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# Preface

My purpose in putting this book together is to make more people aware of the work of A. P. Morse in regard to his development of a formal language for writing mathematics, his application of that language in set theory and mathematical analysis, and his unique perspective on mathematics. This text is primarily a combination of Morse's book *A Theory of Sets, Second Edition* (1986) and notes for a course on analysis from the early 1950's. It has been supplemented and updated with subsequent course notes and publications of Morse.

Morse provided very little in the way of explanation in his written work. I believe that he thought the careful statements in formal or nearly formal language should speak for themselves. This creates a formidable challenge for those who would try to read his work. As an aid and guide to readers, I have tried to provide some commentary and insight in the Editor's Introduction.

This book is based on both previously published material and unpublished notes, and therefore would not be possible without several permissions. I am grateful to Elsevier Publications for granting me permission to republish the material, slightly altered, from *A Theory of Sets, Second Edition* (1986). Likewise I am grateful to the American Mathematical Society for allowing me to reproduce material from *Web Derivatives* (1973). I am especially thankful for the permission and support shown by the families of Tony Morse and Hewitt Kenyon, in particular Peter Morse and Linda and Emily Kenyon. I am grateful to Robert Arnold for allowing me to use material from his Ph.D. thesis *Plus and Times* (1969).

I wish to thank Douglas Bridges for suggesting the comparison of Morse's integral to the Henstock-Kurzweil integral.

In 1966, Bob Neveln, my friend since junior high school, had taken a year's leave of absence from his undergraduate studies at Caltech to spend a year at Northwestern University where I was attending school. During that period he bought *A Theory of Sets* (1965) and showed it to me. We began studying Morse's work then, and have never looked back. Bob has encouraged me during the years I worked on this project, and has used a parser he developed to check the correctness of the formal syntax.

My wife, Marianna, has not only encouraged me during this period, but kept my life together.

December 15, 2021
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# Editor's Introduction

## I. INTRODUCTORY REMARKS

### Biographical Sketch

A. P. Morse was born in 1911. He completed a Ph.D. under C. R. Adams at Brown in 1937. During a two-year stay at the Institute for Advanced Study he proved what has come to be known as the Morse-Sard Theorem. The following is from his paper† on the subject.

"...consider the following statement.

*If* $m \geq 0, n \geq 1$, *R is an open subset of* $E_n$, *and f is a function on R to* $E_1$ *of class* $C^m$ *then f transforms is critical set into a set of (linear) measure zero.*

H. Whitney has shown the statement false if $m = n - 1 \geq 1$; M. Morse and A. Sard, in an unpublished paper, have shown, on the other hand, that it is true providing $m$ is the greatest integer in $n + (n - 3)^2/16$. This last result contains as a special case the fact that the statement is true for $m = n = 1, 2, 3, 4, 5, 6$. Thus it is natural to ask: Is the statement true in case $m = n$? Theorem 4.3 of the present paper answers this question in the affirmative."

In 1939 he joined the faculty at Berkeley where he came into contact with Tarski and began to investigate ways to write mathematics carefully. He developed a formal language and a unique system of logic and set theory, and he used these foundations to give a formal treatment of the usual number systems, metric spaces, measures, measure integration, and the derivatives of set functions. These formal treatments were presented in the classes that he taught.

In 1965 Academic Press published *A Theory of Sets* which included Morse's work on formal language, logic and set theory. A second edition was published in 1986.

Morse's Ph.D. students included Trevor McMinn, Herbert Federer, Woodrow Bledsoe, Hewitt Kenyon, David Peterson, and Robert Arnold. Morse's publications, listed in Appendix D, include articles on differentiation theory, measure theory, and other topics in analysis. These publications make it clear that Morse was first and foremost an analyst. His work on foundations was necessary because he wanted to express theorems of analysis with formality and lack of ambiguity.

Morse retired from Berkeley in 1972, and died in 1984 before the second edition of his book had been completed.

### Background for This Text

By the early 1950s Morse had prepared typewritten chapters covering the topics listed above. It appears that these chapters (which will be referred to as "the Notes") were intended for book format, but, as far as I know, there was never an attempt to publish them. They were used by Morse as the basis for course 210A-B which he taught at Berkeley in the 1951-52 academic year. The course was ostensibly about analysis, but the Notes begin with a description of a formal mathematical language, followed by an axiomatic development of logic and set theory within the formal language. The set theory is used to define the higher level mathematical concepts. Throughout the development, each definition and theorem is expressed in his formal language. A formal expression of higher mathematics had not been done before, not by Frege, not by Russell and Whitehead, not by Hilbert, not by Bourbaki‡, not by anyone. No one saw it as a feasible task. It was believed that any such undertaking would necessitate texts that would be tediously difficult to write and even more difficult to read. By incorporating definitions into the formal language, Morse was able to express

† *The behavior of a function on its critical set*, Annals of Mathematics, Vol. 40, No. 1, January 1939.

‡ "But formalized mathematics cannot in practice be written down in full, and therefore we must have confidence in what might be called the common sense of the mathematician: a confidence analogous to that accorded by a calculator or an engineer to a formula or a numerical table without any awareness of the existence of Peano's axioms, and which ultimately is based on the knowledge that it has never been contradicted by facts.", N. Bourbaki, *Elements of Mathematics, Theory of Sets*, Addison-Wesley, 1968, p. 11.

high level concepts succinctly while maintaining strict syntactic formality. The syntactical flexibility of his formal language meant that he was able to emulate many conventional mathematical notations, which greatly enhances the readability of his formal texts.

Morse's adherence to a formal language for mathematical expression was born from a desire to eliminate ambiguity. In some sense, trying to explain the formal expressions using everyday English only risks introducing the ambiguity he was trying to avoid. But there is another aspect to the carefully worked out formalism that is not readily apparent. It is that the precise meaning and correctness of some important theorems may depend on fine points in the definitions leading up to such theorems. For example, certain theorems about sums of limits, sums of sums, and sums of integrals depend on a fine point regarding the topology of the complex plane with plus and minus infinity appended. There is a level of mathematical scrutiny here is that not found in other texts.

Many of Morse's publications dealt with the topics in the Notes, and up to his death he worked on developing new insights and approaches, generalizing results, and polishing earlier writings. This text is an attempt to update the Notes to bring them to a uniform level of formality and to take into account subsequent writings by Morse. This text is completely formal in the sense that every axiom, definition, and theorem is stated in the formal language, adhering strictly to the rules of syntax for that language. As such each of those statements can be read and parsed using a computer program.

There are two additional senses of complete formality that are not found in this text. The first is having completely formal proofs. Based on the first edition of Morse's book, this would require that each step in a proof could be validated by the use of one of the rules of inference, resulting in granular proofs of great length. Morse gave a formal proof to only the first theorem of logic in his book. In the second edition of his book, Morse included a few derived rules of inference which would go a small way toward shortening such long proofs. Bob Neveln has greatly enlarged on this idea by using over 1000 derived rules of inference, and most notably having derived rules based on a deduction theorem, to check formal proofs with a computer program. In any case, the proofs in this text are essentially as they were written down by Morse, and therefore are not completely formal. However every statement that occurs in each proof is completely formal with regard to syntax.

The second sense of formality not found in this text is in regard to the metamathematical developments in Chapter 0. Here the notions of symbol, expression, variable, constant, formula, and theorem are explained. While the language used is carefully crafted and has an appearance of semi-formality, it is not formal. It seems fair to say that Morse thought of the metamathematics as prior to and distinct from the mathematical developments rather than as a part of the mathematical development.

There are three reasons why I have undertaken this task. The first is that, in the history of formal mathematics, this is a unique achievement that is filled with mathematical nuance and insight. The list below highlights a few of Morse's unique approaches.

The second reason is that it is hard to fully appreciate Morse's formal language without seeing it used in a variety of contexts. While the book describes the formal language and uses it for logic and set theory, it is not evident how to apply the formality to higher mathematics.

The third reason is that I believe Morse wanted this material to be published. During the last two years of his life, we spoke on the phone on a daily basis. In our many conversations, he never talked about the Notes, much less his desire to update or publish them. However, on page 46 of the second edition of his book, Morse added the following comment about his Theory of Notation: "It will become increasingly important to *correctly* understand 0.37 especially if our approach here to Set Theory is to be carried forward into aspects of Elementary Analysis, Metrics, Measure, Linear Measure and Total Variation, Integration, Covering and Differentiation." (cf. Remark following 1.4.16 of this text.) This quote contains a listing of chapter titles from the Notes, and I find it convincing evidence that, up to the time of his death, Morse was thinking about these topics being presented as a whole.

Reading any technical work presents challenges, including accustoming one's self to an author's choice of notation. This text is especially demanding in this regard because Morse's notation always has very precise meaning and because there is a great deal of it. Readers who will appreciate this precision will likely recognize themselves immediately. Trevor J. McMinn made the following observation.

"Readers at various levels of mathematical education may well profit from this elegantly handled and bold-spirited enterprise. A scholar who already largely understands the author's objectives is still apt to discover much that is original and ingenious in his way of attaining them. A student perplexed by fundamental questions stands a good chance of finding them answered. One less interested in the foundations of set theory then in its superstructure should be amply rewarded for the effort spent in learning necessary preliminaries by the impetus given to his understanding of the subsequent beautiful edifice."†

**Unusual and Unique Features**

The following is a short list of unusual or unique features of Morse's work.

**0. Numbering.** Morse's numbering of chapters and theorems starting with zero instead of one. Zero is the most important entity in his mathematical universe. Zero is the first significant thing that is defined and begins its life as the prototypical false statement in the logic chapter. In the set theory chapter, zero takes on the role of the empty set and, later in the chapter, as the first natural number, the first ordinal, and the first cardinal. In Chapter 3, this same zero becomes the zero of the set of real numbers and the zero of the set of complex numbers.

