = t_0 < t_1 < \ldots < t_p = b$, where $t_0 = a, t_1 = a + 1, \ldots, t_p = a + p$. P^* contains all points of $[a, b]_\mathbb{Z}$ and $[t_{i-1}, t_i)_\mathbb{Z} = \{t_{i-1}\}$ for each $i \in \{1, 2, \ldots, p\}$. Then

$$U(f, P^*) = \sum_{i=1}^{p} M_i(t_i - t_{i-1} = \sum_{i=1}^{p} f(t_{i-1}),$$

$$L(f, P^*) = \sum_{i=1}^{p} m_i(t_i - t_{i-1}) = \sum_{i=1}^{p} f(t_{i-1}).$$

So $U(f, P^*) = L(f, P^*) = p_i = \sum_{i=1}^{p} f(t_{i-1}) = \sum_{k=a}^{b-1} f(k)$, and it follows from Theorem 1.2 that f is Δ-integrable from a to b and (1.8) holds. This completes the proof. □

The concept of ∇-integral (nabla integral) on time scales can be described briefly as follows:

Definition 1.4 (see [112, 113]) Let $P : a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $(a, b]_\mathbb{T}$ and f be a real-valued bounded function on $(a, b]_\mathbb{T}$. Now let

$$M' = \sup\{f(t) : t \in (a, b]_\mathbb{T}\}, \quad m' = \inf\{f(t) : t \in (a, b]_\mathbb{T}\},$$
$$M_i' = \sup\{f(t) : t \in (t_{i-1}, t_i]_\mathbb{T}\}, \quad m_i' = \inf\{f(t) : t \in (t_{i-1}, t_i]_\mathbb{T}\}.$$

We call the sums

$$U'(f, P) = \sum_{i=1}^{n} M_i'(t_i - t_{i-1}) \quad \textit{and} \quad L'(f, P) = \sum_{i=1}^{n} m_i'(t_i - t_{i-1})$$

respectively as the upper and lower Darboux ∇-sums of f.

It follows that

$$m'(b-a) \leqslant L'(f, P) \leqslant U'(f, P) \leqslant M'(b-a).$$

Definition 1.5 (see [112, 113]) The numbers

$$U'(f) = \inf\{U'(f, P) : P \text{ is a partition of } (a, b]_{\mathbb{T}}\}$$

and

$$L'(f) = \sup\{L'(f, P) : P \text{ is a partition of } (a, b]_{\mathbb{T}}\}$$

are called the upper and lower Darboux ∇-integrals of f from a to b, respectively.

$U'(f)$ and $L'(f)$ are finite and the inequality $L'(f) \leqslant U'(f)$ holds.

Definition 1.6 (see [112, 113]) f is said to be ∇-integrable (nabla integrable) from a to b (or on $(a, b]_{\mathbb{T}}$) if $L'(f) = U'(f)$. In this case we write $\int_a^b f(t)\nabla t$ for this common value. We call this integral the Darboux ∇-integral.

Definition 1.7 (see [112, 113]) A Riemann ∇-sum of f associated with the partition P is a sum of the form

$$s' = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

where $\xi_i \in (t_{i-1}, t_i]_{\mathbb{T}}$ for $i = 1, 2, \ldots, n$. The function f is Riemann ∇-integrable from a to b (or on $(a, b]_{\mathbb{T}}$) if there exists a number I' with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S' - I'| < \varepsilon$ for every Riemann ∇-sum S' of f associated with a partition $P \in \mathfrak{G}_\delta$, independent of the way in which we choose the points $\xi_i \in (t_{i-1}, t_i]_{\mathbb{T}}, i = 1, 2, \ldots, n$, where $\mathfrak{G}_\delta$ denotes as above the set of all partitions P of $(a, b]_{\mathbb{T}}$ possessing the property indicated in Lemma 1.3 (note that the inequality $t_{i-1} < t_i$ with $\rho(t_i) = t_{i-1}$ is equivalent to $t_{i-1} < t_i$ with $\sigma(t_{i-1}) = t_i$). The number I' is the Riemann ∇-integral of f from a to b.

Remark 1.1 A bounded function f on $(a, b]_{\mathbb{T}}$ is Riemann Δ-integrable if and only if it is (Darboux) ∇-integrable, in this case the values of the integrals equal.

