Studies in Systems, Decision and Control 248

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Cooperative Control of Multi-agent Systems

A Scale-Free Protocol Design

Springer

Studies in Systems, Decision and Control

Volume 248

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ISSN 2198-4182 ISSN 2198-4190 (electronic)
Studies in Systems, Decision and Control
ISBN 978-3-031-12953-7 ISBN 978-3-031-12954-4 (eBook)
https://doi.org/10.1007/978-3-031-12954-4

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

This book is dedicated to:
My family: Yuan, Xinning, Yuan's and my parents

(Zhenwei Liu)

My family

(Donya Nojavanzadeh)

My children: Ingmar, Ula, Dmitri, and Mirabella

(Ali Saberi)

and
the glory of man.

Acknowledgements

We would like to express our sincere gratitude to our collaborators, Anton A. Stoorvogel, Saeed Lotfifard, Meirong Zhang, and Dmitri Saberi, for their contributions in publishing the papers that form the partial results in this book.

The writing of this monograph was supported in part by the National Natural Science Foundation of China under Grant 62273084.

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October 2022

Contents

Chapter 1
Introduction

1.1 Cooperative Control of Multi-agent Systems

Synchronization problem of multi-agent systems (MAS) has become a hot topic among researchers in recent years. Cooperative control of MAS is used in practical application such as robot networks, autonomous vehicles, distributed sensor networks, swarming, flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory by local interaction among agents. See [9, 13, 50, 94, 124, 132, 167] and references therein.

We briefly review several important research directions in MAS. Firstly, we point out that there are two type of MAS: homogeneous (i.e. agents are identical) and heterogeneous (i.e. agents are non-identical). State synchronization inherently requires homogeneous MAS. On the other hand, for a heterogeneous MAS, generically, it is more reasonable to consider output synchronization since the dimensions of states and their physical interpretation may be different.

For homogeneous MAS state synchronization based on diffusive full-state coupling has been studied where the agent dynamics progress from single- and double-integrator dynamics (e.g. [114, 122, 123]) to more general dynamics (e.g. [136, 153, 164]). State synchronization based on diffusive partial-state coupling has also been considered, including static design [85, 86], dynamic design [46, 137, 138, 146, 155], and the design with localized communication [22, 136]. For MAS with discrete-time agents, earlier work can be found in [27, 39, 55, 114, 123, 154] for essentially first and second-order agents, and in [40, 53, 58, 158, 160, 177, 188, 192] for higher-order agents.

In heterogeneous MAS, if the agents have absolute measurements of their own dynamics in addition to relative information from the network, they are said to be introspective, otherwise, they are called non-introspective. For heterogeneous MAS with non-introspective agents, it is well-known that one needs to regulate outputs of the agents to a priori given trajectory generated by a so-called exosystem (see [34, 165]). Other works on synchronization of MAS with non-introspective agents can be found in the literature as [35, 36]. On the other hand, for MAS with introspective

Z. Liu et al., *Cooperative Control of Multi-agent Systems*, Studies in Systems, Decision and Control 248, https://doi.org/10.1007/978-3-031-12954-4_1

agents, one can achieve output and regulated output synchronization. Most of the literature for heterogeneous MAS with introspective agents are based on modifying the agent dynamics via local feedback to achieve some form of homogeneity. There have been many results for synchronization of heterogeneous networks with introspective agents, see for instance [17, 47, 56, 95, 121, 174].

In practical applications, the network dynamics are not-ideal and may be subject to delays. Time delays may afflict system performance or even lead to instability. As discussed in [15], two types of delays have been considered in the literature: input delays and communication delays. The former encapsulate the processing time to execute an input for each agent, whereas the latter can be considered as the time it takes to transmit information from an origin agent to its destination. It is worthwhile to point out that packet drops in exchanging information can be considered as special case of communication delay, because re-sending packets after they were dropped can be easily done but just having time delay in the data transmission channels. Some research work has been done for both constant and time-varying input delay, specifically with the objective of deriving an upper bound on the input delays such that agents can still achieve synchronization; see, for example [12, 61, 62, 73, 85, 114, 152, 170, 180]. In the case of communication delay, some research work has been done; see [18, 31, 49, 98, 99, 152, 169, 183]. Time-varying communication delays for a general multi-agent system have been considered in [142]. As it is well-known that in order to withstand large communication delays one needs to preserve diffusiveness (namely to ensure the invariance of the synchronization manifold). This can be achieved with two methods:

1. The first method is the standard state/output synchronization by regulating the states/outputs to a constant trajectory. This method is intensively utilized in the literature [15]. A notable phenomenon in this case is that the final consensus is constant where in many practical problems this would be the case; see for example [111].
2. The second method is to consider delayed state/output synchronization which is introduced in [18, 19, 21, 74, 101] to allow non-constant or dynamic desired output/state trajectory.