**1. Definitions and complete formality.** As stated earlier, no other mathematical text achieves the level of formality of Morse's book. Strict syntactic formality is made possible by the inclusion of definitions as part of the formal language. Morse worked out a set of syntactic rules governing the construction of definitions.

**2. The unification of logic and set theory.** While logic is ostensibly about statements and set theory is ostensibly about objects (sets), Morse maintained no distinction between these. Syntactically, this means that every well-formed expression is both a term and a formula. Semantically, it means that every well-formed expression both expresses a statement and refers to a set. The unification simplifies the syntactical development and leads to a rather striking notion of truth given at the beginning of the set theory chapter.

**3. Schemators and schematic expressions.** Morse's schematic expressions are essentially predicates and functionals. Their presence produces something approaching second-order logic, without the ability to quantify over these expressions. Standard treatments of first-order predicate logic require axiom schemes and theorem schemes. Using schematic expressions, these are simply axioms and theorems, respectively. Schemators and schematic expressions are the basis for all bound variable notations such as quantifiers, limits, sums, and integrals.

**4. Theory of Notation.** The Theory of Notation is a specific application of the syntactic rules governing definitions and consists of a collection of definitional schemes designed to achieve two primary goals. The first is to minimize the need for parentheses in expressions involving binarians (infix operators). For example the Theory of Notation allows '$(p \to q \wedge r)$' in place of '$(p \to (q \wedge r))$'. The second is to provide useful variations of expressions involving bound variables. For example, given that '$\bigwedge x \underline{\mathrm{u}} x$' means for each $x$, $\underline{\mathrm{u}} x$ is true, there is automatically a form '$\bigwedge x \in A \underline{\mathrm{u}} x$' which means for each $x$ in $A$, $\underline{\mathrm{u}} x$ is true.

**5. Sets and points.** Morse used the word "set" to refer any set or class and used the word "point" to refer to a set that is a member of the universe. In this introduction, the word "set" will be used in Morse's general sense, reserving the word "point" for times when a distinction is necessary.

**6. Ordered pairs.** Morse and David C. Peterson constructed an ordered pair with the property

$$((a, b) = (c, d) \leftrightarrow a = c \wedge b = d),$$

regardless of whether or not $a$, $b$, $c$, or $d$ are points. As a result, Morse was able to use ordered pairs to enhance the Theory of Notation.

**7. Function notation.** In place of the customary '$f(x)$' to express the value of the function $f$ at $x$, Morse used '$\mathbf{.}\, f x$'. The customary notation violates Morse's rules of syntax because it starts a form with a

† A. P. Morse, *A Theory of Sets, Second Edition*, Academic Press (1986), p. xii.

variable and because it uses a left parenthesis in a form as something other than the first symbol. Morse noted that his notation has the advantage of being less cumbersome than the customary notation.

**8. Directed infinities.** For reasons to be explained later, Morse included an infinite number in every direction of the complex plane, even though, with the exception of plus and minus infinity, these numbers are not used for limits, sums, or integrals.

**9. Plus and times.** Robert Arnold, in his Ph.D. dissertation† under Morse's supervision, developed a generalized notion of plus and times that applies to number-valued functions and certain sets of numbers, and iterating upward to higher functions and sets.

**10. Runs and limits.** Morse, along with Hewitt Kenyon, developed a generalization of a directed set called a run. In this text, runs are the basis for all limits.

**11. Metrics and measures.** Morse's concept of metric is what most authors call a pseudometric. This is part of a pattern to use the simpler terminology for the more general case. Morse's concept of measure is what most authors refer to as an outer measure. That is, under Morse's concept, a measure is an extended-real-valued function defined on the set of all subsets of a given point.

**12. Sums and products.** Morse's primary notion for sums and products is unordered. He defined an ordered sum, but it is never used in this text.

**13. Integration.** Morse's approach to integration begins with an abstract yet relatively simple initial notion, proving general results at this level. The concept is then refined in two separate steps, where the final result is still more general than measure integration.

**14. Differentiation.** Early in Chapter 9, a very simple and elegant definition of the derivative of one set function with respect to another is presented. This definition is used throughout the chapter without modification. Conditions are given to guarantee the existence of the derivative, along with further conditions to ensure that the integral of the derivative is equal to the original function and conditions to ensure that the derivative of the integral is the original function. In Chapter 10, these results are applied to complex-valued functions of a real variable. Here a concept is introduced , not seen in textbooks, of the derivative of one point function with respect to another, generalizing the usual concept of the derivative and implicitly suggesting that this concept is worth studying.

A theme to several of Morse's innovations in set theory is the generalization of a concept as it appears in Zermelo-Fraenkel set theory (ZF) to the more extensive theory of points and sets (i.e., sets and classes). This theme can be seen in the treatments of the axioms of choice, ordered pairs, inductive definitions, and runs. In both editions of Morse's book, the axiom of choice involved a primitive operator 'mel' which selected an element from an nonempty set. In this text, An $x \,\underline{\mathrm{u}}\, x$ is used to select a set having a given property. In ZF the ordered pair has the property that

$$((a, b) = (c, d) \leftrightarrow a = c \wedge b = d)$$

This property holds for Morse's ordered pair where the coordinates need not be points. In 2.102, Morse states the Ordinary Induction Theorems. Indeed, these are exactly the results that hold in ZF, though in this setting they are stronger than in the setting of ZF. The General Induction Theorem, on the other hand, allows the inductive definition of a relation in which the vertical sections need not be points. Finally, runs are much like filter bases, where the vertical sections of the run correspond to the elements of the filter base. The difference is that the vertical sections need not be points.

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† Robert S. Arnold, *Plus and Times*, Ph.D. Thesis, University of California at Berkeley, 1969.

# II. EXPLANATORY COMMENTS

## Chapter 0. Language and Inference

Chapter 0 is metamathematical in character. Morse's goals for this chapter include

0. Laying out the elements of the formal language (*symbols* and *expressions*)
1. Describing the well-formed expressions of the language, here known simply as *formulas*
2. Describing when variables are *free* or *indicial* in a formula
3. Developing the Theory of Notation
4. Listing the Rules of Inference

**Unification: Syntax and Semantics.**

As the topics listed above indicate, Chapter 0 is about building of the elements of the language. The usual development of a formal language involves constructing well-formed expressions called *terms* (expressions which refer to objects) and well-formed expressions called *formulas* to express statements. Here there is only one type of well-formed expression, called a formula. Syntactically no distinction is made between expressions that may refer to objects and expressions that may assert a proposition. This lack of syntactic distinction simplifies the syntactic development, but also naturally raises questions about the semantics of the resulting formalism. The unification of terms and formulas is mirrored by a semantic unification of statements and sets, that is, of logic and set theory. Under Morse's semantic unification, there is no difference between the conjunction of two things (considered as statements) and their intersection (considered as sets). Similarly there is no difference between the disjunction of two things and their union. The empty set, 0, is the same thing as the statement False, and the universal set, U, is the same thing as the statement True. Semantically, every well-formed expression is both a statement and a set, although in most contexts one of the two meanings will be more applicable than the other. Certain formulas have a distinctive statementlike character. These all derive from the formula '$(x \in y)$' which will turn out to have either the value U or 0, depending on whether or not $x$ is an element of $y$.

**Symbols and Expressions.**

The first task is the identification of symbols and of expressions as linear arrays of symbols. There are two kinds of symbols: *constants* and *variables*. Constants are symbols such as the left and right parentheses, the plus sign, summation symbol, and integral symbol. Morse has also arranged things so that combinations of nonitalic Latin letters also constitute constant symbols. For example 'dmn' and 'rng' are constants, as are 'sng', 'relation', 'function', and a host of others. These word-like symbols are easy to generate and aid in the recollection of their intended use. The Index of Constants, Part I shows which combinations appear in this text.

Among the variables are italicized Latin letters and subscripted or superscripted versions of these. Any subscript or superscript is part of the symbol itself. Thus no symbol, such as a variable, can be a subscript of another symbol and retain its identity as a separate symbol. While '$A_x$' is a variable, the subscript '$x$' is not subject to replacement as a variable. The linearity of expressions, similar to that of programming languages, is perhaps the most significant constraint in trying to emulate conventional notation.

There is only one type of variable. As a result of the lack of distinction between terms and formulas, there are not different types of variables for statements versus objects. Nor are there, for example, different types of variables for classes versus sets (or, as Morse referred to them, sets versus points).

**Schematic Expressions.**

Much of the expressive power of any formal language derives from the quantified expressions in the language, based on predicates and functionals. For example, the predicate '$(x < 2)$' is dependent on the variable '$x$', while the functional '$(2 + y)$' is dependent on the variable '$y$'. Predicate logic requires a means to deal with such predicates in a general way, using predicate variables. Predicate or functional variables are represented in Morse's language using *schemators.*

A schemator is one of '$\underline{\mathrm{u}}$', '$\underline{\mathrm{v}}$', or '$\underline{\mathrm{w}}$', or a symbol derived from one of them by adding one or more primes. Thus the schemators are

$$\begin{array}{lllll} `\underline{\mathrm{u}}\text{'}, & `\underline{\mathrm{u}}'\text{'}, & `\underline{\mathrm{u}}''\text{'}, & `\underline{\mathrm{u}}'''\text{'}, & \text{etc.} \\ `\underline{\mathrm{v}}\text{'}, & `\underline{\mathrm{v}}'\text{'}, & `\underline{\mathrm{v}}''\text{'}, & `\underline{\mathrm{v}}'''\text{'}, & \text{etc.} \\ `\underline{\mathrm{w}}\text{'}, & `\underline{\mathrm{w}}'\text{'}, & `\underline{\mathrm{w}}''\text{'}, & `\underline{\mathrm{w}}'''\text{'}, & \text{etc.} \end{array}$$

Each schemator is a constant. Predicates or functionals are represented by *schematic expressions.*

A *schematic expression* is a schemator followed by a concatenation of variables, where the number of variables is one more than the number of primes in the schemator. The following are examples of schematic expressions.