For the special case $\mathbb{T} = \mathbb{R}$, the Riemann ∇-integral, as in the case of the ∇-integral, coincides with the usual Riemann integral. For the case $\mathbb{T} = \mathbb{Z}$, it follows that $\int_a^b f(t)\nabla t = \sum_{k=a+1}^{b} f(k)$, $a < b$. Comparing this with (1.8) shows that the delta and nabla integrals are in general different.

Remark 1.2 In the concept of the Δ-integral, the subintervals of a partition $P : a = t_0 < t_1 < \ldots < t_n = b$ were adopted as the intervals $[t_{i-1}, t_i)_{\mathbb{T}}, i = 1, 2, \ldots, n$. In [110, 111], in the definition of the Δ-integral, the intervals $[t_{i-1}, \rho(t_i)]_{\mathbb{T}}, i = 1, 2, \ldots, n$, were adopted instead of the intervals $[t_{i-1}, t_i)_{\mathbb{T}}, i = 1, 2, \ldots, n$. It can be shown that these two representations of the intervals are equivalent (see [113]).

1.1.2 Some Fundamental Results of the Riemann Integral

In this section, some properties of the Riemann delta integral are presented which also hold for the Riemann nabla integral.

Theorem 1.9 (see [112, 113]) *Let f be Δ-integrable on $[a, b)_{\mathbb{T}}$ and let M and m be its supremum and infimum on $[a, b)_{\mathbb{T}}$, respectively. Let, further, $\varphi : \mathbb{R} \to \mathbb{R}$ be a function defined on $[m, M]$ such that there exists a positive constant B with $|\varphi(x) - \varphi(y)| \leqslant B|x - y|$ for all x and y in $[m, M]$ (this condition is called as the Lipschitz condition). Then the composite function $h(t) = \varphi(f(t))$ is Δ-integrable on $[a, b)_{\mathbb{T}}$.*

Proof For an arbitrary $\varepsilon > 0$. By Theorem 1.3, there exists a partition $P : a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b)_{\mathbb{T}}$ such that $U(f, P) - L(f, P) < \varepsilon/B$. Let M_i and m_i be the supremum and infimum of f on $[t_{i-1}, t_i)_{\mathbb{T}}$, respectively, and let M_i^* and m_i^* be the corresponding numbers for h. By the condition on φ, we have, for all s and τ in $[t_{i-1}, t_i)_{\mathbb{T}}$,

$$\begin{aligned} h(s) - h(\tau) \leqslant& |h(s) - h(\tau)| = |\varphi(f(s)) - \varphi(f(\tau))| \\ \leqslant& B|f(s) - f(\tau)| \leqslant B(M_i - m_i). \end{aligned}$$

Hence $M_i^* - m_i^* \leq B(M_i - m_i)$ because there exist two sequences (s_k) and (τ_k) of points in $[t_{i-1}, t_i)_{\mathbb{T}}$ such that $h(s_k) \to M_i^*$ and $h(\tau_k) \to m_i^*$ as $k \to \infty$. Consequently,

$$\begin{aligned} U(h, P) - L(h, P) &= \sum_{i=1}^{n} (M_i^* - m_i^*)(t_i - t_{i-1}) \leqslant B \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &= B[U(f, P) - L(f, P)] < B\frac{\varepsilon}{B} = \varepsilon \end{aligned}$$

and h is Δ-integrable on $[a, b]_{\mathbb{T}}$ by Theorem 1.3. This completes the proof. □

The following theorem is more general than Theorem 1.9 and can be shown similar to the case $\mathbb{T} = \mathbb{R}$; we will not repeat the proofs here.

Theorem 1.10 (see [112, 113]) *Let f be Δ-integrable on $[a, b)_{\mathbb{T}}$ and let M and m be its supremum and infimum on $[a, b)_{\mathbb{T}}$, respectively. Let, further, $\varphi : \mathbb{R} \to \mathbb{R}$*

be a continuous function on $[m, M]$*. Then the composite function* $h = \varphi \circ f$ *is* Δ*-integrable on* $[a, b)_{\mathbb{T}}$*.*

Corollary 1.1 (see [112, 113]) *If* f *is* Δ*-integrable on* $[a, b)_{\mathbb{T}}$*, then for an arbitrary positive number* α*, the function* $|f|^{\alpha}$ *is* Δ*-integrable on* $[a, b)_{\mathbb{T}}$*.*

