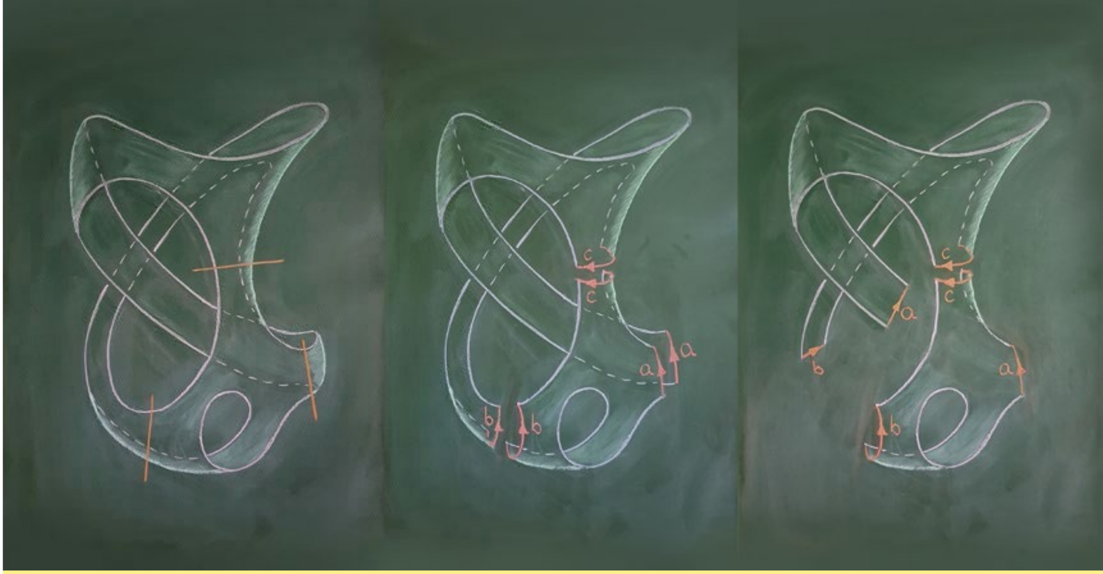

Ton Marar

A Ludic Journey into Geometric Topology

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To Agnaldo Aricê Caldas Farias, teacher

Foreword

In the history of Western thought, geometry goes hand in hand with philosophy. From a handful of techniques invented, according to Herodotus, around 1300 BC by the land surveyors of the Nile valley, over centuries it became the bridge between the world of ideas and the world of things. As a whole, this discipline is multifaceted, uncovering a spectrum of theories, all of which allude to the need to represent and study physical space in its most variegated aspects.

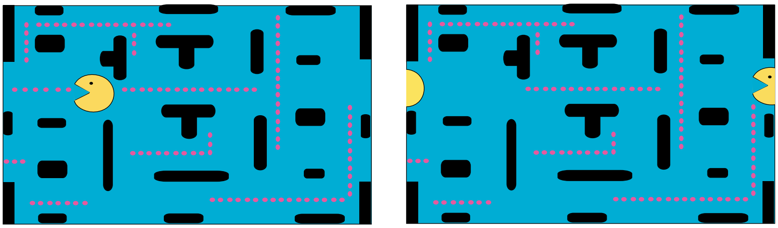
Ton Marar, the author of this precious text, leads us by the hand through the intricate paths of this ancient science, which is nowadays indispensable for understanding our universe. In these modern mirabilia, the inquisitive but curious reader will find many gems, such as the classification of Platonic solids and that of surfaces, the concept of orientability, and many other ideas and suggestions for future journeys into fascinating but certainly impervious territory.

São Carlos, Brazil
February 2022

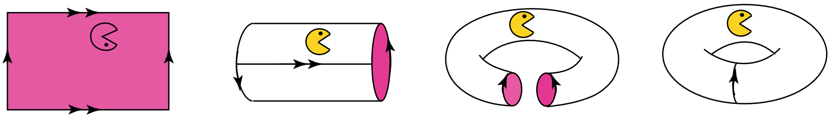
Igor Mencattini

Preface

In the famous video game of the 1980s, the character Pacman moves on a rectangular screen in two perpendicular directions, down or up and left or right. Pacman's universe is two-dimensional, and he does not know what it is like to go forward or backward from the screen. Furthermore, when Pacman crosses the left edge of the screen, he appears at the same height on the right edge. Similarly, it happens when he crosses the horizontal edges.



Pacman's two-dimensional world is immersed in a surface without a boundary, an endless surface shaped like a donut. This surface is called a torus.



How would it be possible for Pacman to understand the toric shape of his world? For him, the third dimension is an esoteric space. He cannot get out of his two-dimensional world and enjoy it in three-dimensional space, like we do. Pacman is confined to his two-dimensional world and the only chance for him to understand the shape of his world is by deduction, an intellectual activity beyond physical sensation.

Our situation is not very different. We know little about the shape of the universe in which we live. Is our universe infinite? If it is finite, it must have no boundary; otherwise, what would be beyond the boundary?

