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Alma L. Albujer · Magdalena Caballero ·
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Developments in Lorentzian Geometry

GeLoCor 2021, Cordoba, Spain, February 1–5

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Preface

In 2001, researchers from several universities with a common interest in Lorentz Geometry met on Benalmádena in what was called “Meeting on Lorentzian Geometry”. After this first and successful meeting, the organizers decided to make this one the first of a biennial series of conferences devoted to presenting and discussing the latest advances on Lorentzian Geometry. Since then, what ended up being called *International Meeting on Lorentzian Geometry* has grown at an impressive pace. Currently, ten regular meetings have been held: Benalmádena 2001 (Spain), Murcia 2003 (Spain), Castelldefels 2005 (Spain), Santiago de Compostela 2007 (Spain), Martina Franca 2009 (Italy), Granada 2011 (Spain), São Paulo 2013 (Brazil), Málaga 2016 (Spain), Warsaw 2018 (Poland) and Córdoba 2021 (Spain). Moreover, a special edition on *Lorentzian and conformal Geometry* was held in Greifswald (Germany) in 2014 in honour of Prof. Helga Baum.

The Department of Mathematics of the University of Córdoba had the pleasure to organize in 2021 the X International Meeting on Lorentzian Geometry (**GeLoCor**). Despite the new challenge presented by the COVID-19 pandemic, the meeting was a complete success from the point of view of the organization and the participation. Full information about **GeLoCor** meeting can be found at http://www.uco.es/gelocor/.[1]

Talks presented at the **GeLoCor** conference dealt with assorted topics in Lorentzian and Differential Geometry, Mathematical Relativity and Theoretical Physics. This assortment has been carried over to this Proceedings Book where the interested reader will be able to find contributions representing the subjects just mentioned.

In the realm of Lorentzian Geometry, the contribution of **Amir B. Aazami** classifies the Lorentzian metrics in dimension 3 admitting a timelike Killing vector field using a 3-dimensional version of the Newman-Penrose formalism. He also considers the global existence of 3-dimensional Lorentzian manifolds whose Ricci tensor has a prescribed algebraic structure.

[1] A YouTube channel with a selection of the presentations is available at https://www.youtube.com/channel/UCrSEpDE_tgfZ-dqUsJ5d9Tw.

Also in the context of Lorentzian geometry, **Manuel Gutiérrez** and **Benjamín Olea** obtain conditions for a totally umbilic null hypersurface of a Lorentzian manifold to be contained in a *generalized null cone*. In addition, it is proven when a co-dimension 2 submanifold of a null hypersurface of a Lorentzian manifold is a leaf of a integrable *screen distribution* constructed from a *rigging vector* related to the null hypersurface.

Continuing with the study of null hypersurfaces of Lorentzian manifolds, the contribution of **Matias Navarro**, **Oscar Palmas** and **Didier A. Solis** considers the geometry of null hypersurfaces assuming the *screen conformal*, *screen quasi-conformal*, *null screen isoparametric* and *null Einstein* hypotheses applied to the generalized Robertson-Walker spacetimes.

In the field of Differential Geometry, **Dmitri Alekseevsky**, **Vicente Cortés**, and **Thomas Leistner** study semi-Riemannian cones admitting a parallel totally isotropic distribution of rank two. They establish the existence of two canonical vector fields on the base manifold of the cones satisfying a prescribed system of differential equations. They use the existence conditions of these canonical vector fields to obtain a local characterization of the afore-said cones.

Naoya Ando proves that an almost nilpotent structure of an oriented neutral 4-dimensional manifold is parallel if and only if its corresponding section is horizontal with respect to the connection induced by the Levi-Civita connection of the neutral manifold.

Also within Differential Geometry, **María Ferreiro-Subrido** presents a number of results about Bochner-flat para-Kähler surfaces. She computes the covariant derivative of the associated almost paracomplex structure, constructs Bochner-flat para-Kähler surfaces with non-constant scalar curvature and gives a characterization of a restricted family of Bochner-flat para Kähler surfaces.

Miguel A. Javaloyes and **Enrique Pendás-Recondo** explain how the notion of *null hypersurface* used in Lorentzian Geometry can be extended to the framework of cone structures and Lorentz-Finsler spaces. They pay close attention to properties like smoothness, foliations by cone geodesics and time minimization.

Miguel A. Javaloyes, **Miguel Sánchez** and **Fidel Villaseñor** study the relation between *anisotropic connections*, the *metric non-linear connection* and *Finsler connections* for pseudo-Finsler spaces.

Adela Latorre and **Luis Ugarte** provide a condition guaranteeing that any *small deformation* of a compact pseudo-Kähler manifold is cohomologically pseudo-Kähler.

To close the contributions devoted to Differential Geometry, **Andrea Seppi** and **Enrico Trebeschi** perform a very detailed study of the *half-space* model of the *pseudo-hyperbolic space* defined in a general pseudo-Riemannian manifold. Their study include the analysis of the geodesic equations, the classification of the totally geodesic submanifolds and the description of the isometry group.

In the field of Mathematical Relativity, **Gregory J. Galloway** and **Eric Ling** obtain new existence results for constant mean curvature Cauchy hypersurfaces in spacetimes of arbitrary dimension in the context of Dilts and Holst conjecture.

Melanie Graf and **Christina Sormani** show how to construct area and volume estimates for spacetimes with only mild assumptions on energy conditions. The estimates are very general and provide an important tool for the analysis of the convergence of data developments obtained from a converging sequence of initial data sets.

Still within the realm of Mathematical Relativity, **Stacey G. Harris** presents a detailed description of his project to characterize the future causal boundary of spacetimes whose main property is the existence of a *reasonable* class of observers. In particular she provides a set of observable conditions which should guarantee that the future causal boundary is spacelike.

To finish with the contributions dealing with Mathematical Relativity, **Philippe G. LeFloch** presents techniques developed by him and his collaborators to study the global properties of solutions of Einstein-matter systems. These techniques have applications to the study of solutions of the Einstein field equations representing *cyclic cosmologies*.

In Theoretical Physics, **Rodrigo Ávalos** presents work of him and his collaborators about the definition of energy for certain fourth-order gravity theories, its positivity properties and its relation to *Q-curvature* in the conformally invariant case.

Martín de la Rosa presents some new results regarding curves which are critical points of the action determined either by the curvature or by the torsion in certain 3-dimensional spacetimes (namely, generalized FLRW and static spacetimes).

Finally **Ángel Murcia** introduces the concept of *ε-contact metric structures*, investigates some of their properties and uses them to construct solutions of six-dimensional supergravity. Special attention is paid to a particular class of ε-contact structures, referred to as *null contact structures*, which have not been considered in the preceding literature. This contribution combines topics of both differential geometry and Theoretical Physics.