Proof In fact, it is sufficient to consider the continuous function $\varphi(x) = |x|^{\alpha}$ and apply Theorem 1.10. □

Theorem 1.11 (see [112, 113]) *Let* f *be a bounded function that is* Δ*-integrable on* $[a, b)_{\mathbb{T}}$*. Then* f *is* Δ*-integrable on every subinterval* $[c, d)_{\mathbb{T}}$ *of* $[a, b)_{\mathbb{T}}$*.*

Proof Let $\varepsilon > 0$ and P be a partition of $[a, b)_{\mathbb{T}}$ such that $U(f, P) - L(f, P) < \varepsilon$. Adding to P the points c and d, we get a new partition P' of $[a, b)_{\mathbb{T}}$. Then by Lemma 1.1 we also have $U(h, P') - L(h, P') < \varepsilon$. Now consider the partition P'' of $[c, d)_{\mathbb{T}}$ consisting of all points of P' belonging to $[c, d]_{\mathbb{T}}$. For upper and lower Δ-sums $\widetilde{U}$ and $\widetilde{L}$ of f on $[c, d)_{\mathbb{T}}$ associated with this partition P'', we have $\widetilde{U} - \widetilde{L} \leqslant U(f, P') - L(f, P')$. So, $\widetilde{U} - \widetilde{L} < \varepsilon$ and by Theorem 1.3 the function f is Δ-integrable on $[c, d)_{\mathbb{T}}$. □

The proof of the following theorem is similar to the case $\mathbb{T} = \mathbb{R}$.

Theorem 1.12 (see [112, 113]) *Let* f *and* g *be* Δ*-integrable functions on* $[a, b)_{\mathbb{T}}$ *and* c *be a real number. Then*

(i) cf *is* Δ*-integrable and* $\int_a^b cf(t)\Delta t = c\int_a^b f(t)\Delta t$;
(ii) $f + g$ *is* Δ*-integrable and* $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$.

Theorem 1.13 (see [112, 113]) *Let* f *and* g *be* Δ*-integrable functions on* $[a, b)_{\mathbb{T}}$*. Then their product* fg *is* Δ*-integrable on* $[a, b)_{\mathbb{T}}$*.*

Proof We claim that if f is Δ-integrable on $[a, b)_{\mathbb{T}}$, then f^2 is Δ-integrable on $[a, b)_{\mathbb{T}}$. In fact, $f^2(t) = \varphi(f(t))$ with $\varphi(x) = x^2$, and φ satisfies the Lipschitz condition on any finite interval $[m, M]$. Therefore f^2 is integrable by Theorem 1.9. Now the desired result follows from the identity $4fg = (f + g)^2 - (f - g)^2$ by Theorem 1.12. This completes the proof. □

Following the proof of the case $\mathbb{T} = \mathbb{R}$, the following theorem is immediate:

Theorem 1.14 (see [112, 113]) *Let* f *be a function defined on* $[a, b)_{\mathbb{T}}$ *and let* $c \in \mathbb{T}$ *with* $a < c < b$*. If* f *is* Δ*-integrable from* a *to* c *and from* c *to* b*, then* f *is* Δ*-integrable from* a *to* b *and*

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$$

Theorem 1.15 (see [112, 113]) *If* f *and* g *are* Δ*-integrable on* $[a, b)_{\mathbb{T}}$ *and if* $f(t) \leqslant g(t)$ *for all* $t \in [a, b)_{\mathbb{T}}$*, then* $\int_a^b f(t)\Delta t \leqslant \int_a^b g(t)\Delta t$*.*

Proof By Theorem 1.12, $h = g - f$ is Δ-integrable on $[a, b)_{\mathbb{T}}$. Since $h(t) \geqslant 0$ for all $t \in [a, b)_{\mathbb{T}}$, it is clear that $L(h, P) \geqslant 0$ for all partitions P of $[a, b)_{\mathbb{T}}$ and so $\int_a^b h(t)\Delta t = L(h) \geqslant 0$. Applying Theorem 1.12 again, we see that $\int_a^b g(t)\Delta t - \int_a^b f(t)\Delta t = \int_a^b h(t)\Delta t \geqslant 0$. The proof is completed. □