Actuator saturation is also pretty common and indeed is ubiquitous in engineering applications. Semi-global state and output synchronization in presence of input saturation have been studied in the literature (see for example [180]). Compared with semi-global results, global synchronization has been studied for single-integrator and more generally neutrally stable agents in [26, 30, 57, 82, 93, 173, 176].

Performance is one of the key elements in designing consensus protocols in practical applications. The asymptotic convergence rate is defined in [171] as an indicator for the performance. The convergence rate is an important element in designing protocols. Typically, the communication topology through which the agents communicate is a deciding factor in establishing the convergence rate of the protocol. More explicitly, the convergence rate of various existing protocols for continuous-time MAS with undirected communication graphs depends on the second smallest eigenvalue of Laplacian matrix of the associated communication graph, also known as the algebraic connectivity of the graph. In discrete-time MAS, the largest modulus of the eigenvalues of *Perron matrix* of the associated communication graph plays the same role in the asymptotic convergence rate, see for example [141, 171, 177]. In fact, for a certain class of undirected graphs the algebraic connectivity decreases with an increase in network size. The recent thesis [148] describes this effect and covers some non-exhaustive classes of graphs. In [13, Chap. 16], a similar conclusion has been drawn for a directed circulant graph.

Synchronization and almost synchronization in presence of external disturbances are studied in the literature, where three classes of disturbances have been considered namely:

1. Disturbances and measurement noise with known frequencies.
2. Deterministic disturbances with finite power.
3. Stochastic disturbances with bounded variance.

For disturbances and measurement noise with known frequencies, it is shown in [181, 182] that actually exact synchronization is achievable. This is shown in [181] for heterogeneous MAS with minimum-phase and non-introspective agents and networks with time-varying directed communication graphs. Then, [182] extended this results for non-minimum phase agents utilizing localized information exchange.

For deterministic disturbances with finite power, the notion of H_∞ almost synchronization[1] is introduced in [116] for homogeneous MAS with non-introspective agents utilizing additional communication exchange. The goal of H_∞ almost synchronization is to reduce the impact of disturbances on the synchronization error to an arbitrarily degree of accuracy (expressed in the H_∞ norm). This work was extended later in [117, 179, 184] to heterogeneous MAS with non-introspective agents and without the additional communication and for network with time-varying graphs. H_∞ almost synchronization via static protocols is studied in [144] for MAS with passive and passifiable agents. Necessary and sufficient solvability conditions are provided in [143] for H_∞ almost synchronization of homogeneous networks with non-introspective agents without additional communication exchange.

In the case of stochastic disturbances with bounded variance, the concept of stochastic almost synchronization is introduced in [185] in presence of both stochastic disturbance and disturbance with known frequency. The idea of stochastic almost synchronization is to reduce the stochastic RMS norm of synchronization error arbitrary small in the presence of colored stochastic disturbances that can be modeled as the output of linear time invariant systems driven by white noise with unit power spectral intensities. By augmenting this model with agent model one can essentially assume that stochastic disturbance is white noise with unit power spectral intensity. In this case, utilizing linear protocols, the stochastic RMS norm of synchronization error equals to the H_2 norm of the transfer function from disturbance to the synchronization error, as such one can formulate the stochastic almost synchronization equivalently

[1] The term "almost synchronization" has been selected in connection with the concept of almost disturbance decoupling (see e.g. [115]) where the problem is to find a family of controllers to reduce the noise sensitivity to any arbitrary degree.

in a deterministic framework to reduce the H_2 norm of the transfer function from disturbance to synchronization error arbitrary small. This deterministic approach is referred to as almost H_2 synchronization problem which is equivalent to stochastic almost synchronization problem. Recent work on H_2 almost synchronization problem is [143] which provides necessary and sufficient conditions for solvability of H_2 almost synchronization for homogeneous networks with non-introspective agents in the absence of additional communication exchange. H_2 almost synchronization via static protocols is studied in [144] for MAS with passive and passifiable agents.

Most of the proposed protocols in the literature for synchronization of MAS require some knowledge of the communication network such as bounds on the spectrum of the associated Laplacian matrix or the number of agents. As it is pointed out in [145, 149, 150], these protocols suffer from *scale fragility* wherein stability properties are lost for large-scale networks or when the communication graph changes.

In the past few years, the authors of this monograph have worked on developing scale-free protocol design for various cases of MAS problems. The "scale-free" design has the following features:

- The proposed protocols are designed solely based on the knowledge of agent models and do not depend on information about the communication network such as the spectrum of the associated Laplacian matrix.
- The designs do not require knowledge of the size of network, i.e., the number of agents. That is to say, the universal dynamical protocols work for any communication network as long as it contains a spanning tree.

The primary focus of this book is on the problem of achieving scale-free design for MAS. The audience for this book includes practicing engineers, graduate students, and researchers in the field of MAS. The contents of this book are drawn from the research of the authors and their coworkers which also includes application of our scale-free design to power grid systems. Thus it bears the signature of the authors and has a recognizable identity and a coherence of point of view which can be characterized as a structural view in both the analysis and design.