'$\underline{\mathrm{u}}\,x$', '$\underline{\mathrm{w}}\,z$', '$\underline{\mathrm{v}}'\,st$', '$\underline{\mathrm{u}}''\,aaa$', '$\underline{\mathrm{w}}'''\,xx'x''x'''$'

**Replacement and Schematic Replacement.**

The analysis of expressions is enabled largely through the concepts of *replacement* and *schematic replacement.* Replacement is the replacing of a symbol in some expression by another expression. For example 'some $xY$ tan' is an expression with four symbols. If the fourth symbol, 'tan' is replaced by '$(t < +$' the result is 'some $xY(t < +$', which has seven symbols. If '$x$' is replaced by 'dmn $x$' in '$(x \in x)$', the result is '$(\mathrm{dmn}\, x \in \mathrm{dmn}\, x)$'. To illustrate the point made earlier about subscripts, if '$x$' is replaced by '3' in '$A_x \subset x$', the result is '$A_x \subset 3$'.

The concept of schematic replacement involves replacing a schematic expression that occurs in some expression by an expression. For this to be successful, the schemator that initiates the schematic expression must occur in the first expression only as the initial symbol of the schematic expression to be replaced. For example, consider the expression 'vs $\underline{\mathrm{u}}'\,yy \in$ hs $\underline{\mathrm{u}}'\,xy$'. In this expression neither '$\underline{\mathrm{u}}'\,yy$' nor '$\underline{\mathrm{u}}'\,xy$ may be schematically replaced, because they are not the same.

**Forms, Formulas, and Definitions.**

The next step is to identify which expressions are formulas. In any system, this process is accomplished by beginning with a collection of expressions called *forms* that are accepted as well-formed and inductively via substitution building up more complex expressions. The typical development begins with a small number of primitive (undefined) term forms and/or formula forms. In Morse's language there are a large number of forms from which the formulas are constructed. While a few of these forms are primitive, most of them enter the language through definitions. In this regard, it is important to understand Morse's treatment of definitions. On page 17 of *A Theory of Sets, Second Edition*, Morse stated:

> "Definitions are especially important for three reasons. Definitions generate forms; real definitions are more than mere shorthand devices; real definitions are accepted as theorems. Accordingly, definitions should be made with care."

Definitions also play a role in helping establish which symbols are constants. Any symbol appearing on the left side of a definition and not on the right side is a constant and is said to be *fixed* by the definition.†

As was noted above, all the well-formed expressions in most formal systems are constructed from a few primitive forms. In such a setting, a definition provides a shorthand replacement for a longer formal expression. Thus, any expression involving the defined term is not one of the well-formed expressions. The definition tells us how to replace the defined term to obtain an actual formal expression. Such a definition is itself a metamathematical statement, conveying this rule for replacement. It is this treatment of definitions that Morse alluded to as "mere shorthand devices." The following two quotes exemplify this approach.

† This approach to constants and variables has the unfortunate consequence that a symbol used as a variable may later be fixed as a constant in a definition. This problem is easily avoided by specifying all variables at the start and saying that all other symbols are constants.

"...the definitions are no part of our subject, but are, strictly speaking, mere typographical conveniences."[†]

"Strictly speaking, the notation "$P \vee Q$" is not part of our symbolic logic, and if one were being really careful one would always write '$\sim(\sim P \sim Q)$'. However, it is very handy to write "$P \vee Q$" instead, and to agree that whenever "$P \vee Q$" appears it is really "$\sim(\sim P \sim Q)$" which occurs. Similar considerations apply to our omission of parentheses. Strictly speaking, it is not permissible to omit parentheses, and we omit them only with the understanding that in any precise formulation they would actually be present.

To put it another way, omission of parentheses and replacement of "$\sim(\sim P \sim Q)$" by "$P \vee Q$" are not part of our symbolic logic, but only a convenience which we permit ourselves in talking about it."[‡]

In Morse's system, the defined terms (and, for that matter, the definitions themselves) are part of the formal mathematical language. Each definition expands the formal mathematical language. The left-hand side of a definition, called the *definiendum*, introduces a new form into the language. It is the collection of forms that determines the formulas of the language. As a result, every definition, axiom, and theorem is a *completely formal* expression in his language.

Following the usual approach, it is indeed not feasible to write higher mathematics using completely formal expressions. One must, at the very least, adopt shorthand devices as abbreviations of the formal expressions, and having lost the formal battle on the first day, one soon succumbs to all manner of abusive notation, comforted by the thought that everything could, in principle, be replaced by formal expressions. Following Morse's approach of including each new defined form as part of the formal language, it is possible and feasible to express higher level concepts and results with complete formality.

The fact that all defined forms are part of the formal language allows us to shorten formulas, which in turn helps to make them readable. However, the type of shortcuts (i.e. forms) that are allowed can make a significant difference in enhancing the readability of the language. Suppose that all defined forms are of the Polish style of one new constant followed by some variables. For example, in sentential logic we may begin with two undefined forms, implication and negation, expressed as

| Undefined Form | Meaning |
|---|---|
| C$pq$ | $p$ implies $q$ |
| N$p$ | not $p$ |

We could then introduce definitions for conjunction, disjunction, and equivalence as follows.

$$(\mathrm{K}pq \equiv \mathrm{N\,C}p\mathrm{N}q)$$
$$(\mathrm{A}pq \equiv \mathrm{C\,N}pq)$$
$$(\mathrm{E}pq \equiv \mathrm{K\,C}pq\mathrm{C}qp)$$

The resulting language may be shortened, but it still could be hard to read. The problem of readability can be addressed by using a different style of forms such as

| Undefined Form | Meaning |
|---|---|
| $(p \rightarrow q)$ | $p$ implies $q$ |
| $\sim p$ | not $p$ |

followed by definitions for conjunction, disjunction, and equivalence as follows.

$$((p \wedge q) \equiv \sim(p \rightarrow \sim q))$$
$$((p \vee q) \equiv (\sim p \rightarrow q))$$
$$((p \leftrightarrow q) \equiv ((p \rightarrow q) \wedge (q \rightarrow p))$$

† A. N. Whitehead and B. Ruseell, *Principia Mathematica to *56*, Cambridge (1967), p. 11.

‡ J. Barkley Rosser, *Logic for Mathematicians, Second Edition*, Dover (2008) p. 14.

These revised forms are generally considered easier to read because they mirror the language we use when expressing the concepts they convey. We say "$p$ implies $q$", not "implies $pq$". We say "$p$ and $q$", not "and $pq$, etc.

There are other types of forms that we might want to have to aid in reading expressions. These include such forms as

$$(p \wedge q \rightarrow r)$$

$$\int\limits_0^1 x^2 dx$$

$$\sum_{x \in A} x^2$$

$$\lim_{x \to 0} x^2$$

The first of the expressions above is notable for its omission of parentheses. Considered as a single form, it includes two logical operators in combination with three variables and a pair of outer parentheses.

The next three forms all involve dummy variables and nonlinear notation. In the integral we find the limits of the integral in nonlinear positions as well as the exponent 2. In addition to those issues, we must inquire about the nature of the letter '$d$' in this hoped for form. It does not appear to be a variable, but how would it be determined, other than by foreknowledge, that it is not a variable. Presumably, if we made a definition with this "form" as the definiendum, the '$d$' would appear in the definiendum but not on the right side of the definition. Thus '$d$' would be fixed as a constant by the definition. This would have the unfortunate consequence that we could never again use '$d$' as a variable, not to mention that we should hope it was not used before as a variable. In fact, Morse declared that the italic Latin letters are all variables, so it should not become fixed by a definition. This means that the '$d$' cannot appear italic in this position, but must be changed in some way such as to upright. To solve all of these issues with the integral, we end up with the following form.

$$\int ab\underline{\mathrm{u}}x\,\mathrm{d}x$$

To implement this for the particular case shown above, we require a notation for exponentiation. Suppose we use '${}^{\bullet}xn$' to represent $x$ raised to the $n$ power. Then the integral would be represented as follows.

$$\int 0\ 1^{\bullet}x2\,\mathrm{d}x$$

Notice in the above that it is necessary to allow some space between 0 and 1, because if there were no space between them, the combination would become a single symbol.

The other two hoped for forms are handled similarly as follows.

$$\sum x \in A\underline{\mathrm{u}}x$$

$$\lim xa\underline{\mathrm{u}}x$$

An alternative possibility for this last form is the following.

$$\lim x \rightarrow a\underline{\mathrm{u}}x$$

A question arises as to whether there are any restrictions on how a form may be designed, other than the issue of linearity. If many forms are constructed with a variety of designs, what difficulties might arise? One possible problem is that when building formulas using a variety of forms, it might be possible to construct the same formula in two different ways, thus making the resulting formula ambiguous. As an illustration, let us consider a type of ambiguity that occurs in conventional mathematical notation. Let '$f(x)$' denote the value of the function $f$ at the point $x$ and let '$\mathbf{D}\,f$' denote the derivative of the function $f$. Then the expression '$\mathbf{D}\,f(x)$' has two readings. The first results from replacing '$f$' by '$f(x)$' in '$\mathbf{D}\,f$'. The second results from replacing '$f$' by '$\mathbf{D}\,f$' in '$f(x)$'.

The problem in the previous example is that the form '$f(x)$' begins with a variable ('$f$'), which is a formula by itself. Even if we restrict our forms to be of the Polish type, it is possible to encounter ambiguity. Suppose '$+xz$' and '$+xyz$' are forms. Then the expression '$+x+abz$' would have two readings. The first results from replacing '$y$' by '$+ab$' in '$+xyz$'. The second results from replacing '$z$' by '$+abz$' in '$+xz$'. The problem with this example is that one of the two forms is an initial segment of the other, which is the same type of problem as in the '$f(x)$' example.