Theorem 1.16 (see [112, 113]) *If f is Δ-integrable on $[a, b)_{\mathbb{T}}$, then $|f|$ is Δ-integrable on $[a, b)_{\mathbb{T}}$ and*

$$\left| \int_a^b f(t)\Delta t \right| \leqslant \int_a^b |f(t)|\Delta t. \tag{1.9}$$

Proof The result easily follows from Theorem 1.15 provided we know $|f|$ is Δ-integrable on $[a, b)_{\mathbb{T}}$. In fact, $-|f| \leqslant f \leqslant |f|$ and so $-\int_a^b |f(t)|\Delta t \leqslant \int_a^b f(t)\Delta t \leqslant \int_a^b |f(t)|\Delta t$, which implies (1.9). We now prove that $|f|$ is Δ-integrable. Consider the function $\varphi(x) = |x|$. This function satisfies a Lipschitz condition on any interval. Further, we have $|f(t)| = \varphi(f(t))$. Therefore, $|f|$ is Δ-integrable by Theorem 1.9. This completes the proof. □

Corollary 1.2 (see [112, 113]) *Let f and g be Δ-integrable on $[a, b)_{\mathbb{T}}$. Then*

$$\left| \int_a^b f(t)g(t)\Delta t \right| \leqslant \int_a^b |f(t)g(t)|\Delta t \leqslant \Big(\sup_{t\in[a,b)_{\mathbb{T}}} |f(t)|\Big) \int_a^b |g(t)|\Delta t.$$

For the case $\mathbb{T} = \mathbb{R}$, the following theorem is immediate:

Theorem 1.17 *Let (f_k) be a sequence of Δ-integrable functions on $[a, b)_{\mathbb{T}}$, and suppose that $f_k \to f$ uniformly on $[a, b)_{\mathbb{T}}$ for a function f defined on $[a, b)_{\mathbb{T}}$. Then f is Δ-integrable from a to b and $\int_a^b f(t)\Delta t = \lim\limits_{k\to\infty} \int_a^b f_k(t)\Delta t$.*

The statement for the series of functions of Theorem 1.17 can be stated as follows:

Theorem 1.18 (see [112, 113]) *Suppose that $\sum\limits_{k=1}^{\infty} g_k$ is a series of Δ-integrable functions g_k on $[a, b)_{\mathbb{T}}$ that converges uniformly to g on $[a, b)_{\mathbb{T}}$. Then g is Δ-integrable and $\int_a^b g(t)\Delta t = \sum\limits_{k=1}^{\infty} \int_a^b g_k(t)\Delta t$.*

1.1.3 Fundamental Theorems of Calculus

In this subsection, we will present two versions of the fundamental theorem of calculus. Before this, some basic definitions concerning the time scales and some mean value theorems for derivatives on time scales will be presented.

Theorem 1.19 (see [112, 113]) *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is Δ-differentiable on $[a,b)_{\mathbb{T}}$ (the differentiability at a is understood as right-sided) and satisfies $f(a) = f(b)$. Then there exist $\xi, \tau \in [a,b)_{\mathbb{T}}$ such that $f^{\Delta}(\tau) \le 0 \le f^{\Delta}(\xi)$.*

Proof Since the function f is continuous on the compact set $[a,b]_{\mathbb{T}}$, f assumes its minimum m and its maximum M. Therefore there exist $\xi, \tau \in [a,b]_{\mathbb{T}}$ such that $m = f(\xi)$ and $M = f(\tau)$. Since $f(a) = f(b)$, we may assume that $\xi, \tau \in [a,b)_{\mathbb{T}}$. Hence $f^{\Delta}(\tau) \leqslant 0$ and $f^{\Delta}(\xi) \geqslant 0$. This completes proof. □

Theorem 1.20 (see [112, 113], Mean Value Theorem) *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ which is Δ-differentiable on $[a,b)_{\mathbb{T}}$. Then there exist $\xi, \tau \in [a,b)_{\mathbb{T}}$ such that*

$$f^{\Delta}(\tau) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f^{\Delta}(\xi).$$

Proof By applying Theorem 1.19 to the function

$$\varphi(t) = f(t) - f(a) - \frac{f(b) - f(a)}{b - a}(t - a),$$

we can obtain the desired result immediately. □

Corollary 1.3 (see [112, 113]) *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is Δ-differentiable on $[a,b)_{\mathbb{T}}$. If $f^{\Delta}(t) = 0$ for all $t \in [a,b)_{\mathbb{T}}$, then f is a constant function on $[a,b]_{\mathbb{T}}$.*

Corollary 1.4 (see [112, 113]) *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is Δ-differentiable on $[a,b)_{\mathbb{T}}$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a,b]_{\mathbb{T}}$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geqslant 0$, and $f^{\Delta}(t) \leqslant 0$ for all $t \in [a,b)_{\mathbb{T}}$, respectively.*

In the similar way, the analogue of ∇-derivative of Theorem 1.20 can be obtained.