We believe that the contributions just described provide a timely snapshot of important current research topics, thus, making the present volume of interest to researchers and students.

We thank all those who made possible this volume: first of all the contributors whose work was already summarized in the previous paragraphs and whose complete scientific results are to be unveiled in the pages lying ahead. In addition, the anonymous referees have played an important role behind the scenes to shape the final form of all the contributions and render a volume of high scientific rigour. Finally, Springer Nature has given us the chance to publish the contents of this volume in their *Springer Proceedings in Mathematics and Statistics* book series. It has been for

us a honour and a pleasure to put together all the pieces that make this volume and we hope it will aid future scientific research.

Córdoba, Spain
January 2022

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Contents

Semi-Riemannian Cones with Parallel Null Planes

Dmitri Alekseevsky, Vicente Cortés, and Thomas Leistner

Abstract We study semi-Riemannian cones admitting a parallel totally isotropic distribution of rank two. We give a local classification of the base manifolds of such holonomy.

Keywords Pseudo-Riemannian manifolds · Metric cones · Special holonomy

1 Introduction

By Gallot's theorem [9] the cone over any complete Riemannian manifold is either flat or irreducible. Moreover, the irreducible cones are Ricci-flat and the possible holonomy groups can be easily read off from Berger's classification [4].

In the semi-Riemannian setting, the systematic study of this circle of ideas was initiated in [1] and the situation turned out to be considerably more involved. First of all, for indefinite metrics one needs to replace the notion of irreducibility by indecomposability. A semi-Riemannian manifold is called *indecomposable* if its holonomy representation does not admit any proper non-degenerate invariant subspace. In the Riemannian setting the notions of indecomposability and irreducibility coincide. By the splitting theorems of de Rham [5] and Wu [14], indecomposable semi-Riemannian manifolds do not admit a decomposition as a semi-Riemannian

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product. A classification of indecomposable holonomy groups is only known in the Lorentzian case [3, 10] and for small index under extra assumptions such as a parallel Kähler structure of index 2 [6, 7]. Gallot's theorem can then be generalized to the statement that the cone over a compact semi-Riemannian manifold is either flat or indecomposable, compare [1, Theorem 6.1] and [11, Proposition 4.1]. However the cone over a non-compact complete semi-Riemannian manifold can be decomposable and the geometry of the base manifolds of decomposable cones was described in [1] and [12].

More recently, the holonomy and geometric structure of indecomposable cones was studied in detail in [2]. Any such cone is either irreducible or admits a parallel totally isotropic distribution. Note that the rank of the distribution is bounded by the index of the cone metric. The irreducible case and the case when the distribution is of rank one were covered in [2]. In this paper we consider the situation in which the parallel totally isotropic distribution is of rank two. Our main result is a local classification of the most general form of the metric of the base manifold of the cone.

2 The Induced Structure on the Base

Let (M, g) be a semi-Riemannnian manifold, where we assume $\dim M > 1$ to exclude trivial cases. The *time-like cone over the base* (M, g) or just the *cone over* (M, g) is the manifold $\widehat{M} := \mathbb{R}^+ \times M$ with the metric

$$\widehat{g} := -\mathrm{d}r^2 + r^2 g. \tag{1}$$

We denote by

$$\xi = r\frac{\partial}{\partial r}$$

the *Euler vector field*. The Levi-Civita connection $\widehat{\nabla}$ of $\widehat{g}$ reduces to the Levi-Civita connection ∇ of g in the following way

$$\widehat{\nabla}\xi = \mathrm{Id}, \qquad \widehat{\nabla}_X Y = \nabla_X Y + g(X, Y)\xi, \tag{2}$$

where here and in the following formulas $X, Y, Z \in \mathfrak{X}(M)$.

Since we do not make any assumption about the signature of the base manifold, the following also applies to spacelike cones by multiplying a spacelike cone metric by -1 to obtain a time-like cone.

If a semi-Riemannian manifold $(\widehat{M}, \widehat{g})$ admits a a parallel totally null 2-plane bundle $\widehat{\mathbf{P}}$, then locally there are two null vector fields χ and ζ that are orthogonal to each other and such that

$$\widehat{\nabla}\chi = \alpha \otimes \chi + \mu \otimes \zeta, \qquad \widehat{\nabla}\zeta = \beta \otimes \chi + \nu \otimes \zeta, \tag{3}$$

for 1-forms α, β, μ and ν.

If $(\widehat{M}, \widehat{g})$ is a timelike cone with a parallel null 2-plane bundle $\widehat{\mathbf{P}}$, we can intersect $\widehat{\mathbf{P}}$ with $\xi^\perp$, where ξ is the Euler vector field. A subset of $\widehat{M} = \mathbb{R}^{>0} \times M$ will be called *conical* if it is of the form $\widehat{M}_0 = \mathbb{R}^{>0} \times M_0$ for some subset $M_0 \subset M$.

Lemma 1 *On a conical open dense subset in $\widehat{M}$ the intersection $\widehat{\mathbf{P}} \cap \xi^\perp$ is a null-line bundle $\mathbf{L}$ invariant under the flow of ξ. In particular, $\mathbf{L}$ admits local sections, defined on conical open sets, invariant under the flow of ξ and descends to a null line distribution on an open dense subset of M.*

Proof For this and the following proofs, we note that

$$\left[\xi, \Gamma(\xi^\perp)\right] \subset \Gamma(\xi^\perp) \quad \text{and} \quad \left[\xi, \Gamma(\widehat{\mathbf{P}})\right] \subset \Gamma(\widehat{\mathbf{P}}).$$

This implies that the dimension of the fibres of $\widehat{\mathbf{P}} \cap \xi^\perp$ is constant on the integral curves of ξ. At each point $p \in \widehat{M}$, $\xi^\perp|_p$ is a hyperplane and $\widehat{\mathbf{P}}|_p$ a 2-plane in $T_p\widehat{M}$. Hence their intersection has dimension one or two. Now let us assume that, over an open set $U \subset \widehat{M}$, $\widehat{\mathbf{P}} \cap \xi^\perp$ is of rank 2, i.e. that $\widehat{\mathbf{P}} \subset \xi^\perp$. Hence $\widehat{\mathbf{P}} \cap \xi^\perp$ a distribution of 2-planes spanned by vector fields V_1 and V_2 on U that are tangential to M. Then formulae (2) and (3) give us

$$TM \ni \widehat{\nabla}_X V_i = \nabla_X V_i + g(X, V_i)\xi,$$

for all $X \in TM$. Hence, on U it is $g(X, V_i) = 0$ for all $X \in TM$ which is impossible. Consequently, the conical open set over which the fibres of $\widehat{\mathbf{P}} \cap \xi^\perp$ are one-dimensional is dense and $\widehat{\mathbf{P}} \cap \xi^\perp$ restricts to a line bundle $\mathbf{L}$ over that set. □

Now we project $\widehat{\mathbf{P}}$ to $\xi^\perp$.