It is clear that, using the Polish-style syntax, ambiguity can be avoided by requiring that no two different forms begin with the same constant. We are interested in finding rules of construction that will allow a greater variety than just the Polish style.

Let us first look at additional examples of ambiguity. Suppose we have the two forms '$|x|$' and '$|xyz|$'. Then the expression '$||x|y|z||$' would have two readings. The first results from replacing '$x$' by '$|x|$' and '$z$' by '$|z|$' in '$|xyz|$'. The second results from replacing '$x$' by '$|xyz|$' in '$|x|$' and then replacing $y$' by '$|y|$' in the result.

For the next example, suppose we have the two forms '$|x$', and '$||x$'. Then the expression '$|||x$' would have two readings. The first results from replacing '$x$' by '$|x$' in '$||x$'. The second results from replacing '$x$' by '$||x$' in '$|x$'.

In both of the last two examples, neither of the forms under consideration is the initial segment of the other form under consideration. What these examples have in common is that a constant symbol that is the initial symbol of a form appears in a form in a position other than the initial position. This situation requires special handling under Morse's rules for forms. Clearly if we want to have the usual form for absolute value, we must allow this situation to occur, but when it does, special rules apply. For the most part, ambiguity can be avoided if two rules are followed when creating forms: (1) don't use an symbol that is an initial symbol in a form as a noninitial symbol in a form and (2) don't allow one form to be the initial segment of a longer form (after appropriate change of variables). Notice that both of these rules are global in that they concern the collection of forms as a whole and not just an individual form. There are additional rules about the number of times a variable can appear in a form and coordinating the appearances of variables in a form with the appearances of schematic expressions in the form. Following Morse's rules about forms ensures that each formula has a unique reading and no formula is the initial segment of a longer formula.†

Besides the left parenthesis, which is the initial symbol for a great many forms, we cite three other symbols to exemplify the use of the same symbol as the initial symbol in multiple forms. The first example was mentioned earlier; both of the forms '$\bigwedge x\,\underline{\mathrm{u}}\,x$' and '$\bigwedge x \in A\,\underline{\mathrm{u}}\,x$' begin with the same symbol '$\bigwedge$'. The second example is the symbol 'The'. This symbol occurs in many forms such as

'The domain of $R$'
'The range of $R$'
'The cardinality of $A$'
'The $x\,\underline{\mathrm{u}}\,x$'
'The $x \in A\,\underline{\mathrm{u}}\,x$'

The third example, '$\int$', is the initial symbol used to designate each of the following integrals in Chapter 7.

† See Appendix A.

| Definition | Integral | Definition | Integral |
|---|---|---|---|
| 7.0.1 | $\int \#M\,\underline{\mathrm{u}}x\varphi\ \mathrm{d}'x$ | 7.197.1 | $\int A;\,\underline{\mathrm{u}}x\ \mathrm{d}x$ |
| 7.1 | $\int \#MA;\,\underline{\mathrm{u}}x\varphi\ \mathrm{d}'x$ | 7.197.2 | $\int ab\underline{\mathrm{u}}x\ \mathrm{d}x$ |
| 7.11.2 | $\int \#M\,\underline{\mathrm{u}}x\varphi\ \mathrm{d}x$ | 7.198.0 | $\int \underline{\mathrm{u}}xg\,\partial\,x$ |
| 7.11.3 | $\int \#MA;\,\underline{\mathrm{u}}x\varphi\ \mathrm{d}x$ | 7.198.1 | $\int \underline{\mathrm{u}}xg\ \mathrm{dil}\,x$ |
| 7.19.0 | $\int \underline{\mathrm{u}}x\varphi\ \mathrm{d}'x$ | 7.198.2 | $\int A;\,\underline{\mathrm{u}}xg\,\partial\,x$ |
| 7.19.1 | $\int A;\,\underline{\mathrm{u}}x\varphi\ \mathrm{d}'x$ | 7.198.3 | $\int ab\underline{\mathrm{u}}xg\,\partial\,x$ |
| 7.19.2 | $\int \underline{\mathrm{u}}x\varphi\ \mathrm{d}x$ | 7.199.0 | $\int \underline{\mathrm{u}}x\,\partial\,x$ |
| 7.19.3 | $\int A;\,\underline{\mathrm{u}}x\varphi\ \mathrm{d}x$ | 7.199.1 | $\int A;\,\underline{\mathrm{u}}x\,\partial\,x$ |
| 7.196 | $\int ab\underline{\mathrm{u}}x\varphi\ \mathrm{d}x$ | 7.199.2 | $\int ab\underline{\mathrm{u}}x\,\partial\,x$ |
| 7.197.0 | $\int \underline{\mathrm{u}}x\ \mathrm{d}x$ | | |

To understand how the forms of each of these integrals is distinguished from the others in terms of readability, it helps to analyze them in terms of what constants occur in what positions. In this analysis, a schematic expression occupies a single variable position. In four of these forms, the second symbol is the constant, '#'. Of these four, two have a semi-colon and two do not. In each of those two pairs, one of the forms has the symbol 'd′' and one has the symbol 'd'. This distinguishes these four forms from one another and from the remaining 15 forms. In all of these four forms, the variable '$M$' stands for an unspecified method of integration.

In the 15 remaining forms, the method of integration is implicitly understood to be integration by refinement, and thus neither the constant '#' nor the variable '$M$' appears. The last seven of the remaining 15 forms have either the symbol '$\partial$' or the symbol 'dil'. These are all used for Riemann-type (including Riemann-Stieltjes) integrals. Of these seven, two have a semi-colon, and 7.198.2 has two variable positions between ';' and '$\partial$', while 7.199.1 has only one variable position. 7.198.1 is clearly unique because of the 'dil'. Of the remaining four forms, 7.198.0 has two variable positions between '$\int$' and '$\partial$', while 7.198.3 has four, 7.199.0 has one, and 7.199.2 has three.

If we look at the remaining eight forms, we see that two of them contain the symbol 'd′'. Of the remaining six forms, two of them have a semi-colon. This leaves four forms whose only nonschemator constants are the integral sign and the symbol 'd'. What distinguishes these four forms from one another is the number of variable positions between the integral sign and the 'd'. In 7.19.2, there are two variable positions, '$\underline{\mathrm{u}}x$' and '$\varphi$'; in 7.196 there are four; in 7.197.0 there is one; and in 7.197.2 there are three. Because of the different numbers, that is the placement of 'd' with respect to '$\int$', no one of these four can be an initial segment of another one of the four.

Summarizing the foregoing, formulas are constructed using all forms, both primitive and defined. In order to maintain complete formality, all forms and all definitions are part of the formal language. In order to enhance the readability of the formal language, a great deal of flexibility is allowed in the design of forms.

**Free and Indicial Variables.**

Some textbooks in logic require that every variable appearing in a statement be bound. That is, there are no free variables in a statement. This is not the case here. A formula such as

$$\text{‘}(x \cup x = x)\text{’},$$

which states that the union of a set with itself is equal to the set, is a theorem, meaning that this is a fact no matter what $x$ is. The variable '$x$' is free in this theorem. An inductive procedure is used to determine whether a variable is free in a formula. First, a variable is free in a form if and only if it appears less than twice in the form. One consequence of this is that every variable not appearing in a form is free in the form. Second, if a formula $B$ is obtained from a form $A$ by replacing free variables by formulas and simultaneously schematically replacing schematic expressions by formulas, then a variable is free in $B$ if and only if it is free in $A$ and free in each of the replacing formulas.

Bound variables appear in formulas such as

$$\text{‘}\bigwedge x(x \cup x = x)\text{’}$$

i.e., for each $x$, $x$ union $x$ equals $x$, and

$$‘\bigvee y(0 \in y)’$$

i.e., for some $y$, zero belongs to $y$. The variable ‘$x$’ is bound in the first of these two formulas, and the variable ‘$y$’ is bound in the second. When the variable is bound in the entire formula, Morse referred to the variable as being *indicial* in the formula. The two formulas above exemplify indicial variables. In the following formula

$$‘(\bigvee x(x < 1) \to \bigvee z(z < 2))’$$

no variable is indicial, although ‘$x$’ and ‘$z$’ are both bound.

Based on the examples above, it may appear that the concepts of indicial and bound are difficult to explain precisely. However, Morse's treatment makes it relatively easy. Consider first the concept of indicial. A variable is indicial in a form if any only if it appears more than once in the form. For any formula $F$, if there is a form $G$, such that $G$ is not a variable or a schematic expression and $F$ can be obtained from $G$ by replacing free variables by formulas and simultaneously schematically replacing schematic expressions by formulas, then a variable $\alpha$ is indicial in $F$ if and only if $\alpha$ is indicial in $G$ and does not appear in any of the formulas replacing free variables.

A variable $\alpha$ is bound in a formula $F$ if and only if there is a formula $G$ such that $\alpha$ does not appear in $G$ and $F$ can be obtained from $G$ by simultaneously replacing variables free in $G$ by formulas in which $\alpha$ is indicial.

Indicial variables are used in significant ways outside of the chapter on logic. Concepts expressed with forms having indicial variables include the classifier in set theory, indefinite and definite descriptions, limits, summation, and integration.

**Theory of Notation.**

Morse worked very hard to create forms that would (1) adhere to the technical requirements of his syntax, (2) facilitate the reading and writing of mathematical expressions, and (3) conform to common mathematical practice whenever feasible. Much of this was accomplished in his Theory of Notation which is primarily concerned with two kinds of constants: *binarians* and *notarians*.