Theorem 1.21 (see [112, 113]) *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is ∇ differentiable on $(a,b]_{\mathbb{T}}$. Then there exist $\xi, \tau \in (a,b]_{\mathbb{T}}$ such that*

$$f^{\nabla}(\tau) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f^{\nabla}(\xi).$$

For Δ-predifferentiable functions, there is a generalization of Theorem 1.20 as follows:

Definition 1.8 (see [112, 113]) A function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-predifferentiable (with region of Δ-differentiation D), provided that the following conditions hold:

(i) f is continuous on $\mathbb{T}$;
(ii) $D \subset \mathbb{T}^k$, $\mathbb{T}^k - D$ is countable and contains no right-scattered elements of $\mathbb{T}$;
(iii) f is Δ-differentiable at each $t \in D$.

Theorem 1.22 (see [112, 113]) *Let f and g be real-valued functions defined on $\mathbb{T}$. Suppose both f and g are Δ-predifferentiable with region of Δ-differentiation D. Then $|f^{\Delta}(t)| \leqslant g^{\Delta}(t)$ for all $t \in D$ implies $|f(r) - f(s)| \leqslant g(r) - g(s)$ for all $r, s \in \mathbb{T}, r \geqslant s$.*

In Theorem 1.22 by letting $f = 0$, we have the following:

Corollary 1.5 (see [112, 113]) *Let $g : \mathbb{T} \to \mathbb{R}$ be a Δ-predifferentiable function with region of Δ-differentiation D. If $g^{\Delta}(t) \geqslant 0$ for all $t \in D$, then g is nondecreasing on $\mathbb{T}$.*

Theorem 1.23 (see [112, 113]) *Let f be a continuous function on $[a, b]_{\mathbb{T}} \subset \mathbb{T}$ that is Δ-predifferentiable on $[a, b)_{\mathbb{T}}$ with region of Δ-differentiation $D \subset [a, b)_{\mathbb{T}}$. Suppose $f(a) = f(b)$. Then there exist $\xi, \tau \in D$ such that $f^{\Delta}(\tau) \leqslant 0 \leqslant f^{\Delta}(\xi)$.*

Proof If f is a constant function, then $f^{\Delta}(t) = 0$ for all $t \in [a, b)_{\mathbb{T}}$ and, therefore, the theorem holds in this case. Now suppose that f is not constant. To prove that there exists $\tau \in D$ such that $f^{\Delta}(\tau) \leqslant 0$, we suppose the contrary: let $f^{\Delta}(t) > 0$ for all $t \in D$. Applying Corollary 1.5 to the function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$, we get that f is nondecreasing on $[a, b]_{\mathbb{T}}$. But this gives a contradiction, since $f(a) = f(b)$ and f is nonconstant. Therefore the desired point $\tau \in D$ exists. Similarly, considering the function $-f$, we can prove that there exists $\xi \in D$ such that $f^{\Delta}(\xi) \geqslant 0$. This completes the proof. □

Based on Theorem 1.23, similar to the proof of Theorem 1.20, the following generalization is immediate:

Theorem 1.24 (see [112, 113]) *Let f be a continuous function on $[a, b]_{\mathbb{T}} \subset \mathbb{T}$ that is Δ-predifferentiable on $[a, b)_{\mathbb{T}}$ with region of Δ-differentiation $D \subset [a, b)_{\mathbb{T}}$. Then there exist $\xi, \tau \in D$ such that $(b - a)f^{\Delta}(\tau) \leqslant f(b) - f(a) \leqslant (b - a)f^{\Delta}(\xi)$.*

Let $[a, b]_{\mathbb{T}}$ be a closed bounded interval in $\mathbb{T}$. A function $F : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is called a Δ-antiderivative of $f : [a, b)_{\mathbb{T}} \to \mathbb{R}$ provided F is continuous on $[a, b]_{\mathbb{T}}$ and Δ-differentiable on $[a, b)_{\mathbb{T}}$, and $F^{\Delta}(t) = f(t)$ for all $t \in [a, b)_{\mathbb{T}}$.