Lemma 2 *The projection $\mathrm{pr}_{\xi^\perp}(\widehat{\mathbf{P}}) \subset \xi^\perp$ is an involutive 2-plane distribution $\mathbf{P}$ on $\widehat{M}$ and descends to an involutive 2-plane distribution on M_0.*

Proof First note that the fibres of $\mathrm{pr}_{\xi^\perp}(\widehat{\mathbf{P}})$ have dimension 2 because $\widehat{\mathbf{P}} \cap \mathbb{R} \cdot \xi = \{0\}$. Hence, $\mathbf{P} := \mathrm{pr}_{\xi^\perp}(\widehat{\mathbf{P}}) \subset \xi^\perp$ is a 2-plane distribution.

Clearly the projection of a vector field V on $\widehat{M}$ to $\xi^\perp$ is given as

$$\mathrm{pr}_{\xi^\perp}(V) = V + r^{-2}\widehat{g}(V, \xi)\xi.$$

By a calculation using $\widehat{\nabla}\xi = \mathrm{Id}$ we obtain for all $V_1, V_2 \in \mathfrak{X}(\widehat{M})$:

$$\left[\mathrm{pr}_{\xi^\perp}(V_1), \mathrm{pr}_{\xi^\perp}(V_2)\right] = \mathrm{pr}_{\xi^\perp}\left([V_1, V_2] + r^{-2}\widehat{g}(V_2, \xi)[V_1, \xi] - r^{-2}\widehat{g}(V_1, \xi)[V_2, \xi]\right).$$

Since the distribution $\widehat{\mathbf{P}}$ is invariant under ξ, parallel and hence involutive, the right-hand side is a section of $\mathbf{P}$ for all sections V_1, V_2 of $\widehat{\mathbf{P}}$. This proves the involutivity of $\mathbf{P}$. The distribution $\mathbf{P}$ descends to M due to the invariance under ξ. □

Moreover we obtain:

Lemma 3 *There exist local sections V of* $\mathbf{L}$ *and Z of* $\mathbf{P}$, *defined on a conical open set, such that V and*

$$\zeta = \xi + Z$$

locally span $\widehat{\mathbf{P}}$ *and satisfy*

$$[\xi, V] = 0 \text{ and } [\xi, Z] = 0.$$

The vector fields V and Z descend to local vector fields on M.

Proof We have already seen that there exists a non-vanishing section V of $\mathbf{L}$ over a conical open set such that $[\xi, V] = 0$. In the following we always work locally over conical open sets. Every section of $\widehat{\mathbf{P}}$ that is nowhere a multiple of V is of the form $f\xi + Z$ for Z a (possibly vanishing) local section of $\mathbf{P}$ and f a non-vanishing local function on $\widehat{M}$. Hence, by multiplying with $1/f$ we can assume that we have a section

$$\hat{\zeta} = \xi + \hat{Z}$$

of $\widehat{\mathbf{P}}$. We will now use the freedom to add multiples of V to $\hat{Z}$ without leaving $\widehat{\mathbf{P}}$, in order to find a $Z = \hat{Z} + \varphi V$ for which we have $[\xi, Z] = 0$. Indeed, writing

$$\nabla_\xi \hat{\zeta} = fV + h\hat{\zeta}$$

with functions f and h, we compute

$$[\xi, \hat{Z}] = [\xi, \hat{\zeta}] = fV + (h-1)\hat{\zeta}.$$

Since $[\xi, \hat{Z}]$ belongs to $\xi^\perp$, we must have that $h \equiv 1$ and

$$[\xi, \hat{Z}] = fV.$$

Now if we fix a solution φ of

$$\mathrm{d}\varphi(\xi) + f = 0,$$

and set $Z = \hat{Z} + \varphi V$ we get

$$[\xi, Z] = 0.$$

Clearly, since V is a section of $\widehat{\mathbf{P}}$, the vector field

$$\zeta := \xi + Z = \hat{\zeta} + \varphi V,$$

is also a section in $\widehat{\mathbf{P}}$ that is still linearly independent of V and therefore Z is a section of $\mathbf{P}$ that locally descends to M. □

Theorem 1 *Let $(\widehat{M}, \widehat{g})$ be a timelike cone over a semi-Riemannian manifold (M, g). If the cone admits a parallel distribution of totally null 2-planes field, then the base (M, g) admits locally two vector fields V and Z such that*

$$g(V, V) = 0, \quad g(Z, Z) = 1, \quad g(V, Z) = 0, \tag{4}$$

and

$$\nabla_X V = \alpha(X)V + g(X, V)Z, \tag{5}$$

$$\nabla_X Z = -X + \beta(X)V + g(X, Z)Z, \tag{6}$$

for all $X \in TM$, with 1-forms α and β on M.

Conversely, each pair of vector fields V and Z on M satisfying relations (4), (5) and (6) defines a parallel distribution of totally null 2-planes on the cone.

Proof First assume that the cone admits a parallel totally null 2-plane $\widehat{\mathbf{P}}$ which is spanned by V and $\zeta = \xi + Z$ as in Lemma 3. Equation (4) are implied by $\widehat{\mathbf{P}}$ being totally null. Moreover, Eq. (3) with $\chi = V$ and $X \in TM$ become

$$\widehat{\nabla}_X V \;=\; \nabla_X V + g(X, V)\xi = \alpha(X)V + \mu(X)(\xi + Z), \tag{7}$$

$$\widehat{\nabla}_X \zeta \;=\; X + \nabla_X Z + g(X, Z)\xi = \beta(X)V + \nu(X)(\xi + Z), \tag{8}$$

and imply

$$\mu(X) = g(X, V),$$
$$\nu(X) = g(X, Z),$$

as well as Eqs. (5) and (6), but still with r-dependent 1-forms α and β. Hence, it remains to show that α and β, when restricted to $\xi^{\perp}$, are invariant under the flow of ξ and therefore descend to 1-forms on M, i.e., that

$$\mathcal{L}_\xi \alpha|_{\xi^\perp} = \mathcal{L}_\xi \beta|_{\xi^\perp} = 0.$$

But from of Eq. (7) we get

$$\begin{aligned} 0 &= \hat{R}(\xi, X)V \\ &= (\mathcal{L}_\xi \alpha)(X)V + \alpha(X)V + g(X, V)(\xi + Z) - (\nabla_X V + g(X, V)\xi) \\ &= (\mathcal{L}_\xi \alpha)(X)V. \end{aligned}$$

This proves that $\mathcal{L}_\xi \alpha|_{\xi^\perp} = 0$. Analogously we get

$$0 = \hat{R}(\xi, X)\zeta = (\mathcal{L}_\xi \beta)(X)V$$

and again $\mathcal{L}_\xi \beta|_{\xi^\perp} = 0$.