Binarians are the infix operators in expressions such as ‘$(p \to q)$’, ‘$(x = y)$’, ‘$(A \cap B)$’, ‘$(x + y)$’, and ‘$(x, y)$’. Here the binarians are ‘$\to$’, ‘$=$’, ‘$\cap$’, ‘$+$’, and ‘,’. Each binarian is assigned a *type* which is an integer value indicating its level in a precedence ordering. This allows for the removal of many parentheses and greatly improves the readability of many expressions such as

$$‘(m \in x \wedge n \in x \to m + n \in x)’$$

which otherwise would need to be expressed as

$$‘(((m \in x) \wedge (n \in x)) \to ((m + n) \in x))’,$$

The first expression with just one pair of parentheses is a completely formal expression in Morse's language and not merely a shorthand for the longer expression with six pairs of parentheses. In fact this expression is derived from the form

$$‘(x \in x' \wedge x'' \in x''' \to x'''' + x''''' \in x'''''')’$$

which contains the constants ‘(’, ‘$\in$’, ‘$\wedge$’, ‘$\to$’, and ‘)’. The binarians are listed in 0.30 under their respective precedences.

The Theory of Notation also treats expressions having multiple binarians of the same type. For example, the binarians of type 6 are verb-like and are used to express a relation between two objects, as in

$$(a = b) \quad (x \in A) \quad (B \subset C) \quad (x < y).$$

When these appear in a string without parentheses, they are understood to be parenthesized and separated by "and". For example

$$`(x \in A \subset B = C)'$$

is explained as

$$`((x \in A) \wedge (A \subset B) \wedge (B = C))'.$$

This treatment also applies to '$\leftrightarrow$' (if and only if). Thus

$$`(p \leftrightarrow q \leftrightarrow r)'$$

is explained as

$$`((p \leftrightarrow q) \wedge (q \leftrightarrow r))'.$$

However, Morse made a special point to use a different treatment with '$\rightarrow$' (implies). Here

$$`(p \rightarrow q \rightarrow r)'$$

is explained as

$$`((p \rightarrow q) \wedge (p \rightarrow r))'.$$

Presumably, the purpose of this is to recognize that the second implication may to some extent be relying on the hypothesis from the first implication.

The form '$\sim x$' is used to mean the negation of $x$ as well as the complement of $x$. The form '$(x \backsim y)$' is used to denote the set difference between $x$ and $y$. Note that '$\sim$' and '$\backsim$' are different symbols. Morse used the second symbol to set up a negating convention illustrated by the following three examples.

$$`((x \backsim\in y) \leftrightarrow \sim(x \in y))'$$
$$`((x \backsim\subset y) \leftrightarrow \sim(x \subset y))'$$
$$`((x \backsim< y) \leftrightarrow \sim(x < y))'$$

Morse used the form '$(x \neq y)$', but this is the only instance in this text of Morse using the slash to negate a relational operator.

The second kind of symbol treated in the Theory of Notation is notarian. Section 0.50 classifies several expressions as being one of class 0 through class 6. A notarian is the initial symbol of any one of these expressions. In most cases, the expression itself is the notarian, i.e., a single symbol. In the definitional schema in this section, each notatian appears as the first symbol in a form that has one or more indicial variables. For example the universal and existential quantifiers, '$\bigwedge$' and '$\bigvee$' are notarians. They appear in the forms '$\bigwedge x \underline{\mathrm{u}} x$' and '$\bigvee x \underline{\mathrm{u}} x$' which express "for each $x$, $\underline{\mathrm{u}} x$ is true" and "for some $x$, $\underline{\mathrm{u}} x$ is true", respectively. The schematic expression '$\underline{\mathrm{u}} x$' occurs in each of these forms. Morse's Schematic Substitution rule of inference governs the replacement of schematic expressions by specific formulas.

Other forms beginning with a notarian include

$$`\mathsf{E}\, x \underline{\mathrm{u}} x', \quad `\measuredangle x \underline{\mathrm{u}} x', \quad `\sum x \underline{\mathrm{u}} x'$$

The first of these represents the set of all $x$ such that $\underline{\mathrm{u}} x$ is true; the second respresents the function that assigns to each $x$ the value $\underline{\mathrm{u}} x$; and the third represents the sum over all $x$ of $\underline{\mathrm{u}} x$. The Theory of Notation provides for three variations of forms for each of the expressions of class 0 through 3.

The first of these is the unconditional variation which is usually the expression followed by '$x \underline{\mathrm{u}} x$', such as the examples in the previous paragraph.

The second of these is the conditional variation. These look like

$$`\bigwedge x; \underline{\mathrm{v}} x \underline{\mathrm{u}} x', \quad `\mathsf{E}\, x; \underline{\mathrm{v}} x \underline{\mathrm{u}} x', \quad `\measuredangle x; \underline{\mathrm{v}} x \underline{\mathrm{u}} x', \quad `\sum x; \underline{\mathrm{v}} x \underline{\mathrm{u}} x'$$

In each of these, the '; $\underline{\mathrm{v}} x$' part of the expression is to be read "subject to the condition that $\underline{\mathrm{v}} x$ is true" or "such that $\underline{\mathrm{v}} x$". Usually the conditional variation is defined in terms of the unconditional variation as in

$$(\bigwedge x; \underline{\mathrm{v}}\, x \underline{\mathrm{u}}\, x \equiv \bigwedge x(0 \in \underline{\mathrm{v}}\, x \rightarrow \underline{\mathrm{u}}\, x)).$$

However, it may happen that the unconditional variation is defined in terms of the conditional variation, as in

$$(\sup x \underline{\mathrm{u}}\, x \equiv \sup x; (x = x) \underline{\mathrm{u}}\, x).$$

The third of these is the restricted variation, and looks like the following.

$$\text{‘}\bigwedge x \in A \underline{\mathrm{u}}\, x\text{’}, \quad \text{‘}\mathsf{E}\, x \subset y \underline{\mathrm{u}}\, x\text{’}, \quad \text{‘}\measuredangle x < 2 \underline{\mathrm{u}}\, x\text{’}, \quad \text{‘}\sum x \in A \cup B \underline{\mathrm{u}}\, x\text{’}$$

These are all defined using the conditional variation as shown in the two following examples.

$$(\bigwedge x \in A \underline{\mathrm{u}}\, x \equiv \bigwedge x; (x \in A) \underline{\mathrm{u}}\, x)$$
$$(\sum x \in A \cup B \underline{\mathrm{u}}\, x \equiv \sum x; (x \in A \cup B) \underline{\mathrm{u}}\, x)$$

In addition to the three variations above, there are also multi-variable variations such as the following.

$$\bigwedge x \cap y; \underline{\mathrm{u}}'\, xy \underline{\mathrm{v}}'\, xy$$
$$\bigwedge x, y; \underline{\mathrm{u}}'\, xy \underline{\mathrm{v}}'\, xy$$
$$\bigwedge x, y \in A \cap B \underline{\mathrm{u}}'\, xy$$

It may be helpful to observe that we cannot have ‘$\bigwedge xy \underline{\mathrm{u}}'\, xy$’ as a multi-varialble form because the first three symvols are already a formula.

The following comments about the Theory of Notation are very technical. At the end of the Theory of Notation, in 0.65 through 0.69, Morse said that he was trying to tie up some loose ends. In particular, he was trying to establish that each binarian and each notarian is a constant, and he was trying to establish a canonical form for each binarian and each notarian. To these ends, 0.65 fixes as a constant each notarian not of class 5. However, classes are assigned to expressions, while a notarian is a symbol. In many cases, the expressions that are assigned a class are indeed symbols, but in some cases they are not. For example ‘far $R$’ is an expression of class 1 which has as its first symbol the notarian ‘far’. As a matter of fact, there is only one expression of class 5 and it is the notarian ‘$|$’. Therefore, every other notarian is *not* of class 5. For example, ‘far’ is a notarian of no class. Thus 0.65 fixes as a constant each notarian except ‘$|$’. In addition, 0.65 fixes the semicolon as a constant. Each binarian is fixed as a constant by 0.66, and along with the binarians, the notarian ‘$|$’ is fixed by 0.66.

One instance of 0.65 is duplicated in 0.0.1 for the notarian ‘$\bigwedge$’. Note that 0.0.0 also fixes the binarian ‘$\rightarrow$’ along with the left and right parentheses. The orienting definitions 0.0.0 and 0.0.1 along with orienting definitions 2.0 are made so as not to depend on the Theory of Notation.

In 0.67, a canonical form is established for each expression of class 0, 1, or 2. In 0.68, a canonical form is established for each expression of class 3. It is worthwhile noting that the notarian ‘alm’ appears as the first symbol of two different expressions of class 1.

In 0.69, a canonical form is established for the sole expression (symbol, in this case) of class 5. This is a special case as it involves what Morse called a *flanker*. A flanker is a symbol that appears in some definiendum as the first symbol and in some definiendum in a position other than the first. The only listed expression of class 5 is the vertical bar used for absolute value. In definition 3.30.2 it appears as the first and last symbol of the defined form for absolute value. Morse's rule for flankers is that in every definiendum in which a flanker occurs, it must be the first and last symbol and have no other appearance. There are further rules about the positions of variables and other constants in such definienda, but these will not be discussed here. The rules for flankers are given in A.7 and in 0.63.

It might appear that Morse needed something like the following to establish a canonical form for each binarian or, more generally, for each *nexus* (string of binarians).

**DEFINITIONAL SCHEMA** We accept as a definition each expression which can be obtained from ‘$((x \rightarrow x') \equiv (x \rightarrow x'))$’ by replacing ‘$\rightarrow$’ by a nexus.

The reason that no such schema occurs here is that this case is covered by 0.48.

There is another aspect of 0.48 that is worth noting. In 0.36 of the first edition of his book, Morse defined adjacency to mean "and" as follows:

$$((xx') \equiv (x \wedge x'))$$
$$((xx'x'') \equiv (x \wedge x' \wedge x''))$$
etc.

In the second edition, these definitions were dropped, but the forms for adjacency were not abandoned. In fact, 0.48 lifts each of these forms. That is, 0.48 includes definitions such as

$$((xx') \equiv (xx'))$$
$$((xx'x'') \equiv (xx'x''))$$
etc.