1.2 Outline

This monograph is written as a collection of the authors' published work. Each chapter presents self-contained technical results and can be read independently. The outline of the remainder of this monograph and its main contributions can be summarized as follows.

- In Chap. 2, we provide some notations and technical preliminaries on linear algebra, signal system norms, graph theory, MAS, and passivity.
- Chapters 3 and 4 address scale-free synchronization of homogeneous and heterogeneous MAS for continuous- and discrete-time networks, respectively.

The developments and results of Chaps. 3 and 4 are based on [102, 103, 107, 108].

- Chapter 5 deals with regulated state synchronization problem of homogeneous continuous- and discrete-time MAS when the agents are subject to unknown, and non-uniform input delays.
  The development and results of this chapter are based on [69–73].
- State synchronization of MAS in the presence of unknown, non-uniform, and arbitrarily large communication delays is considered for homogeneous continuous- and discrete-time MAS in Chaps. 6 and 7, respectively.
  The developments and results of these chapters are based on [65, 66, 68].
- Regulated output synchronization of heterogeneous MAS in presence of unknown, non-uniform, and arbitrarily large communication delays is considered for both continuous- and discrete-time MAS in Chap. 8.
  The development and results of this chapter are based on [67].
- Delayed regulated synchronization of homogeneous and heterogeneous MAS subject to unknown, non-uniform, and arbitrarily large communication delays has been studied in Chaps. 9 and 10 for continuous- and discrete-time MAS, respectively.
  The developments and results of these chapters are based on [101, 105, 110].
- In Chaps. 11 and 12, nonlinear and linear protocols are designed to achieve global regulated state synchronization for homogeneous continuous- and discrete-time MAS with non-introspective agents in presence of input saturation.
  The developments and results of these chapters are partially based on [63, 64, 76, 80].
- State and regulated state synchronization of continuous-time MAS with arbitrary fast convergence is studied in Chap. 13.
  The development and results of this chapter are partially based on [81].
- H_∞ and H_2 almost synchronization of homogeneous and heterogeneous MAS are studied in Chaps. 14 and 15.
  The developments and results of these chapters are based on [77–79, 104, 106, 109].
- State synchronization of MAS via non-collaborative protocol is studied in Chap. 16.
  The development and results of these chapters are partially based on [86].
- Finally, in Chaps. 17 and 18, voltage control of multiterminal HVDC systems and Microgrids are studied utilizing scale-free nonlinear controllers.
  The developments and results of Chaps. 17 and 18 are based on [87, 111, 112].

Chapter 2
Notations and Preliminaries

2.1 Linear Algebra

We denote the set of real numbers by $\mathbb{R}$, integers by $\mathbb{Z}$, non-negative real numbers by $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} | x \geq 0\}$, and the entire complex plane by $\mathbb{C}$. We denote the field of rational functions with real coefficients by $\mathbf{R}(s)$. By $\text{rank}_{\mathcal{K}}$ we denote the rank of a matrix whose entries are in the filed $\mathcal{K}$. We shall write rank only for the case when $\mathcal{K} = \mathbb{R}$, or $\mathcal{K} = \mathbb{C}$. Moreover, we use the term *normalrank* for $\text{rank}_{\mathcal{K}}$ whenever $\mathcal{K} = \mathbf{R}(s)$. Given a matrix $A \in \mathbb{R}^{n\times m}$, A^{T} and A^* denote transpose and conjugate transpose of A respectively, and $\|A\|$ is the induced 2-norm (which has submultiplicative property). The $\text{im}(\cdot)$ denote the image of matrix (vector). For a square matrix $M \in \mathbb{R}^{n\times n}$, we denote the set of eigenvalues of M by $\lambda(M)$ and the smallest singular value of M by $\sigma_{\min}(M)$ while $\rho(M)$ denotes the spectral radius of M. A square matrix M is said to be Hurwitz/Schur stable if all its eigenvalues are in the open left half complex plane/open unit disc. Let $\mathbf{j}$ indicate $\sqrt{-1}$. We denote by $\text{diag}\{A_1, \ldots, A_N\}$, a block-diagonal matrix with $A_1, \ldots, A_N$ as its diagonal elements. I_n denotes the n-dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context. We define:

$$\overline{[t_1, t_2]} = \{t \in \mathbb{Z} : t_1 \leq t \leq t_2\}.$$

Then, we recall matrix's Kronecker product. We denote the Kronecker product between A and B by $A \otimes B$. The Kronecker product is bilinear and associative:

$$\begin{aligned}
A \otimes (B + C) &= A \otimes B + A \otimes C,\\
(A + B) \otimes C &= A \otimes C + B \otimes C,\\
(kA) \otimes B &= A \otimes (kB) = k(A \otimes B),\\
(A \otimes B) \otimes C &= A \otimes (B \otimes C),
\end{aligned}$$