Conversely, if we start with a manifold (M, g) and vector fields satisfying conditions (4)–(6), a straightforward computations shows that the cone admits a parallel null plane spanned by V and $\xi + Z$. □

Corollary 1 *If the cone (1) admits a distribution of parallel totally null 2-planes, then the base (M, g) admits locally a geodesic, shearfree null vector field V.*

Proof Since V is null, Eq. (5) implies that V is geodesic. Recall that a geodesic null vector field is called *shearfree* if

$$\mathcal{L}_V g = \lambda g + \theta \cdot V^\flat,$$

with a function λ and a 1-form θ and where the dot stands for the symmetric product. From (5) and the formula

$$\mathcal{L}_X g = 2(\nabla X^\flat)^{\mathrm{sym}}, \tag{9}$$

where 'sym' denotes the projection onto the symmetric part, we compute

$$\mathcal{L}_V g = 2(\alpha + Z^\flat) \cdot V^\flat,$$

i.e., the shear free condition is satisfied with $\lambda = 0$. □

Remark 1 We can change the basis of $\mathrm{span}(V, Z)$ to V', Z' such that V' is still null and orthogonal to Z' and such that Z' is a unit vector field,

$$(V, Z) \longmapsto (V' = \mathrm{e}^f V, Z' = Z + hV).$$

Then the 1-forms α and β transform as

$$\begin{aligned}
\alpha &\longmapsto \alpha' = \alpha + \mathrm{d}f - hV^\flat, \\
\beta &\longmapsto \beta' = e^{-f}(\beta + h\alpha + \mathrm{d}h - hZ^\flat - h^2 V^\flat).
\end{aligned}$$

3 Consequences of the Fundamental Equations

Let (M, g) be a semi-Riemannian manifold endowed with two pointwise linearly independent vector fields V, Z which satisfy (4)–(6).

Proposition 1 *The fundamental equations (4)–(6) imply*

$$dV^\flat = (\alpha - Z^\flat) \wedge V^\flat, \tag{10}$$

$$dZ^\flat = \beta \wedge V^\flat, \tag{11}$$

$$[Z, V] = (\alpha(Z) - \beta(V) + 1)V, \tag{12}$$

$$\mathcal{L}_V g = 2(\alpha + Z^\flat)V^\flat, \tag{13}$$

$$\mathcal{L}_Z g = -2g + 2(Z^\flat)^2 + 2\beta V^\flat, \tag{14}$$

where we are using the symmetric product of 1*-forms in the last two formulas.*

Proof Since ∇ is torsion-free, the differential of any 1-form φ is given by

$$\mathrm{d}\varphi(X, Y) = (\nabla_X \varphi)Y - (\nabla_Y \varphi)X, \quad X, Y \in \mathfrak{X}(M).$$

Now (10) and (11) follow immediately from (5) and (6). Using again that ∇ is torsion-free, the fundamental equations easily imply (12). Similarly, the last two formulas follow from (9). □

Corollary 2 *We have*

$$\mathcal{L}_V V^\flat = \alpha(V) V^\flat, \tag{15}$$
$$\mathcal{L}_Z V^\flat = (\alpha(Z) - 1)\, V^\flat, \tag{16}$$
$$\mathcal{L}_V Z^\flat = \beta(V) V^\flat. \tag{17}$$

The vector fields Z and V commute if and only if

$$\beta(V) = \alpha(Z) + 1. \tag{18}$$

Proof The first three formulas are obtained from Cartan's formula for the Lie derivative to the Eqs. (10) and (11). Alternatively one can use (12)–(14). The last assertion follows from Eq. (12). □

Corollary 3 *By multiplying V with a function we can locally assume that*

$$\mathrm{d}V^\flat = 0, \tag{19}$$

that is

$$\alpha = Z^\flat + f_\alpha V^\flat$$

for some function f_α. The latter equation implies

$$\alpha(Z) = 1, \quad \alpha(V) = 0, \quad \mathcal{L}_V V^\flat = 0, \quad \mathcal{L}_Z V^\flat = 0.$$

By adding a functional multiple of V to Z we can further locally assume that

$$\beta(V) = 2,$$

which implies $\mathcal{L}_V Z^\flat = 2V^\flat$ and is equivalent to $[Z, V] = 0$.

Proof By Eq. (10) and the Frobenius theorem, the hyperplane distribution $V^\perp$ is integrable, which locally implies that a functional multiple of $V^\flat$ is closed. The equations and the second statement follow from the transformation formulae for α and β in Remark 1 and Corollary 2. □

Corollary 4 *With the normalisation that* $\mathrm{d}V^\flat = 0$, *the leaves of the integrable distribution* $V^\perp$ *are totally geodesic and the vector field* V *preserves the tensor field* $g|_{V^\perp \times V^\perp}$.

Proof For $X, Y \in V^\perp$ we have

$$g(\nabla_X Y, V) = -g(Y, \nabla_X V),$$

and because of $\mathrm{d}V^\flat = 0$,

$$g(Y, \nabla_X V) = \frac{1}{2}(\mathcal{L}_V g)(X, Y).$$

Using Eq. (13) for $X, Y \in V^\perp$ we get $(\mathcal{L}_V g)(X, Y) = 0$ and hence $g(\nabla_X Y, V) = 0$, which means that the leaves of $V^\perp$ are totally geodesic. □

4 The Local Form of the Metric on the Base

In the following we will assume all of the above equations. By (19), locally, there exists a function u such that $\mathrm{d}u = V^\flat$. The function u is constant on each leaf L of the distribution $V^\perp$. Locally, we can decompose M as $M = L \times \mathbb{R}$, such that u corresponds to the coordinate on the $\mathbb{R}$-factor and the leafs of $V^\perp$ are the hypersurfaces $L_u = L \times \{u\}$. Since the vector fields V and Z commute and are tangent to $V^\perp$, we can further decompose each leaf of $V^\perp$ locally as $L_u \cong L = M_0 \times \mathbb{R} \times \mathbb{R}$, such that $V = \partial_t$, $Z = \partial_s$ are the coordinate vector fields tangent to the first and second $\mathbb{R}$-factor, respectively.