There is a statement at the end of the article *Plus and Times* that gives a hint about his intentions in this regard.

"...it is possible, with proper choice of $x$ and $y$, for

$$(xy)$$

to consistently be either an instance of multiplication of scalars, or function evaluation, or composition, or a rather general inner product, or matrix multiplication, or the left application of a matrix to a vector, or the right application of a matrix to a vector. The operation we here have in mind is not universally associative."

Writing two symbols next to each other is probably the most used and abused notation in all of mathematics. Morse made an effort to formally justify many such usages. No attempt has been made in this text to carry out Morse's suggested plan.

It is worth noting that '$\int$' is not a notarian even though it is the initial symbol of bound variable forms such as 7.0.1, 7.1, 7.11.2, and several others shown above. None of the forms for integrals have the structure needed to be included in the Theory of Notation.

**Rules of Inference.**

Chapter 0 also includes a listing of the Rules of Inference. There are six rules: a rule to initiate an axiom or definition, a *modus ponens* rule for detachment, three substitution rules for free variables, schematic expressions, and indicial (bound) variables, respectively, and a rule of universalization.

## Chapter 1. Logic

**Foundations.**

The logic is based on two primitive forms:

| | |
|---|---|
| $(p \to q)$ | implication |
| $\bigwedge x \,\underline{\mathrm{u}}\, x$ | universal quantification |

All of the axioms for logic, except one, involve only these two forms. Axiom 1.3.5 involves the form

| | |
|---|---|
| $\sim p$ | negation |

which in turn comes from two definitions

$$(0 \equiv \bigwedge xx)$$
$$(\sim p \equiv (p \rightarrow 0))$$

These two primitive forms, the two definitions above, the six axioms in 1.3, and the rules of inference comprise the foundations for logic.

**Notations.**

The notations for sentential logic are shown in the table below. The negation symbol and binary logical connectives are in common use.

| Notation | Meaning | Alternate Explanation | Truth Condition |
|---|---|---|---|
| 0 | zero | Falsity | never |
| U | U | Truth | always |
| $(p \rightarrow q)$ | $p$ implies $q$ | If $p$ then $q$ | $p$ is false or $q$ is true |
| $(p \wedge q)$ | $p$ and $q$ | The conjunction of $p$ and $q$ | both $p$ and $q$ are true |
| $(p \vee q)$ | $p$ or $q$ | The disjunction of $p$ and $q$ | $p$ is true or $q$ is true, or both |
| $(p \leftrightarrow q)$ | $p$ if and only if $q$ | $p$ is equivalent to $q$ | both are true or both are false |
| $\sim p$ | not $p$ | The negation of $p$ | $p$ is false |

The predicate logic treats the quantifiers "for each" and "for some".

| Notation | Meaning | Conventional Notation |
|---|---|---|
| $\bigwedge x \underline{\mathrm{u}} x$ | For each $x$, $\underline{\mathrm{u}} x$ is true | $(\forall x) \underline{\mathrm{u}} x$ |
| $\bigvee x \underline{\mathrm{u}} x$ | For some $x$, $\underline{\mathrm{u}} x$ is true | $(\exists x) \underline{\mathrm{u}} x$ |

The symbols '$\bigwedge$' and '$\bigvee$' were used by the Polish logicians Lukasiewicz and Lesniewski and emphasize the relationship with '$\wedge$' and '$\vee$'. In these expressions, '$\underline{\mathrm{u}} x$' is to be thought of as a predicate about the object $x$. For example

$$(x + 1 = 3 \rightarrow x = 2)$$

An unusual feature of the sentential logic is that it is in part based on the statement of falsity, denoted by '0', which in turn is defined via the universal quantifier. This definition is possible because of the unification of terms and formulas. In definition 1.0.3

$$(0 \equiv \bigwedge xx)$$

'0' is defined to mean that for each $x$, $x$ is true. This definition will acquire a very different interpretation in the set theory chapter.

**Theorems.**

The sentential logic is standard in that the true propositions, i.e., the theorems, are the tautologies. The theorems of predicate logic are essentially those of classical first-order predicate logic.

Most of the theorems in this text take the form '$(p \rightarrow q)$'. That is, there is a group of one or more hypotheses, followed by one or more conclusions. The logical symbols determine the sentence structure of nearly every theorem. There are exceptions such as Theorem 2.6

$$(x \,\text{is a set})$$

and Theorem 2.7

$$(0 \in \mathrm{U})$$

in which no logical symbol occurs. The first step the reader must take in learning to read the formal mathematics is to understand the logical structure of statements.

The Definitional Axioms for Logic in 1.1 were given by Morse so that any definitions needed to state axioms are listed as axioms as well. It would have been relatively easy to avoid these two definitional axioms. They were given because they are used to define '$\sim p$' which in turn appears in Axiom 1.3.5. By rewriting 1.3.5 as

$$(((p \rightarrow \bigwedge x x) \rightarrow p) \rightarrow p)$$

the definitional axioms could have been avoided. However, the way the axioms are stated uses only concepts that would appear in a more standard approach. Besides that, these definitional axioms are useful for the axioms in Chapter 2.

## Chapter 2. Set Theory

**Foundations.**

Two primitive forms are introduced for set theory:

| | |
|---|---|
| $(x \in y)$ | $x$ belongs to $y$ |
| $\text{An}\, x \underline{\text{u}} x$ | An $x$ such that $\underline{\text{u}} x$ |

Morse's axioms for set theory are highly primitive in the sense that the only defined set-theoretical concepts involved are subset, equality, and singleton. Morse's axioms of choice are notable in this context. Usually the axiom of choice is a statement about certain functions, where the concept of a function is the result of a long chain of definitions. Here the primitive form '$\text{An}\, x \underline{\text{u}} x$' is used to express choice. There are three axioms.

$$(\underline{\text{u}} x \rightarrow \underline{\text{u}}\, \text{An}\, x \underline{\text{u}} x)$$
$$(\sim \bigvee x \underline{\text{u}} x \rightarrow (\text{An}\, x \underline{\text{u}} x = \text{U}))$$
$$(\bigwedge x (\underline{\text{u}} x \leftrightarrow \underline{\text{v}} x) \rightarrow (\text{An}\, x \underline{\text{u}} x = \text{An}\, x \underline{\text{v}} x))$$

The first of these states that if $\underline{\text{u}}$ holds for $x$, then it holds for $\text{An}\, x \underline{\text{u}} x$. Another statement of this is

$$(\underline{\text{u}} z \wedge y = \text{An}\, x \underline{\text{u}} x \rightarrow \underline{\text{u}} y).$$

The second states that if $\underline{\text{u}}$ does not hold for any $x$, then $\text{An}\, x \underline{\text{u}} x$ equals the universe. The third states that equivalent predicates produce the same result from the 'An' operator.

Morse's set theory is a theory of sets and not just of points, i.e. not just of sets that are members of other sets. The distinguishing property of a point is its ability to be a member of another set. Sets that fail to belong to another set, do so because they are too large. Among the sets that are not points is the universe, U. The universe was defined in Chapter 1, Logic, as the true statement. A set is a point if and only if it belongs to U.

**Unification of Logic and Set Theory.**

In this chapter the unification of logic and set theory is developed in full. All the forms from the logic chapter take on a set-theoretical meaning in this chapter. As might be expected, '$(p \wedge q)$' means $p$ intersect $q$; '$(p \vee q)$' means $p$ union $q$; '$\sim p$' means the complement of $p$; '$\bigwedge x \underline{\text{u}} x$' means the intersection over all $x$ of $\underline{\text{u}} x$; and '$\bigvee x \underline{\text{u}} x$' means the union over all $x$ of $\underline{\text{u}} x$. Other forms such as '$(p \rightarrow q)$' and '$(p \leftrightarrow q)$' have meanings implied by meanings already cited. Thus, '$(p \rightarrow q)$' means $(\sim p \cup q)$. The false statement '0' means the empty set, and the true statement 'U' means the universe.

The unification of logic and set theory raises two questions:

(1) When is a set true?

(2) What set does a statement equal?

Axiom 2.5.0 (the axiom of truth) answers the first question, while Axioms 2.5.1, 2.5.2, and 2.5.3 collectively answer the second question. Let us discuss the axiom of truth further. Clearly Morse wanted the empty set to be false and the universe to be true. Thus some means was needed to determine the truth or falsity of all other sets. Suppose only the empty set is false and every other set is true. This would cause a problem because the conjunction (intersection) of two disjoint nonempty sets (true sets) would be false. Similarly it would not work to declare the universe as the only true set. Apparently what does work is to single out some point $z$ and declare any set $x$ to be true if and only if $(z \in x)$. The obvious choice for such a $z$ is the empty set.

A unification along the lines described above cannot be carried out in Zermelo-Fraenkel (ZF) set theory. This can be seen as follows. The empty set exists in ZF, and unification demands that the empty set is also a statement. Because every statement has a negation, it follows that the negation of the empty set is a statement. However unification demands that the negation of the empty set is identical to the complement of the empty set, i.e., the universe. This contradicts the fact that there is no universal set in ZF.