Z. Liu et al., *Cooperative Control of Multi-agent Systems*, Studies in Systems, Decision and Control 248, https://doi.org/10.1007/978-3-031-12954-4_2

where A, B and C are matrices and k is a scalar. The following properties of the Kronecker product will be useful:

$$\begin{aligned}(A \otimes B)(C \otimes D) &= (AC) \otimes (BD),\\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1},\\ (A \otimes B)^{\mathrm{T}} &= A^{\mathrm{T}} \otimes B^{\mathrm{T}},\\ (A \otimes B)^{*} &= A^{*} \otimes B^{*}.\end{aligned}$$

Next, in the following, we recall the definitions of invariant zeros and right-invertibility of the linear time-invariant system Σ

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Definition 2.1 $\lambda \in \mathbb{C}$ is called invariant zero of linear system Σ if

$$\operatorname{rank}_{\mathbb{C}} \begin{pmatrix} \lambda I - A & -B \\ C & 0 \end{pmatrix} < \operatorname{normalrank} \begin{pmatrix} sI - A & -B \\ C & 0 \end{pmatrix},$$

where by *normalrank* we mean the rank of a matrix with entries in the field of rational function $\mathbf{R}(s)$.

Definition 2.2 The linear system Σ is right-invertible if, given a smooth reference output y_r, there exists an initial condition $x(0)$ and an input $u(t)$ that ensures $y(t) = y_r(t)$ for all $t \geq 0$.

Remark 2.3 The linear system Σ

- is right-invertible if and only if its transfer function matrix is a surjective rational matrix.
- is right-invertible if and only if the rank of $\begin{pmatrix} sI - A & -B \\ C & 0 \end{pmatrix}$ is $n + p$ for all but finitely many $s \in \mathbb{C}$.

The linear system Σ is at most weakly unstable if all eigenvalues of A are in the closed left half plane. It should be noted that the set of at most weakly unstable agents contains stable agents, neutrally stable agents as well as weakly unstable agents. The related definitions and notations can be found in [34, 130, 163, 165].

2.2 Signal and System Norms

For a deterministic continuous-time signal $v(t)$, the L_2 norm is defined by

$$\|v(t)\|_{L_2} = \left(\int_0^T v^{\mathrm{T}}(t) v^{\mathrm{T}}(t) dt \right)^{\frac{1}{2}}, \tag{2.1}$$

and its *Root Mean Square (RMS)* value is defined by

$$\|v(t)\|_{RMS} = \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T v(t) v(t) dt \right)^{\frac{1}{2}}, \tag{2.2}$$

and for a stochastic signal $v(t)$ which is modeled as wide-sense stationary stochastic process, the $\|v(t)\|_{RMS}$ is given by

$$\|v(t)\|_{RMS} = \left(\mathbf{E}[v^{\mathrm{T}}(t) v(t)] \right)^{\frac{1}{2}}, \tag{2.3}$$

where $\mathbf{E}[\cdot]$ stands for the expectation operation. For stochastic signals that approach wide-sense stationarity as time t goes on to infinity (i.e. for asymptotically wide-sense stationary signals) (2.3) is rewritten as

$$\|v(t)\|_{RMS} = \left(\lim_{t \to \infty} \mathbf{E}[v^{\mathrm{T}}(t) v(t)] \right)^{\frac{1}{2}}. \tag{2.4}$$

For a continuous-time system having a $q \times l$ stable transfer function $G(s)$, the H_2 norm of $G(s)$ is defined as

$$\|G\|_{H_2} = \left(\frac{1}{2\pi} \operatorname{trace} \left[\int_{-\infty}^{+\infty} G(j\omega) G^*(j\omega) d\omega \right] \right)^{\frac{1}{2}}.$$

By Parseval's theorem, $\|G\|_{H_2}$ can be equivalently be defined as

$$\|G\|_{H_2} = \left(\operatorname{trace} \left[\int_0^{+\infty} g(t) g^{\mathrm{T}}(t) dt \right] \right)^{\frac{1}{2}},$$

where $g(t)$ is the weighting function or unit impulse (Dirac distribution) response matrix of $G(s)$, as such for single-input single-output system $\|G\|_{H_2} = \|g\|_{L_2}$. The H_2 norm of $G(s)$, can be interpreted as the RMS value of the output when the given system is driven by independent zero mean white noise with unit power spectral density. Note that the H_2 norm of a stable transfer function $G(s)$ is finite if and only if it is strictly proper. The H_∞ norm of $G(s)$ is defined as

$$\|G\|_{H_\infty} := \sup_{\omega} \sigma_{\max}[G(j\omega)],$$

where $\sigma_{\max}$ is the largest singular value of $G(j\omega)$. Let $\omega(t)$ and $z(t)$ be energy signals which are respectively the input and the corresponding output of the given system. Then, the H_∞ norm of $G(s)$ turns out to coincide with its RMS gain, namely

$$\|G\|_{H_\infty} = \|G\|_{RMS\,gain} = \sup_{\|\omega\|_{RMS}\neq 0} \frac{\|z\|_{RMS}}{\|\omega\|_{RMS}}.$$