Let us denote by $\mathbf{P}$ the integrable distribution spanned by V and Z. Notice that by (11) the distribution $\mathbf{P}^\perp = Z^\perp \cap V^\perp$ is also integrable, in virtue of the Frobenius theorem. So we can assume that the level sets of s are tangent to $\mathbf{P}^\perp$. Finally, the decomposition $M = L \times \mathbb{R}$ can be chosen such that the decomposition $L_u = M_0 \times \mathbb{R} \times \mathbb{R}$ is independent of u, that is the vector field ∂_u commutes with V, Z and with the canonical lift of vector fields of M_0.

Theorem 2 *Let* (M, g) *be a semi-Riemannian manifold such that the cone* $(\widehat{M}, \widehat{g})$ *admits a parallel totally null distribution of* 2*-planes. In terms of the above local decomposition* $M = M_0 \times \mathbb{R}^3$ *we have*

$$g = \mathrm{d}s^2 + e^{-2s} g_0(u) + 2\,\mathrm{d}u\,\eta, \tag{20}$$

for some 1-form η *on* M *such that* $\eta(\partial_t)$ *is nowhere vanishing and a family of metrics* $g_0(u)$ *on* M_0 *depending on* u.

Proof The restriction of the metric to a leaf $N = M_0 \times \mathbb{R} \times \{(s, u)\}$ of $\mathbf{P}^\perp$ is degenerate with kernel $V = \partial_t \in \mathbf{P}^\perp$ and invariant under the flow of V, see (13). Since M_0 is transversal to V, we see that $g|_N = g_0(u, s)$ for some family of metrics on M_0 depending on u and s. The flow of $Z = \partial_s$ is a 1-parameter family of homotheties of weight -2, see (14). This shows that $g_0(u, s) = e^{-2s} g_0(u)$ for some 1-parameter family of metrics $g_0(u)$. It follows that on the leafs $L_u = M_0 \times \mathbb{R} \times \mathbb{R} \times \{u\}$ of $V^\perp$ the metric is of the form $\mathrm{d}s^2 + e^{-2s} g_0(u)$. Finally, on M we obtain the general form (20) with $\eta(\partial_t) \neq 0$, in view of the non-degeneracy of g. □

It remains to determine the necessary and sufficient conditions for the data $g_0(u)$ and η ensuring that the cone over (M, g) as in (20) admits a parallel totally null distribution of 2-planes. Let M_0 be a manifold and let us denote the standard coordinates on $\mathbb{R}^3$ by (t, s, u).

Theorem 3 *For any 1-form η on $M := M_0 \times \mathbb{R}^3$ such that $\eta_t := \eta(\partial_t) \neq 0$ and any family of semi-Riemannian metrics $g_0(u)$ on M_0 the tensor field*

$$g = \mathrm{d}s^2 + e^{-2s} g_0(u) + 2\,\mathrm{d}u\,\eta,$$

cf. (20), is a semi-Riemannian metric on M such that the vector fields $V = \partial_t$ and $Z = \partial_s$ satisfy (4). The covariant derivatives of V and Z are given by (5) and (6) for some 1-forms $\alpha = Z^\flat + f_\alpha V^\flat$ and β such that f_α is a function on M and $\beta(V) = 2$, if and only if the coefficients of η solve the following system of first order partial differential equations:

$$\partial_t \eta_t = \partial_s \eta_t = X\eta_t = \partial_t \eta(X) = 0, \quad \partial_t \eta_s = 2\eta_t, \quad \partial_s \eta(X) - X\eta_s = -2\eta(X) \tag{21}$$

for all $X \in \mathfrak{X}(M_0)$. Then α and β are determined by

$$f_\alpha = \frac{1}{\eta_t^2}\partial_t \eta_u - \frac{2}{\eta_t}\eta_s, \quad \beta(Z) = \frac{1}{\eta_t}\partial_s \eta_s, \quad \beta(X) = \frac{1}{2\eta_t}(X\eta_s + \partial_s \eta(X) + 2\eta(X)),$$
$$\beta(\partial_u) = \frac{1}{\eta_t}(\partial_s \eta_u - \eta_s^2 + 2\eta_u).$$

Proof We denote by X the canonical lift of a vector field on M_0. Then X, V, Z and ∂_u commute and using the Koszul formula we obtain

$$
\begin{aligned}
&g(\nabla_V V, X) = g(\nabla_V V, V) = g(\nabla_V V, Z) = 0, \quad g(\nabla_V V, \partial_u) = \partial_t \eta_t,\\
&g(\nabla_Z V, X) = g(\nabla_Z V, V) = g(\nabla_Z V, Z) = 0, \quad 2g(\nabla_Z V, \partial_u) = \partial_s \eta_t + \partial_t \eta_s,\\
&g(\nabla_X V, X) = g(\nabla_X V, V) = g(\nabla_X V, Z) = 0, \quad 2g(\nabla_X V, \partial_u) = X\eta_t + \partial_t \eta(X),\\
&2g(\nabla_{\partial_u} V, X) = \partial_t \eta(X) - X\eta_t, \quad g(\nabla_{\partial_u} V, V) = 0, \quad 2g(\nabla_{\partial_u} V, Z) = \partial_t \eta_s - \partial_s \eta_t,\\
&g(\nabla_{\partial_u} V, \partial_u) = \partial_t \eta_u,\\
&g(\nabla_V Z, X) = g(\nabla_V Z, V) = g(\nabla_V Z, Z) = 0, \quad 2g(\nabla_V Z, \partial_u) = \partial_t \eta_s + \partial_s \eta_t,\\
&g(\nabla_Z Z, X) = g(\nabla_Z Z, V) = g(\nabla_Z Z, Z) = 0, \quad g(\nabla_Z Z, \partial_u) = \partial_s \eta_s,\\
&g(\nabla_X Z, X) = -g(X, X), \quad g(\nabla_X Z, V) = g(\nabla_X Z, Z) = 0,\\
&2g(\nabla_X Z, \partial_u) = X\eta_s + \partial_s \eta(X),\\
&2g(\nabla_{\partial_u} Z, X) = \partial_s \eta(X) - X\eta_s, \quad 2g(\nabla_{\partial_u} Z, V) = \partial_s \eta_t - \partial_t \eta_s, \quad g(\nabla_{\partial_u} Z, Z) = 0,\\
&g(\nabla_{\partial_u} Z, \partial_u) = \partial_s \eta_u.
\end{aligned}
$$

Comparing with (5), (6) we obtain the above formulas for α and β and the following system for η:

$$
\begin{aligned}
&\partial_t \eta_t = 0, \quad \partial_s \eta_t + \partial_t \eta_s = 2\eta_t, \quad X\eta_t + \partial_t \eta(X) = 0, \quad \partial_t \eta(X) - X\eta_t = 0,\\
&\partial_t \eta_s - \partial_s \eta_t = 2\eta_t,\\
&\partial_s \eta(X) - X\eta_s = -2\eta(X)
\end{aligned}
$$

for all $X \in \mathfrak{X}(M_0)$. This system can be brought to the form (21). □

For convenience we denote a system of local coordinates on M_0 by $(x^i)_{i=1,\dots,n_0}$ and denote by x the corresponding coordinate vector, where $n_0 = \dim M_0$. The general solution of (21) is obtained as follows.