Theorem 1.25 (see [112, 113]) *(Fundamental theorem of calculus I). Let f be Δ-integrable function on $[a, b)_{\mathbb{T}}$. If f has a Δ-antiderivative $F : [a, b]_{\mathbb{T}} \to \mathbb{R}$, then*

$$\int_a^b f(t)\Delta t = F(b) - F(a). \tag{1.10}$$

Proof Let $\varepsilon > 0$. By Theorem 1.3, there exists a partition $P : a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b)_{\mathbb{T}}$ such that

$$U(f, P) - L(f, P) < \varepsilon. \tag{1.11}$$

Applying Theorem 1.20 to $F : [t_{i-1}, t_i]_{\mathbb{T}} \to \mathbb{R}$ for each $i = 1, 2, \ldots, n$, we obtain $\xi_i, \tau_i \in [t_{i-1}, t_i)_{\mathbb{T}}$ such that $(t_i - t_{i-1}) f(\tau_i) \leqslant F(t_i) - F(t_{i-1}) \leqslant (t_i - t_{i-1}) f(\xi_i)$. Hence summing we have $\sum_{i=1}^{n} (t_i - t_{i-1}) f(\tau_i) \leqslant F(b) - F(a) \leqslant \sum_{i=1}^{n} (t_i - t_{i-1}) f(\xi_i)$. So the estimate

$$L(f, P) \leqslant F(b) - F(a) \leqslant U(f, P) \tag{1.12}$$

follows. Since we have $L(f, P) \leqslant \int_a^b f(t)\Delta t \leqslant U(f, P)$ for all partitions P of $[a, b)_{\mathbb{T}}$, inequalities (1.11) and (1.12) imply that $|\int_a^b f(t)\Delta t - [F(b) - F(a)]| < \varepsilon$. Since ε is arbitrary, (1.10) holds. This completes the proof. □

Theorem 1.26 ([112, 113], Integration by Parts) *Let μ and ν be continuous functions on $[a, b]_{\mathbb{T}}$ that are Δ-differentiable on $[a, b)_{\mathbb{T}}$. If u^{Δ} and v^{Δ} are integrable from a to b, then*

$$\int_a^b u^{\Delta}(t) v^{\Delta}(t)\Delta t + \int_a^b u(\sigma(t)) v^{\Delta}(t)\Delta t = u(b)v(a). \tag{1.13}$$

Proof Let $F = uv$; then $F^{\Delta}(t) = u^{\Delta}(t)v(t) + u(\sigma(t))v^{\Delta}(t)$ and F^{Δ} is Δ-integrable. Now Theorem 1.25 shows that $\int_a^b F^{\Delta}(t)\Delta t = F(b) - F(a) = u(b)v(b) - u(a)v(a)$ and so (1.13) holds. The proof is completed. □

Theorem 1.27 (see [112, 113], Fundamental Theorem of Calculus II) *Let f be a function which is Δ-integrable from a to b. For $t \in [a, b]_{\mathbb{T}}$, let $F(t) = \int_a^t f(s)\Delta s$. Then F is continuous on $[a, b]_{\mathbb{T}}$. Further, let $t_0 \in [a, b)_{\mathbb{T}}$ and let f be arbitrary at t_0 if t_0 is right-scattered, and let f be continuous at t_0 if t_0 is right-dense. Then F is Δ-differentiable at t_0 and $F^{\Delta}(t_0) = f(t_0)$.*

Proof Choose $B > 0$ such that $|f(t)| \leqslant B$ for all $t \in [a, b)_{\mathbb{T}}$. If $t, \tau \in [a, b]_{\mathbb{T}}$ and $|t - \tau| < \varepsilon/B$ where $t < \tau$, say, then

$$|F(\tau) - F(t)| = \left| \int_t^{\tau} f(s)\Delta s \right| \leqslant \int_t^{\tau} |f(s)|\Delta s \leqslant \int_t^{\tau} B\Delta s = B(\tau - t) < \varepsilon.$$