In Morse's unification there is no syntactical basis to distinguish terms from formulas. However, from a semantic perspective, terms and formulas can, to some extent, be distinguished. Axiom 2.5.5 says

$$((t \in \mathrm{U}) \to ((t \in (x \in y)) \leftrightarrow (x \in y))).$$

The form '$(x \in y)$' is usually thought of as expressing a statement that $x$ is a member of $y$. In Morse's unified treatment, '$(x \in y)$' also names a set, but Axiom 2.5.5 ensures that the set named is either the empty set or the universe, i.e. either the prototypical false statement '0' or the prototypical true statement 'U'. Morse referred to this as the Kronecker character of '$(x \in y)$'. As a result of Axioms 2.5.2, 2.5.3, and 2.5.4 and the definitions of the pertinent forms, many other statementlike forms have this Kronecker character; for example, '$(x \subset y)$', '$(x \supset y)$', '$(x = y)$', 'One $x\,\underline{\mathrm{u}}x$', 'relation is $R$', 'function is $f$','univalent is $f$', 'upon $A$ is $f$', 'nest is $N$', 'wellordered is $N$', 'strung is $N$', 'ordinal is $N$', 'Induced $Rxy\,\underline{\mathrm{u}}'\,xy$', and many others. This Kronecker character of each of these concepts derives from the Kronecker character of the membership relation, and thus the forms of Chapter 1, Logic, do not generally have this character. However, if the components of a logical statement have the Kronecker character, then that character will be preserved by the logical connectives. For example, if '$p$' and '$q$' are replaced by formulas having the Kronecker character, then the same replacements in '$(p \wedge q)$', '$(p \vee q)$', and '$(p \to q)$' will also have the character.

**Union and Intersection of a Family.**

Morse used the notations '$\nabla A$' and '$\sqcap A$' to represent, respectively, the union and intersection of the set $A$. The symbol '$\nabla$' is a stylized Greek letter sigma and the symbol '$\sqcap$' is a stylized Greek letter pi. Respectively, they suggest sum and product, which is a carry-over from the early notations of writing the union of two sets using the plus sign and the intersection of two sets as their product.

**Singletons.**

Morse defined '$\mathrm{sng}\,x$', read "singleton $x$", to denote the set with the sole member $x$, if $x$ is a point, and 0 otherwise. While this is the form of singleton that Morse used most often, he also defined a second form, '$\mathrm{sngl}\,x$', which denotes U in case $x$ is not a point. This second form is also denoted by '$\{x\}$'.

**The Classifier.**

In 2.33.0, the form '$\mathsf{E}\,x\,\underline{\mathrm{u}}x$' is used to refer to the set of all points $x$ such that $\underline{\mathrm{u}}x$ is true. The more common notation for this concept is '$\{x : \underline{\mathrm{u}}x\}$' which Morse included in 2.33.2, but never uses. The symbol '$\mathsf{E}$' for the classifier was used by Polish logicians and derives from the French word for set, *ensemble.* The form '$\mathsf{E}\,x\,\underline{\mathrm{u}}x$' is technically advantageous because the classifier '$\mathsf{E}$' can be designated as a notarian, making available the such-that and restricted variations of '$\mathsf{E}\,x\,\underline{\mathrm{u}}x$'. Thus, for example, '$\mathsf{E}\,x \in A\,\underline{\mathrm{u}}x$' means the set of all $x$ in $A$ such that $\underline{\mathrm{u}}x$ is true.

It is instructive to consider the Theorem of Classification (2.35)

$$(x \in \mathsf{E}\,x\,\underline{\mathrm{u}}x \leftrightarrow \underline{\mathrm{u}}x \wedge x \in \mathrm{U})\,.$$

In this theorem, the variable '$x$' appears in both free and bound positions. The confusion that is potentially generated could be avoided by the following formulation.

$$(y \in \mathsf{E}\,x\,\underline{\mathrm{u}}x \leftrightarrow \underline{\mathrm{u}}y \wedge y \in \mathrm{U})$$

While this may be easier to read, it is not directly usable in Morse's system. The reason for this is that his rule of inference, 0.27 Schematic Substitution, (and more fundamentally the concept of schematic replacement upon which it is based) requires all instances of the schemator (in this case '$\underline{\mathrm{u}}$') to appear in the same schematic expression. In this second formulation, '$\underline{\mathrm{u}}\,x$' and '$\underline{\mathrm{u}}\,y$' are different schematic expressions, and therefore schematic substitution cannot be applied.

It might be noted that one of the axioms of choice, namely Axiom 2.5.8, does not exhibit schematic uniformity, and thus is not directly amenable to schematic replacement. In 2.107.0 and 2.107.1, there are schematically uniform versions that are consequences of the axioms.

The theorem of classification is similar to the following axiom-scheme of Kelley.†

"**II Classification axiom-scheme** An axiom results if in the following '$\alpha$' and '$\beta$' are replaced by variables, '$A$' by a formula $\mathcal{A}$ and '$B$' by the formula obtained from $\mathcal{A}$ by replacing each occurrence of the variable which replaced $\alpha$ by the variable which replaced $\beta$:

For each $\beta$, $\beta \in \{\alpha : A\}$ if and only if $\beta$ is a set and $B$."

Beside the fact that the second occurrences of '$\alpha$' and '$\beta$' should be enclosed in single quotes, Kelley failed to require that the variable that relaces '$\beta$' is free in $\mathcal{A}$. Consider the following example. Suppose '$x$' replaces '$\alpha$', '$y$', replaces '$\beta$', and 'For some $y, y \neq x$' replaces '$A$'. In this example, '$B$' is replaced by 'For some $y$, $y \neq y$', and the alleged axiom is

For each $y$, $y \in \{x :$ For some $y$, $y \neq x\}$ if and only if $y$ is a set and For some $y$, $y \neq y$.

The classifier is used to define the set of all subsets of a set $A$ and the set of all supersets of $A$.

$$(\mathrm{sb}\, A \equiv \mathsf{E}\, B(B \subset A))$$
$$(\mathrm{sp}\, A \equiv \mathsf{E}\, B(A \subset B))$$

**Ordered Pairs.**

In 2.57.1, Morse defined an ordered pair with several interesting properties, the most significant of which is that it can be applied to all sets, not just points. As a result, was able to state theorem 2.61.2.

$$(a, b = c, d \leftrightarrow a = c \wedge b = d)\ .$$

That is, two ordered pairs are equal if and only if their first coordinates are equal and their second coordinates are equal. This theorem is also interesting from a Theory of Notation perspective, because the parentheses around the ordered pairs are missing. The parentheses in the ordered pair notation are usually considered an intrinsic part of the notation. Here the parentheses are merely grouping symbols, which may be omitted in some circumstances according to the Theory of Notation. If Morse had not included the comma of the ordered pair as one of his binarians, then such dropping of parentheses would not be possible. However, the ordered pair plays a significant role in the Theory of Notation regarding notarians, so that it is advantageous for the comma to be a binarian. Morse also defined the forms '$\mathrm{crd}'\, p$' and '$\mathrm{crd}''\, p$' which allow the extraction of the first and second coordinates from an ordered pair $p$.

**Indefinite and Definite Descriptions.**

One of the primitive forms in the set theory chapter is the indefinite description '$\mathrm{An}\, x\, \underline{\mathrm{u}}\, x$', which is to be read "an $x$ such that $\underline{\mathrm{u}}\, x$ is true." One of the axioms provides that if $\underline{\mathrm{u}}\, x$ is true for some $x$, and if $(y = \mathrm{An}\, x\, \underline{\mathrm{u}}\, x)$, then $\underline{\mathrm{u}}\, y$ is true. Another axiom provides that if there is no $x$ for which $\underline{\mathrm{u}}\, x$ is true, then $(\mathrm{An}\, x\, \underline{\mathrm{u}}\, x = \mathrm{U})$. Herein the first of these cases will be referred to as the legitimate case and the second as the anomalous case. The legitimate case may also result in $(\mathrm{An}\, x\, \underline{\mathrm{u}}\, x = \mathrm{U})$. For example, $(\mathrm{An}\, x(x = \mathrm{U}) = \mathrm{U})$.

In 2.71.0 Morse made the following definition

† J. L. Kelley, *General Topology*, Springer-Verlag, 1975, p. 253.

$$(\text{One}\,x\,\underline{\mathrm{u}}x \equiv \bigvee y \bigwedge x(0 \in \underline{\mathrm{u}}x \leftrightarrow x = y))^{\dagger}$$

so that '$\text{One}\,x\,\underline{\mathrm{u}}x$' means there is exactly one $x$ such that $\underline{\mathrm{u}}x$ is true, i.e., $(0 \in \underline{\mathrm{u}}x)$. In 2.71.2 the definite description '$\text{The}\,x\,\underline{\mathrm{u}}x$' is defined as follows.

$$(\text{The}\,x\,\underline{\mathrm{u}}x \equiv (\text{One}\,x\,\underline{\mathrm{u}}x \rightarrow \bigwedge x;\ \underline{\mathrm{u}}xx)).$$

The definite description also has a legitimate case and an anomalous case. In the legitimate case, there is exactly one $x$ such that $\underline{\mathrm{u}}x$ is true. In this case

$$(\text{One}\,x\,\underline{\mathrm{u}}x = \mathrm{U})$$

and

$$\begin{aligned}(\text{The}\,x\,\underline{\mathrm{u}}x &= (\text{One}\,x\,\underline{\mathrm{u}}x \rightarrow \bigwedge x;\ \underline{\mathrm{u}}xx)\\ &= (\mathrm{U} \rightarrow \bigwedge x;\ \underline{\mathrm{u}}xx)\\ &= \sim\mathrm{U} \cup \bigwedge x;\ \underline{\mathrm{u}}xx\\ &= 0 \cup \bigwedge x;\ \underline{\mathrm{u}}xx\\ &= \bigwedge x;\ \underline{\mathrm{u}}xx).\end{aligned}$$

Thus in the legitimate case, $\text{The}\,x\,\underline{\mathrm{u}}x$ is equal to the intersection of all $x$ such that $\underline{\mathrm{u}}x$ is true, i.e. the $x$ such that $\underline{\mathrm{u}}x$ is true.