An important property of the H_∞ norm is that it is sub-multiplicative. That is for transfer functions G_1 and G_2, we have

$$\|G_1 G_2\|_{H_\infty} \leq \|G_1\|_{H_\infty} \|G_2\|_{H_\infty}.$$

2.3 Graphs

A *weighted graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix. Each pair in $\mathcal{E}$ is called an *edge*, where $a_{ij} > 0$ denotes an edge $(j, i) \in \mathcal{E}$ from node j to node i with weight a_{ij}. Moreover, $a_{ij} = 0$ if there is no edge from node j to node i. We assume there are no self-loops, i.e. we have $a_{ii} = 0$. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A *directed tree* with root r is a subgraph of the graph $\mathcal{G}$ in which there exists a unique path from node r to each node in this subgraph. A *directed spanning tree* is a directed tree containing all the nodes of the graph. See [32].

For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. The Laplacian matrix L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$, i.e. a vector with all entries equal to 1. When graph contains a spanning tree, then it follows from [123, Lemma 3.3] that the Laplacian matrix L has a simple eigenvalue at the origin, with the corresponding right eigenvector $\mathbf{1}$, and all the other eigenvalues are in the open right-half complex plane.

2.4 Multi-agent Systems and Graphs

Consider a homogeneous MAS composed of N identical linear time-invariant agents of the form,

$$\begin{aligned} x_i^+(t) &= Ax_i(t) + Bu_i(t), \\ y_i(t) &= Cx_i(t), \end{aligned} \tag{2.5}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, and $y_i(t) \in \mathbb{R}^p$ are the state, input, and output of agent i for $i = 1, \ldots, N$. In the aforementioned presentation, for continuous-time systems, $x_i^+(t) = \dot{x}_i(t)$ for $t \in \mathbb{R}$; while for discrete-time systems, $x_i^+(t) = x_i(t+1)$ for $t \in \mathbb{Z}$.

For continuous-time agents, each agent $i \in \{1, \ldots, N\}$ has access to the quantity,

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(y_i(t) - y_j(t)) \tag{2.6}$$

where $a_{ij} \geq 0$, and $a_{ii} = 0$ for $i, j \in \{1, \ldots, N\}$. The topology of the communication network can be described by a directed graph (digraph) $\mathcal{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ implies that an edge exists from agent j to i. Agent j is then called a *parent* of agent i, and agent i is called a *child* of agent j. The weight of the edge equals $a_{ij} \geq 0$. It is assumed that there is no self-loops in the graph, i.e. $a_{ii} = 0$. In this context the matrix $A = [a_{ij}]$ is referred to as the adjacency matrix.

The *weighted in-degree* of a vertex i is given by

$$d_{\text{in}}(i) = \sum_{j=1}^{N} a_{ij}.$$

Similarly, the *weighted out-degree* of a vertex i is given by

$$d_{\text{out}}(i) = \sum_{j=1}^{N} a_{ji}.$$

A graph is called *balanced* if for every node we have

$$d_{\text{in}}(i) = d_{\text{out}}(j).$$

Based on the adjacency matrix and the weighted in-degree, we can associate a Laplacian matrix L to a graph,

$$L = \text{diag}\{d_{\text{in}}(1), d_{\text{in}}(2), \ldots, d_{\text{in}}(N)\} - A.$$

Based on the above definition, it is easily verified that a Laplacian matrix has the property that all the row sums are zero. In terms of the coefficients of L, (2.6) can be rewritten as

$$\zeta_i(t) = \sum_{j=1}^{N} \ell_{ij} y_j(t). \tag{2.7}$$

We denote by $\mathbf{1}_N$ the column vector in $\mathbb{R}^n$ with all elements equal to 1. We will use $\mathbf{1}$ if the dimension is obvious from the context. In light of [167, Corollary 2.37] the following lemma is yield.

Lemma 2.4 *The graph $\mathcal{G}$ describing the communication topology of the network is balanced if and only if the associated Laplacian matrix L has the property that $L + L^{\mathrm{T}} \geq 0$. Moreover, in that case $\mathbf{1}$ is both a left and right eigenvector of the Laplacian matrix L.*

A *directed tree* is a directed subgraph of $\mathcal{G}$, consisting of a subset of the nodes and edges, such that every node has exactly one parent, except a single root node with no parents. In that case, there exists a directed path from the root to every other agent. A *directed spanning tree* is a directed tree that contains all the nodes of $\mathcal{G}$. In that case, the root node with no parents is called a *root agent*.

A directed graph may contain many directed spanning trees, and thus there may be several choices for the root agent. The set of all possible root agents for a graph $\mathcal{G}$ is denoted by $\Pi_{\mathcal{G}}$.

Example 2.5 Figure 2.1 illustrates a directed graph containing multiple directed spanning trees.

We recall from [2] and more specifically [123, Lemma 3.3] the following crucial connection between the graph and its associated Laplacian matrix.