Proposition 2 *Let $f_1 = f_1(u)$ be an arbitrary nowhere vanishing smooth function on the real line equipped with the coordinate u and $f_2 = f_2(x, s, u)$ an arbitrary smooth function on M which does not depend on t. Let $h_i = h_i(x, s, u)$ be a (t-independent) solution of the ordinary differential equation*

$$\partial_s h_i + 2h_i = \partial_i f_2$$

for all $i = 1, \dots, n_0$, where $\partial_i = \partial/\partial x^i$. Then

$$\eta_t := f_1(u), \quad \eta_s := 2tf_1(u) + f_2(x, s, u), \quad \eta(\partial_i) := h_i(x, s, u)$$

solves (21) and every solution is of this form.

Remark 2 Finally we comment on the relation to the Lorentzian metrics that were considered in [2] and arose from the case where the cone $(\widehat{M}, \widehat{g})$ admits a parallel null line: in this case the cone metric $\widehat{g}$ was isometric to the metric $\widetilde{g} = 2\mathrm{d}u\mathrm{d}v + u^2 g_0$ with a Lorentzian metric g_0 and g was isometric to $g = \mathrm{d}s^2 + \mathrm{e}^{2s} g_0$. Then [2, Theorem 1.3] states that if the holonomy of the cone is not equal to $\mathfrak{hol}(g_0) \ltimes \mathbb{R}^{1,n-1}$, then

g_0 admits a parallel null vector field. It is well known (see for example [8, 13]) that locally g_0 is of the form $g_0 = 2\mathrm{d}x\mathrm{d}z + h(z)$, where $h(z)$ is a z-dependent family of Riemannian metrics. Hence, g is of the form

$$g = ds^2 + \mathrm{e}^{2s}h(z) + 2\mathrm{e}^{2s}\mathrm{d}x\mathrm{d}z.$$

This corresponds to the local form in Theorem 3, where x corresponds to t and $2\mathrm{e}^{2s}\mathrm{d}x$ to η, z to u and $h(z)$ to $g_0(u)$.

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Nilpotent Structures of Neutral 4-Manifolds and Light-Like Surfaces

Naoya Ando

Abstract Nilpotent structures of neutral 4-manifolds are analogues of complex structures and paracomplex structures. Nilpotent structures give two-dimensional involutive distributions and the integral surfaces are light-like and analogues of complex curves and paracomplex curves. Light-like surfaces in neutral 4-manifolds with local horizontal lifts are characterized in terms of the curvature tensors and such surfaces are analogues of isotropic minimal surfaces in Riemannian 4-manifolds.

Keywords Nilpotent structure · Neutral 4-manifold · Light-like surface

1 Introduction

The purpose of this paper is to study almost nilpotent structures of neutral 4-manifolds and light-like surfaces in neutral 4-manifolds.

Almost nilpotent structures of neutral 4-manifolds are analogues of almost complex structures of Riemannian 4-manifolds. Almost complex structures on an oriented Riemannian 4-manifold (M, h) which are h-preserving and compatible with the orientation of M correspond to sections of a suitable one of the twistor spaces associated with M. Such an almost complex structure I is parallel with respect to the Levi-Civita connection ∇ of h if and only if the corresponding section Θ is horizontal with respect to the connection $\hat{\nabla}$ of the 2-fold exterior power of the tangent bundle TM induced by ∇. It is known that $\nabla I = 0$ just means that (M, h, I) is a Kähler surface and then I is its complex structure. If (M, h, I) is a Kähler surface, then integral surfaces of involutive I-invariant 2-dimensional distributions are complex curves of (M, I). A complex curve of a Kähler surface is just an isotropic minimal surface compatible with the orientation of the space and equipped with at least one complex point and notice that there exist totally geodesic surfaces in $\mathbb{C}P^2$, $\mathbb{C}H^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}H^1 \times \mathbb{C}H^1$ with no complex points ([1]). In general, an isotropic minimal surface

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in an oriented Riemannian 4-manifold compatible with the orientation of the space is characterized by horizontality of a suitable one of the twistor lifts ([12]). See [7] for the case where the space is S^4. We can refer to [11] for the twistor spaces and isotropic minimal surfaces.

On oriented neutral 4-manifolds, we can consider not only almost complex structures but also almost paracomplex structures. On such a 4-manifold (M, h), almost complex (resp. paracomplex) structures which are h-preserving (resp. h-reversing) and compatible with the orientation of M correspond to sections of a suitable one of the space-like (resp. time-like) twistor spaces associated with M. See [3, 6] for the space-like twistor spaces and [3, 13, 14] for the time-like twistor spaces. For almost complex structures and almost paracomplex structures, we can find analogues of results on almost complex structures of oriented Riemannian 4-manifolds ([3]). In addition, for complex curves of neutral Kähler surfaces and paracomplex curves of paraKähler surfaces, we can find analogues of results on complex curves of Kähler surfaces; for space-like or time-like surfaces in oriented neutral 4-manifolds with zero mean curvature vector which are isotropic and compatible with the orientations of the spaces, we can find analogues of results on isotropic minimal surfaces in oriented Riemannian 4-manifolds compatible with the orientations of the spaces ([3]).