In the anomalous case, there is not exactly one $x$ such that $\underline{\mathrm{u}}x$ is true. That is, either there is no $x$ such that $\underline{\mathrm{u}}x$ is true, or there is more than one $x$ such that $\underline{\mathrm{u}}x$ is true. In either event, $(\text{One}\,x\,\underline{\mathrm{u}}x = 0)$ and $(\text{The}\,x\,\underline{\mathrm{u}}x = \mathrm{U})$. Once again, it is possible that a legitimate case will also result in $(\text{The}\,x\,\underline{\mathrm{u}}x = \mathrm{U})$. For example, $(\text{The}\,x(x = \mathrm{U}) = \mathrm{U})$. However, for the definite description, a further oddity arises, namely that the predicate may be true for the definite description in the anomalous case. For example, $(\text{The}\,x(x = x) = \mathrm{U})$ because there is not a unique $x$ such that $(x = x)$. However, $(\mathrm{U} = \mathrm{U})$.

What is known about both the indefinite and definite descriptions is that if the result is not the universe, then we have a legitimate case. However, the fact that the legitimate case and the anomalous case can return the same value suggests the possibility of ambiguity. This issue is revisited at the end of this discussion of descriptions.

Theorem 2.75.2 says

$$(\text{The}\,x\,\underline{\mathrm{u}}x = (\text{One}\,x\,\underline{\mathrm{u}}x \rightarrow \text{An}\,x\,\underline{\mathrm{u}}x))$$

which gives a very natural alternative way to define the definite description. This was not done here to make it clear that definite descriptions in no way depend on an axiom of choice. A similar technique can be used to define definite description in terms of indefinite description in a nonunified set theory. Below is a nonunified version of $\text{One}\,x\,\underline{\mathrm{u}}x$ followed by this alternate definition.

$$(\text{One}\,x\,\underline{\mathrm{u}}x \equiv \bigvee y \bigwedge x(\underline{\mathrm{u}}x \leftrightarrow x = y))$$
$$(\text{The}\,x\,\underline{\mathrm{u}}x \equiv \text{An}\,x(\text{One}\,x\,\underline{\mathrm{u}}x \wedge \underline{\mathrm{u}}x))$$

The crucial issue regarding descriptions is the treatment of the anomalous cases. These cases have historically been a challenge for formal treatments of descriptions, and it may add perspective to see some of the different ways these cases have been handled.

Hilbert and Bernays introduced an indefinite description operator, one that for any predicate of $x$, $\underline{\mathrm{u}}x$, would return an object $y$ such that $\underline{\mathrm{u}}y$, whenever some such object exists. They used the symbol '$\varepsilon$' for their

† In order to ensure that '$\text{One}\,x\,\underline{\mathrm{u}}x$' has the Kronecker character mentioned earlier, it is necessary to use '$0 \in \underline{\mathrm{u}}x$' in the definition instead of just '$\underline{\mathrm{u}}x$'. All other statement-like forms in Chapters 3 through 10 achieve their Kronecker character without need of this ad hoc device.

operator,[†0] but the examples in this discussion will use the symbol 'An' as found in this text. Before a term of the form 'An $x\,\underline{u}\,x$' could be used, they required that a theorem of the type

$$\bigvee x\,\underline{u}\,x$$

be established. The definite description[†1] was introduced in their text prior to introducing the indefinite description. Paralleling their treatment of indefinite description, they required that two unicity theorems equivalent to a single theorem of the type

$$\mathrm{One}\, x\,\underline{u}\,x$$

be established before a term of the form 'The $x\,\underline{u}\,x$' could be used.

The approach of Hilbert and Bernays will be referred to herein as *conditional.* In placing restrictions on the usage of the indefinite and definite description, Hilbert and Bernays made the syntax of their language dependent on mathematical facts. Such restrictions are far too crippling to enable descriptions to be useful in mathematics. Descriptions so restricted can apply only to very specific cases, whereas in mathematics, it is necessary to be able to use descriptions in situations where the existence is uncertain, such as in limits, sums, derivatives, and integrals. The application of descriptions to limits is discussed below. Furthermore, these restrictions have serious consequences for the syntax of the language. As Quine notes, "It is awkward, in general, to let questions of meaningfulness rest upon casual matters of fact which are not open to any systematic and conclusive method of decision."[†2]

Bourbaki followed Hilbert and Bernays in using indefinite descriptions in logic, and like them, Bourbaki did not use the indefinite description to define the definite description in the manner indicated above. In fact, Bourbaki does not have a definite description at all. Bourbaki is content to note that if a relation is functional (i.e. One $x\,\underline{u}\,x$), then An $x\,\underline{u}\,x$ equals the unique $x$ such that $\underline{u}\,x$ is true.

However, Bourbaki's treatment differs from that of Hilbert and Bernays in that there is no restriction on the use of the indefinite description. Thus for Bourbaki the indefinite description may appear in either a legitimate or anomalous situation. Bourbaki provides no axiom to clarify what the result will be in the anomalous cases. Further comments on Bourbaki's treatment are given in the discussion of limits below.

In Bourbaki's treatment, the question of what happens in the anomalous cases has been left unanswered. In Morse's treatment, every anomalous case has the universe as its value. Thus simply stating that the description is not equal to the universe implies that the description is legitimate. The only means for Bourbaki to assert that a description is legitimate is to state $\bigvee x\,\underline{u}\,x$ for the indefinite description.

The use of an indefinite description, as in Hilbert and Bernays and Bourbaki, has not attracted many adherents, perhaps because of the implications it has for the Axiom of Choice. Perhaps this has also resulted in the lack of followers of this approach even for the definite description. It is the approach of Whitehead and Russell that has been more influential. Their treatment in *Principia Mathematica* was based on avoiding the anomalous cases through the use of a "contextual definition." as indicated below.

> "If $\phi\hat{x}$ is a propositional function, the symbol "$(\iota x)(\phi x)$" is used in our symbolism in such a way that it can always be read as "the $x$ which satisfies $\phi\hat{x}$." But we do not define "$(\iota x)(\phi x)$" as standing for "the $x$ which satisfies $\phi\hat{x}$," thus treating this last phrase as embodying a primitive idea. Every use of "$(\iota x)(\phi x)$," where it apparently occurs as a constituent of a proposition in the place of an object, is defined in terms of the primitive ideas already at hand."[†3]

Consider the definite description 'The $x\,\underline{u}\,x$'. According to Whitehead and Russell, any statement about

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†0 D. Hilbert and P. Bernays, *Grundlagen der Mathematik II*, Springer-Verlag 1970, p,11

†1 D. Hilbert and P. Bernays, *Grundlagen der Mathematik I*, Springer-Verlag, 1968, pp. 392-393

†2 W. Quine, *Mathematical Logic*, Harper and Row (1964), p. 147.

†3 A. N. Whitehead and B. Russell, *Principia Mathematica to ∗56, 2nd Edition, 1927*, Cambridge University Press, 1967, p. 30

this description, e.g. '$\underline{\mathrm{v}}$ The $x\,\underline{\mathrm{u}}x$' is explained as follows

$$(\underline{\mathrm{v}}\ \mathrm{The}\, x\,\underline{\mathrm{u}}x \leftrightarrow \mathrm{One}\, x\,\underline{\mathrm{u}}x \wedge \bigvee x(\underline{\mathrm{u}}x \wedge \underline{\mathrm{v}}x))$$

For example

$$((\mathrm{The}\, x\,\underline{\mathrm{u}}x \in A) \leftrightarrow \mathrm{One}\, x\,\underline{\mathrm{u}}x \wedge \bigvee x(\underline{\mathrm{u}}x \wedge x \in A))$$

Consider the anomalous definite description 'The $x(x \neq x)$' and further consider the proposition containing this description

$$(\mathrm{The}\, x(x \neq x) = \mathrm{The}\, x(x \neq x))$$

According to Whitehead and Russell, we must analyze this proposition as follows.

$$(\mathrm{One}\, x(x \neq x) \wedge \bigvee x(x \neq x \wedge x = x))$$

But this is clearly false, apparently contradicting the general logical proposition

$$(a = a)$$

While this is not truly a logical contradiction because the statement containing the descriptions is not actually a statement of equality, merely an context, it is clearly the intent of Whitehead and Russell to try to preserve our basic understandings of equality and other operators, and this violates that basic understanding.

The approach of Whitehead and Russell will be referred to herein as *contextual.* Kleene discusses this approach with the following comments.

> "In ordinary discourse, improper descriptions hardly occur. If someone speaks about "the $w$ such that $F(w)$" when there isn't a unique such $w$, we usually conclude that he is either confused himself or is attempting to mislead us. We might try to say that any sentence A containing an improper description is simply false. But then we run into the difficulty that ¬A and A ⊃ B would also be false by this criterion, whereas (by our truth tables for ¬ and ⊃) ¬ A and A ⊃ B must be true when A is false. Whitehead and Russell 1910 pp. 69-75 (66-71 in the 1925 ed,; also it's in van Heijenoort 1967) handled this difficulty by requiring the part of a sentence which is declared false because it contains an improper description to be indicated; the truth or falsity of larger parts containing it then to be determined by the usual rules. There is some awkwardness in this; and the problem of the best way to treat descriptions allowing improper ones is still a subject of research.... However, for the brief treatment we give here, we shall follow common usage in avoiding improper descriptions.
>
> But then we have another difficulty, that what a sentence is (or in our treatment of logic, what a formula is) will not always be determinable immediately from the way it is assembled out of its parts (as in our definitions of "formula" ... , but will depend on validity or provability (or valid consequence or deducibility) results."†

Kleene, while criticizing the approach of Whitehead and Russell, falls back on the conditional approach of Hilbert and Bernays. As indicated in Quine's earlier quote, the intrusion of semantics into the realm of syntax is problematic. Bernays, himself, later commented

> "Following usual language one would be inclined to admit an expression $\iota_r A(r)$ as a term only if the unicity formulas $((Er)A(r)$ and $(r)(n)(A(r)\,\&\,A(n) \rightarrow r = n))$ are proved, as it has been done in [Hilbert and Bernays 1934/39]. To this measure there has been objected that the concept of term is complicated by it in a way that in general there is no decision procedure

† S. C. Kleene, *Mathematical Logic*, Dover 2002, p.168