There is no chance of leaving this universe to perceive its shape, as we did with the world of Pacman. We will have to deduce it.

To solve this great mystery, mathematics, or more specifically geometry, is fundamental.

It was easy to determine the shape of Pacman's universe. We just needed to gather some physical information from that world, which implied identifying the vertical edges and the horizontal edges of the rectangle where Pacman lives his life. Then, through an extra dimension, we made the identifications and finally saw the shape of Pacman's two-dimensional world without a boundary.

By analogy, from the three-dimensionality of our universe, it would take at least one extra dimension to be able to *see* its shape: a fourth dimension.

In this book, we are going to show the reader how to develop some sensitivity to see certain three-dimensional objects without boundary, which we call hypersurfaces.

Is our universe a hypersurface? Astrophysicists have the task of describing the cosmos geometrically, and some of them believe that the universe is modeled by a hypersurface, as shown in the article by JP Luminet et al. in Nature 425 (2003), *Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background.*

If one day the hypothesis is confirmed that our universe is in fact a hypersurface and we have a list of all possible shapes of three-dimensional objects, then with some physical information we can finally deduce the shape of the universe, who knows?

In Chapter 1, we deal with mathematical models, which are allegories through which abstract mathematics can be used in interpreting phenomena and solving problems.

Chapter 2 describes how Platonic and Keplerian theories seek to explain the cosmos with a mixture of mathematics and faith.

A brief account of geometries from Felix Klein's point of view is found in Chapter 3.

The next chapter contains an introduction to topology as a kind of geometry, and we present the classification of finite-sized, borderless two-dimensional objects called closed surfaces. These surfaces are divided into two classes, namely orientable and non-orientable. The first ones have two sides, they define an interior and an exterior, while the others, the non-orientable ones, all contain a Möbius strip, which is a truly one-sided surface.

There are several areas of knowledge that make use of the topological classification of closed surfaces. On October 3, 2016, the Nobel Prize in Physics was awarded to a trio of British materials experts. In their work entitled *Topological phase transitions and topological phases of matter,* certain exotic states of matter (beyond solid, liquid and gas) that occur at extreme temperatures are described. In this research, the authors used the topological classification of closed surfaces, and the practical result is a series of new superconducting materials.

From the classification of closed surfaces, we can see the difficulty of obtaining a topological classification of hypersurfaces. These geometric objects live in high-dimensional spaces, at least four, and this is why the fourth dimension is introduced in Chapter 5.

Three-dimensional models of non-orientable closed surfaces are covered in Chapter 6, and the final chapter contains some hypersurface models.

I would like to thank my various students from the mathematics for architecture discipline in the architecture and urbanism course at IAU-USP, São Carlos, Brazil, who for decades motivated me to write this text. Thanks are also due to many of my colleagues, among them Carlos Martins, David Sperling, and Marcelo Suzuki from IAU-USP; Tiago Pereira, Igor Mencattini, Farid Tari, and Ali Tahzibi from ICMC-USP; Flávio Coelho and Paolo Piccione from IME-USP; Stefano Luzzatto from ICTP, Trieste; and Robinson dos Santos and Martin Peters from Springer. The presentation of several chapters improved a great deal after Alessandra Pavesi carefully read the book, to whom I am eternally grateful.

São Carlos, Brazil — Ton Marar
February 2022

Contents

Chapter 1
Mathematical Models

Contrary to the belief that mathematics is a collection of problem-solving techniques, historian and mathematician Morris Kline (1908–1992) provides the following description: *Mathematics is more than a method, an art, and a language. It is a body of knowledge with content that serves the physical and social scientist, the philosopher, the logician, and the artist; content that influences the doctrines of statesmen and theologians; content that satisfies the curiosity of the man who surveys the heavens and the man who muses on the sweetness of musical sounds; and content that has undeniably, if sometimes imperceptibly, shaped the course of modern history* [2, p. 9].

This body of knowledge has been under construction for millennia. Mathematicians participate in this process mainly driven by curiosity, like a climber who climbs a mountain because it is there, although practical questions have also motivated the theoretical development of mathematics from the beginning.

The activity of researchers in mathematics is purely intellectual and, in principle, it is not related to our physical world, as everything happens in the perfect world of ideas. Their conclusions, in the form of theorems, are defended with logical arguments to convince non-believers. The set of arguments is called the proof of the theorem. Often a proof does not stop the curiosity of finding a new one, for the same result, more objective, more beautiful. Beauty that can be quantified by the minimality of arguments.

How can it be that mathematics, being after all a product of thought, independent of experience, is so admirably adapted to the objects of reality? [1] asked the perplexed Albert Einstein (1879–1955).

Using mathematics in everyday situations takes place mainly through the so-called mathematical models, that is, allegories that adapt the real problem to the world of ideas, creating the possibility of dealing with problems scientifically.

Models have accompanied us since childhood and some even entertain us. A broomstick once played the role of a horse. In this case, we have a physical model, which is not very close to a real horse.

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