The space-like (resp. time-like) twistor spaces associated with an oriented neutral 4-manifold (M, h) are fiber bundles such that fibers are hyperboloids of two sheets (resp. one sheet). They are contained in subbundles $\bigwedge^2_{\pm} TM$ of rank 3 in the 2-fold exterior power $\bigwedge^2 TM$ of TM. We can find fiber bundles $U_0(\bigwedge^2_{\pm} TM)$ in $\bigwedge^2_{\pm} TM$ respectively such that fibers are light-like cones. Our main objects of study in the present paper are almost nilpotent structures and they correspond to sections of either $U_0(\bigwedge^2_{+} TM)$ or $U_0(\bigwedge^2_{-} TM)$. We will see that an almost nilpotent structure N is parallel with respect to ∇ if and only if the corresponding section Θ is horizontal with respect to $\hat{\nabla}$. If $\nabla N = 0$, then (h, N) is called a *nilpotent Kähler structure* of M, and M equipped with (h, N) is called a *nilpotent Kähler* 4*-manifold*. Neutral hyperKähler 4-manifolds have almost nilpotent structures parallel with respect to ∇ and we can refer to [10, 15] for neutral hyperKähler 4-manifolds. An almost nilpotent structure N of M gives a light-like 2-plane of the tangent space at each point of M. Therefore we have a light-like two-dimensional distribution $\mathscr{D}$. We will see that $\mathscr{D}$ is involutive if and only if for the section Θ corresponding to N and each tangent vector V of M contained in $\mathscr{D}$, the covariant derivative $\hat{\nabla}_V \Theta$ is given by Θ up to a constant. In particular, if $\nabla N = 0$, then $\mathscr{D}$ is involutive. In the case where $\nabla N = 0$, we can consider integral surfaces of $\mathscr{D}$ to be analogues of complex curves and paracomplex curves. Since $\mathscr{D}$ is light-like, we naturally have interest in light-like surfaces of M. Referring to the discussions on whether $\mathscr{D}$ is involutive, we will study a light-like surface in M with a nonzero horizontal section of a suitable one of the pull-back bundles of $U_0(\bigwedge^2_{\pm} TM)$ on a neighborhood of each point and we will see that a light-like surface in M has such a section if and only if ∇ induces a connection of the surface such that the curvature tensor of $\hat{\nabla}$ vanishes. We can consider light-like surfaces in M with local nonzero horizontal sections as above to be analogues of isotropic minimal surfaces in oriented Riemannian 4-manifolds compatible with the orientations of the spaces.

Remark 1 In [5], nilpotent Kähler structures of an oriented vector bundle E of rank 4 over $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ or $T^2 = S^1 \times S^1$ were studied. Let h be a neutral metric of E. Let ∇ be an h-connection of E, which means $\nabla h = 0$. Suppose that E is over S^1. Then we can find a nowhere zero, horizontal section Θ of $\bigwedge^2_+ E$ ([5]). If Θ is light-like, then Θ gives a nilpotent structure N of E and therefore (h, ∇, N) is a nilpotent Kähler structure of E. Suppose that E is over T^2. Then for a light-like, partially horizontal section Θ of $\bigwedge^2_+ E$, there exists an h-connection ∇' related to ∇ such that h, ∇' and Θ give a nilpotent Kähler structure of E ([5]).

2 Complex Structures and Paracomplex Structures of 4-Dimensional Neutral Vector Spaces

Let X be an oriented 4-dimensional vector space and h_X a neutral metric of X, i.e., an indefinite metric of X with signature (2, 2). Let $\bigwedge^2 X$ be the 2-fold exterior power of X and $\hat{h}_X$ the metric of $\bigwedge^2 X$ induced by h_X:

$$\begin{aligned}&\hat{h}_X(x_1 \wedge x_2, x_3 \wedge x_4)\\&= h_X(x_1, x_3)h_X(x_2, x_4) - h_X(x_1, x_4)h_X(x_2, x_3)\end{aligned}$$

($x_i \in X$). Let $\mathscr{B}_X$ be the set of ordered pseudo-orthonormal bases of X giving the orientation of X. Then $(e_1, e_2, e_3, e_4) \in \mathscr{B}_X$ satisfies

$$h_X(e_i, e_j) = \begin{cases} 1 & (i = j = 1 \text{ or } 2),\\ -1 & (i = j = 3 \text{ or } 4),\\ 0 & (\text{otherwise}). \end{cases}$$

For $(e_1, e_2, e_3, e_4) \in \mathscr{B}_X$, we set

$$\theta_{ij} := e_i \wedge e_j \quad (i, j \in \{1, 2, 3, 4\},\ i \neq j)$$

and

$$\begin{aligned}\Theta_{\pm,1} &:= \frac{1}{\sqrt{2}}(\theta_{12} \pm \theta_{34}),\\ \Theta_{\pm,2} &:= \frac{1}{\sqrt{2}}(\theta_{13} \pm \theta_{42}),\\ \Theta_{\pm,3} &:= \frac{1}{\sqrt{2}}(\theta_{14} \pm \theta_{23}).\end{aligned}$$

Then $\Theta_{\pm,1}$, $\Theta_{\pm,2}$, $\Theta_{\pm,3}$ form a pseudo-orthonormal basis of $\bigwedge^2 X$ and therefore we see that $\hat{h}_X$ has signature (2, 4). Let $\bigwedge^2_+ X$, $\bigwedge^2_- X$ be subspaces of $\bigwedge^2 X$ generated by $\Theta_{-,1}$, $\Theta_{+,2}$, $\Theta_{+,3}$ and $\Theta_{+,1}$, $\Theta_{-,2}$, $\Theta_{-,3}$, respectively. Then by the definitions of $\bigwedge^2_\pm X$, we have

$$\bigwedge^2 X = \bigwedge^2_+ X \oplus \bigwedge^2_- X$$

and we see that $\bigwedge^2_+ X$, $\bigwedge^2_- X$ are orthogonal to each other and that the restriction of $\hat{h}_X$ on each of them has signature (1, 2). In addition, noticing the double covering

$$SO_0(2,2) \longrightarrow SO_0(1,2) \times SO_0(1,2),$$

we see that $\bigwedge^2_\pm X$ do not depend on the choice of $(e_1, e_2, e_3, e_4) \in \mathscr{B}_X$.

We set

$$U_+\left(\bigwedge^2_\pm X\right) := \left\{ \Theta \in \bigwedge^2_\pm X \;\middle|\; \hat{h}_X(\Theta, \Theta) = 1 \right\}.$$

Then each $\Theta \in U_+\left(\bigwedge^2_+ X\right)$ corresponds to a unique h_X-preserving complex structure I of X satisfying

$$\Theta = \frac{1}{\sqrt{2}}(e \wedge I(e) - e^\perp \wedge I(e^\perp)), \tag{1}$$

where e is a space-like and unit vector of X and $e^\perp$ is a time-like vector of X satisfying

$$h_X(e^\perp, e^\perp) = -1, \quad h_X(e, e^\perp) = h_X(I(e), e^\perp) = 0.$$

Then we have $(e, I(e), e^\perp, I(e^\perp)) \in \mathscr{B}_X$, which means that I is compatible with the orientation of X. Conversely, each h_X-preserving complex structure I of X compatible with the orientation corresponds to a unique element of $U_+\left(\bigwedge^2_+ X\right)$ by (1). Hence we have a one-to-one correspondence between $U_+\left(\bigwedge^2_+ X\right)$ and the set of h_X-preserving complex structures of X compatible with the orientation. Similarly, we have a one-to-one correspondence between $U_+\left(\bigwedge^2_- X\right)$ and the set of h_X-preserving complex structures of X which are not compatible with the orientation.