This shows that F is (uniformly) continuous on $[a, b]_{\mathbb{T}}$. Let $t_0 \in [a, b)_{\mathbb{T}}$ be right-scattered. Then, since F is continuous, it is Δ-differentiable at t_0 and we have by Theorems 1.14 and 1.7,

$$F^{\Delta}(t_0) = \lim_{t \to t_0} \frac{F(t) - F(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{1}{t - t_0} \left[\int_a^t f(s)\Delta s - \int_a^{t_0} f(s)\Delta s \right]$$

$$= \lim_{t \to t_0} \frac{1}{t - t_0} \int_{t_0}^{t} f(s) \Delta s.$$

Let $\varepsilon > 0$. Since f is continuous at t_0, there exists $\delta > 0$ such that $s \in [a, b)_{\mathbb{T}}$ and $|s - t_0| < \delta$ imply $|f(s) - f(t_0)| < \varepsilon$. Then

$$\begin{aligned} \left| \frac{1}{t - t_0} \int_{t_0}^{t} f(s)\Delta s - f(t_0) \right| &= \left| \frac{1}{t - t_0} \int_{t_0}^{t} [f(s) - f(t_0)] \Delta s \right| \\ &\leqslant \frac{1}{|t - t_0|} \int_{t_0}^{t} |f(s) - f(t_0)| \Delta s \\ &\leqslant \frac{\varepsilon}{|t - t_0|} \left| \int_{t_0}^{t} \Delta s \right| = \varepsilon \end{aligned}$$

for all $t \in [a, b]_{\mathbb{T}}$ such that $|t - t_0| < \delta$ and $t = t_0$. Hence the desired result follows. □

A function $F : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is said to be a Δ-preantiderivative of $f : [a, b)_{\mathbb{T}} \to \mathbb{R}$ provided F is continuous on $[a, b]_{\mathbb{T}}$ and Δ-predifferentiable on $[a, b)_{\mathbb{T}}$ with region of Δ-differentiation $D \subset [a, b)_{\mathbb{T}}$, and $F^{\Delta}(t) = f(t)$ for all $t \in D$.

The following result is a generalization of Theorem 1.25:

Theorem 1.28 *Let f be a Δ-integrable function on $[a, b)_{\mathbb{T}}$. If f has a Δ-preantiderivative $F : [a, b]_{\mathbb{T}} \to \mathbb{R}$, then $\int_a^b f(t)\Delta t = F(b) - F(a)$.*

Proof The proof is analogous to that of Theorem 1.25 and uses Theorem 1.24. □

Let $\mathbb{T}$ be a time scale and σ and ρ be the forward and backward jump functions on $\mathbb{T}$. The following knowledge is about Δ-measure on time scales and can be found in [69].

Let $\mathfrak{J}_1$ be the family (collection) of all left closed and right open intervals of $\mathbb{T}$ of the form $[a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t < b\}$ with $a, b \in \mathbb{T}$ and $a \leqslant b$. The interval $[a, a)_{\mathbb{T}}$ is understood as the empty set. $\mathfrak{J}_1$ is a semiring of subsets of $\mathbb{T}$. Let $m_1 : \mathfrak{J}_1 \to [0, \infty)$ be the set function on $\mathfrak{J}_1$ (whose values belong to the extended real half-line $[0, \infty)$) that assigns to each interval $[a, b)_{\mathbb{T}}$ its length $b - a$: $m_1([a, b)_{\mathbb{T}}) = b - a$. Then m_1 is a countably additive measure on $\mathfrak{J}_1$. We denote by μ_{Δ} the Carathéodory extension of the set function m_1 associated with family $\mathfrak{J}_1$ (for the Carathéodory extension, see [39]) and call μ_{Δ} the Lebesgue Δ-measure on $\mathbb{T}$.

Now we briefly describe the Carathéodory extension μ_{Δ} of m_1. First, using the pair $(\mathfrak{J}_1, m_1)$, an outer measure m_1^* is generated on the family of all subsets of $\mathbb{T}$ as follows:

Let E be any subset of $\mathbb{T}$. If there exists at least one finite or countable system of intervals $V_j \in \mathfrak{J}_1 (j = 1, 2, \ldots)$ such that $E \subset \bigcup_j V_j$, then we put $m_1^*(E) = \inf \sum\limits_j m_1(V_j)$, where the infimum is taken over all coverings of E by a finite or