Lemma 2.6 *The graph $\mathcal{G}$ describing the communication topology of the network contains a directed spanning tree if and only if the associated Laplacian matrix L has a simple eigenvalue at the origin. Moreover, in that case, the associated right eigenvector is given by* $\mathbf{1}$.

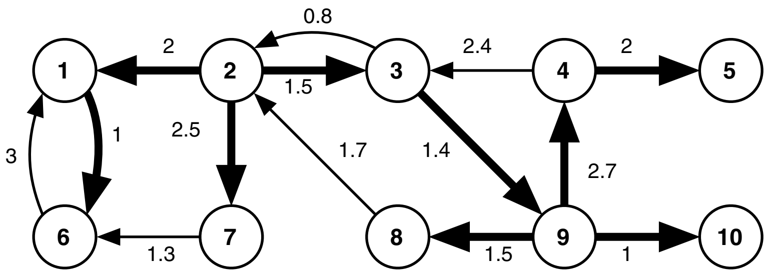


Fig. 2.1 The depicted directed graph contains multiple directed spanning trees, rooted at nodes 2, 3, 4, 8, and 9. One of these, with root node 2, is illustrated by bold arrows

Many problems and definitions in this book assume that the network graph is in some certain sets. As such we introduce the following definitions of sets of graphs.

Definition 2.7 Let $\mathbb{G}^N$ denote the set of directed graphs with N nodes that contain a directed spanning tree.

Definition 2.8 Let $\mathbb{G}^{b,N}$ denote the set of undirected graphs with N nodes which are strongly connected and for which the corresponding Laplacian matrix L is balanced, i.e.,

$$L + L^{\mathrm{T}} \geq 0.$$

Definition 2.9 Let $\mathbb{G}^{u,N}$ denote the set of undirected graphs with N nodes.

Definition 2.10 Given a node set $\mathscr{C}$, we denote by $\mathbb{G}^N_{\mathscr{C}}$ the set of all graphs with N nodes containing the node set $\mathscr{C}$, such that every node of the network graph $\mathcal{G} \in \mathbb{G}^N_{\mathscr{C}}$ is a member of a directed tree which has its root contained in the node set $\mathscr{C}$. We will refer to the node set $\mathscr{C}$ as root set.

Remark 2.11 Note that Definition 2.10 does not require necessarily the existence of directed spanning tree. On the other hand, the main difficulty is the loss of symmetry for the Laplacian matrix. We therefore lose some nice properties which are valid for symmetric Laplacian matrix which makes synchronization analysis more difficult. Some of the existing results showed us this fact. See for instance [178].

For discrete-time agents, each agent $i \in \{1, \ldots, N\}$ has access to the quantity,

$$\zeta_i(t) = \sum_{j=1}^{N} d_{ij}(y_i(t) - y_j(t)), \tag{2.8}$$

where the $d_{ij} \geq 0$. We choose $d_{ii} = 1 - \sum_{j=1, j\neq i}^{N} d_{ij}$ to form a matrix D which is a row-stochastic matrix, such that

$$\sum_{j=1}^{N} d_{ij} = 1$$

with $i, j \in \{1, \ldots, N\}$. The topology of the network can again be described by a directed graph (digraph) $\mathcal{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients d_{ij}, where $j \neq i$. In particular, $d_{ij} > 0$ $(j \neq i)$ implies that an edge exists from agent j to i. Moreover, it is assumed that there is no self-loops in the graph. Agent j is then called a *parent* of agent i, and agent i is called a *child* of agent j. The weight of the edge equals the magnitude of d_{ij}. Clearly, in this case all weights are less than equal to 1, and, additionally, the weighted in-degree has to be less than or equal to 1. Note that the diagonal elements d_{ii} do not affect (2.8) and are chosen to be such that the matrix becomes a row-stochastic matrix.

We have the following result (see [123] for the sufficiency part).

Lemma 2.12 *Assume that the graph $\mathcal{G}$ describing the communication topology of the network contains a directed spanning tree and $d_{ii} > 0$ for $i = 1, \ldots, N$ implies that the row stochastic matrix D has a simple eigenvalue at 1 and all other eigenvalues have amplitude strictly less than 1. Moreover, in that case the associated right eigenvector is given by* $\mathbf{1}$.

Remark 2.13 Note that the condition that $d_{ii} > 0$ for $i = 1, \ldots, N$ is not a necessary condition for the property that the row stochastic matrix D has a simple eigenvalue at 1 and all other eigenvalues have amplitude strictly less than 1. This can be seen from the example,

$$D = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

A necessary condition is that the graph contains a spanning tree and the row stochastic matrix is aperiodic.

Proof of Lemma 2.12. Based on the row stochastic matrix D, we can associate a Laplacian matrix $L = I - D$ where both the Laplacian matrix and the row stochastic matrix D are connected to the same graph but with different weighting of the edges. Based on Lemma 2.6, we know that L has a simple eigenvalue at 0 with associated right eigenvector given by $\mathbf{1}$. Clearly this implies that D has a simple eigenvalue at 1 with associated right eigenvector given by $\mathbf{1}$.