We set

$$U_-\left(\bigwedge^2_\pm X\right) := \left\{ \Theta \in \bigwedge^2_\pm X \;\middle|\; \hat{h}_X(\Theta, \Theta) = -1 \right\}.$$

Then each $\Theta \in U_-\left(\bigwedge^2_+ X\right)$ corresponds to a unique h_X-reversing paracomplex structure J of X satisfying

$$\Theta = \frac{1}{\sqrt{2}}(e \wedge J(e) - e^\perp \wedge J(e^\perp)), \tag{2}$$

where $e, e^{\perp}$ are as above. Then we have $(e, J(e^{\perp}), J(e), e^{\perp}) \notin \mathscr{B}_X$, which means that J is not compatible with the orientation of X. Conversely, each h_X-reversing paracomplex structure J of X which is not compatible with the orientation corresponds to a unique element of $U_-\left(\bigwedge_+^2 X\right)$ by (2). Hence we have a one-to-one correspondence between $U_-\left(\bigwedge_+^2 X\right)$ and the set of h_X-reversing paracomplex structures of X which are not compatible with the orientation. Similarly, we have a one-to-one correspondence between $U_-\left(\bigwedge_-^2 X\right)$ and the set of h_X-reversing paracomplex structures of X compatible with the orientation.

3 Nilpotent Structures of 4-Dimensional Neutral Vector Spaces

In the present paper, our main objects of study are closely related to the light-like cones of $\bigwedge_\pm^2 X$:

$$U_0\left(\bigwedge_\pm^2 X\right) := \left\{ \Theta \in \bigwedge_\pm^2 X \setminus \{0\} \;\middle|\; \hat{h}_X(\Theta, \Theta) = 0 \right\}.$$

For each $\Theta \in U_0\left(\bigwedge_+^2 X\right)$, there exists an element (e_1, e_2, e_3, e_4) of $\mathscr{B}_X$ satisfying

$$\Theta = \Theta_{-,1} + \Theta_{+,3}. \tag{3}$$

We call such a basis as (e_1, e_2, e_3, e_4) an *admissible basis* of Θ. Let G be a subgroup of $SO(2, 2)$ defined by

$$G := \left\{ B = \begin{bmatrix} b_1 & -b_2 & b_3 & b_4 \\ b_2 & b_1 & -b_4 & b_3 \\ b_3 & -b_4 & b_1 & b_2 \\ b_4 & b_3 & -b_2 & b_1 \end{bmatrix} \;\middle|\; \begin{array}{l} b_1, b_2, b_3, b_4 \in \mathbb{R}, \\ b_1^2 + b_2^2 - b_3^2 - b_4^2 = 1 \end{array} \right\}.$$

This is isomorphic to $SU(1, 1)$. Let H be a subset of $SO(2, 2)$ defined by

$$H := \left\{ C(h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{h^2+2}{2} & h & -\frac{h^2}{2} \\ 0 & h & 1 & -h \\ 0 & \frac{h^2}{2} & h & -\frac{h^2-2}{2} \end{bmatrix} \;\middle|\; h \in \mathbb{R} \right\}.$$

We see that H is a subgroup of $SO(2, 2)$. Let (e'_1, e'_2, e'_3, e'_4) be another admissible basis of Θ than (e_1, e_2, e_3, e_4). Then there exist $B \in G$, $h \in \mathbb{R}$ satisfying

$$(e'_1, e'_2, e'_3, e'_4) = (e_1, e_2, e_3, e_4)BC(h). \tag{4}$$

We set

$$\Lambda := \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then we have $\Lambda B = B\Lambda$ for any $B \in G$ and $\Lambda C(h) = C(h)\Lambda$ for any $h \in \mathbb{R}$. Therefore we see that a linear transformation N of X can be defined by

$$(N(e_1), N(e_2), N(e_3), N(e_4)) = (e_1, e_2, e_3, e_4)\Lambda \tag{5}$$

for an admissible basis (e_1, e_2, e_3, e_4) of Θ and that N is determined by Θ and does not depend on the choice of an admissible basis (e_1, e_2, e_3, e_4) of Θ. We call N a *nilpotent structure* of X corresponding to $\Theta \in U_0\left(\bigwedge_+^2 X\right)$. We denote by $\mathcal{N}_{X,+}$ the set of nilpotent structures of X corresponding to the elements of $U_0\left(\bigwedge_+^2 X\right)$. We have

$$\begin{aligned} \Theta &= \frac{1}{\sqrt{2}}(e_1 \wedge N(e_1) - e_3 \wedge N(e_3)) \\ &= \frac{1}{\sqrt{2}}(e_2 \wedge N(e_2) - e_4 \wedge N(e_4)). \end{aligned} \tag{6}$$

We set

$$V_1 := e_1 - e_3, \quad V_2 := e_2 + e_4.$$

Then we have $\Theta = (1/\sqrt{2})V_1 \wedge V_2$. We see that $\operatorname{Im} N$ is generated by light-like vectors V_1, V_2 and that it coincides with $\operatorname{Ker} N$. We have $h_X(N(x), x) = 0$ for any $x \in X$.

For each $\Theta \in U_0\left(\bigwedge_-^2 X\right)$, there exists an element (e_1, e_2, e_3, e_4) of $\mathcal{B}_X$ satisfying

$$\Theta = \Theta_{+,1} + \Theta_{-,3}.$$

We call such a basis as (e_1, e_2, e_3, e_4) an *admissible basis* of Θ. Let (e'_1, e'_2, e'_3, e'_4) be another admissible basis of Θ than (e_1, e_2, e_3, e_4). Then there exist $B \in G, h \in \mathbb{R}$ satisfying

$$(e'_1, e'_2, -e'_3, e'_4) = (e_1, e_2, -e_3, e_4)BC(h).$$

Therefore we see that a linear transformation N of X can be defined by

$$(N(e_1), N(e_2), -N(e_3), N(e_4)) = (e_1, e_2, -e_3, e_4)\Lambda$$

for an admissible basis (e_1, e_2, e_3, e_4) of Θ and that N is determined by Θ and does not depend on the choice of an admissible basis (e_1, e_2, e_3, e_4) of Θ. We call N a *nilpotent structure* of X corresponding to $\Theta \in U_0\left(\bigwedge_-^2 X\right)$. We denote by $\mathcal{N}_{X,-}$ the