From Geršgorin's circle criterion, any eigenvalue of the matrix D is contained in a disc with center d_{ii} and radius $1 - d_{ii}$ for some i. Since $d_{ii} \neq 0$ we find that the only eigenvalue on the unit disc can be 1. Therefore all eigenvalues unequal to 1 have amplitude strictly less than 1.

2.5 Passivity

We define the concept of passivity for continuous- and discrete-time systems.

2.5.1 Continuous-Time System

Consider a general, strictly proper system Σ,

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{2.9}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. We first define passive and passifiable systems.

Definition 2.14 The system (2.9) is called *passive* if the system is square (i.e., $m = p$), and for initial condition $x(0) = 0$, for any input u, and for any $T \geq 0$, we have

$$\int_0^T y^{\mathrm{T}}(t)u(t)\,\mathrm{d}t \geq 0.$$

The system is called *passifiable via static output feedback* if the system is square and there exists a matrix H such that for initial condition $x(0) = 0$, for any input v and for any $T \geq 0$, we have

$$\int_0^T y^{\mathrm{T}}(t)u(t)\,\mathrm{d}t \geq \int_0^T y^{\mathrm{T}}(t)Hy(t)\,\mathrm{d}t.$$

The system is called *passifiable via static input feedforward* if the system is square and there exists a matrix R such that for initial condition $x(0) = 0$, for any input v and for any $T \geq 0$, we have

$$\int_0^T y^{\mathrm{T}}(t)u(t)\,\mathrm{d}t \geq \int_0^T u^{\mathrm{T}}(t)Ru(t)\,\mathrm{d}t.$$

The positive real lemma (see e.g., [5, 166]) gives an easy characterization when systems are passive.

Lemma 2.15 *Assume that (A, B) is controllable and (A, C) is observable with B and C full-column and full-row rank, respectively. The system (2.9) is passive if and only if there exists a matrix $P > 0$ such that*

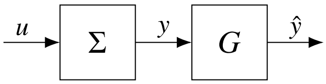
$$G(P) = \begin{pmatrix} PA + A^{\mathrm{T}}P & PB + C^{\mathrm{T}} \\ B^{\mathrm{T}}P - C & -D - D^{\mathrm{T}} \end{pmatrix} \leq 0. \tag{2.10}$$

Remark 2.16 For strictly proper systems the condition (2.10) reduces to

$$\begin{aligned} PA + A^{\mathrm{T}}P &\leq 0, \\ PB &= C^{\mathrm{T}}. \end{aligned} \tag{2.11}$$

Classical passivity requires the system to be square. For non-square systems, G-passivity and G-passifiability has been introduced in [29]. Given a prespecified $m \times p$-matrix G, a system (2.9) is called *G-passive* if the cascade of the system (2.9) with post-compensator $G \in \mathbb{R}^{m\times p}$ as shown in Fig. 2.2 is passive. Similarly, given a prespecified $m \times p$-matrix G, a system (2.9) is called *G-passive* if the cascade of the system (2.9) with post-compensator $G \in \mathbb{R}^{m\times p}$ as shown in Fig. 2.2 is passifiable via static output feedback.

Fig. 2.2 A G-passive system



From the positive-real lemma we almost immediately find a characterization of G-passivity.

Lemma 2.17 *Assume that* (A, B) *is controllable and* (A, C) *is observable with* B *and* GC *full-column and full-row rank, respectively. The system* (2.9) *is* G*-passive if and only if there exists a matrix* $P > 0$ *such that*

$$G(P) = \begin{pmatrix} PA + A^{\mathrm{T}}P & PB + C^{\mathrm{T}}G^{\mathrm{T}} \\ B^{\mathrm{T}}P - GC & -GD - D^{\mathrm{T}}G^{\mathrm{T}} \end{pmatrix} \leq 0. \tag{2.12}$$

Remark 2.18 For strictly proper systems the condition (2.12) reduces to

$$\begin{aligned} PA + A^{\mathrm{T}}P &\leq 0, \\ PB &= C^{\mathrm{T}}G^{\mathrm{T}}. \end{aligned} \tag{2.13}$$

From the literature on squaring down, it follows that such G should be designed, because in general G may introduce invariant zeros in $\mathbb{C}^+$ and in this case the system can never be passive or passifiable.

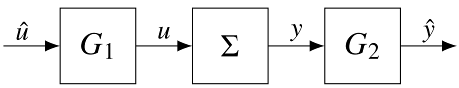
In this book, we consider the more general concepts of *squared-down passive*, *squared-down passifiable via static output feedback* and *squared-down passifiable via static input feedforward* for a non-square system (2.9) based on the idea of *squaring down* in [129]. Since this book is considering MAS with strictly proper systems, we define these concepts for strictly proper systems.

The system (2.9) is called *squared-down passive* if there exists a pre-compensator G_1 and a post-compensator G_2 such that the interconnection in Fig. 2.3 with input $\hat{u}$ and output $\hat{y}$ is passive. Assuming G_1 and G_2 are such that (A, BG_1) is controllable, (A, G_2C) is observable, this is equivalent to the existence of a positive definite matrix P, such that

$$\begin{aligned} PA + A^{\mathrm{T}}P &\leq 0, \\ PBG_1 &= C^{\mathrm{T}}G_2^{\mathrm{T}}. \end{aligned} \tag{2.14}$$

Remark 2.19 Note that when $G_1 = I$, squared-down passivity is similar to G-passivity as used in [29]. However, in [29] a more strict version of passivity is used which requires the system to be asymptotically stable. Our version can also be used for neutrally stable systems.

Fig. 2.3 A squared-down passive system



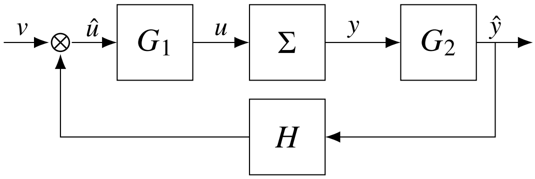


Fig. 2.4 A squared-down passive system via static output feedback

For a square system, i.e., $G_1 = G_2 = I$ squared-down passivity becomes conventional passivity.

Similar to the definition of G-passifiability, system (2.9) is called *squared-down passifiable via static output feedback* if there exists a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$, a post-compensator $G_2 \in \mathbb{R}^{q\times p}$, and a static output feedback

$$\hat{u} = -H\hat{y} + v = -HG_2 y + v, \qquad u = G_1\hat{u} \tag{2.15}$$

such that interconnection of the system (2.9) and the pre-feedback (2.15) is passive with respect to the new input v and output $\hat{y}$ as shown in Fig. 2.4.

Assume G_1 and G_2 are such that (A, BG_1) is controllable, (A, G_2C) is observable with BG_1 and G_2C full-column and full-row rank, respectively. In that case, squared-down passifiable via static output feedback is equivalent to the existence of a matrix H and a positive-definite matrix P such that

$$\begin{aligned} P(A - BG_1HG_2C) + (A - BG_1HG_2C)^{\mathrm{T}}P \le 0,\\ PBG_1 = C^{\mathrm{T}}G_2^{\mathrm{T}}. \end{aligned} \tag{2.16}$$

The system (2.9) is called *squared-down passifiable via static input feedforward* if there exists a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$, a post-compensator $G_2 \in \mathbb{R}^{q\times p}$, and an input feedforward matrix R, which make the interconnection of (2.9) and

$$z = R\hat{u} + \hat{y} = R\hat{u} + G_2 y, \qquad u = G_1\hat{u},$$

with respect to the new input $\hat{u}$ and the new output z, as shown in Fig. 2.5. If (A, BG_1) is controllable and (A, G_2C) is observable with BG_1 and G_2C full-column and full-row rank, respectively, this is equivalent to the existence of a positive definite matrix P such that

$$G(P) = \begin{pmatrix} PA + A^{\mathrm{T}}P & PBG_1 - C^{\mathrm{T}}G_2^{\mathrm{T}} \\ G_1^{\mathrm{T}}B^{\mathrm{T}}P - G_2C & -R - R^{\mathrm{T}} \end{pmatrix} \le 0. \tag{2.17}$$

Note that systems which are squared-down passifiable via static input feedforward are always neutrally stable which follows directly from (2.17).

Remark 2.20 If $G_1 = G_2 = I$ with the choice of $R = aI$, our squared-down passifiability via static input feedforward is reduced to input feedforward passivity as given in [119, Eq. (4)].

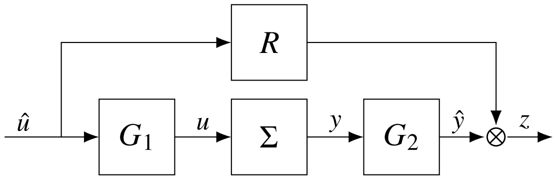


Fig. 2.5 Squared-down passive system via static input feedforward

Finally, we will define a class of agents, which are *squared-down minimum-phase with relative degree 1*. A system (2.9) is called *squared-down minimum-phase with relative degree 1* which uses a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$ and a post-compensator $G_2 \in \mathbb{R}^{q\times p}$ if the square system (A, BG_1, G_2C) is minimum-phase with relative degree 1 ($\det(G_2CBG_1) \neq 0$). Note that for such a system (A, BG_1, G_2C), with input $\hat{u}$, with $u = G_1\hat{u}$, and output $\hat{y} = G_2 y$ there exist non-singular state transformation matrices T_x and T_u with

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = T_x x, \qquad \tilde{u} = T_u \hat{u}$$

such that the dynamics of $\tilde{x}$ is represented by

$$\begin{cases} \dot{\tilde{x}}_1 = A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2, \\ \dot{\tilde{x}}_2 = A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + \tilde{u}, \\ \hat{y} = \tilde{x}_2, \end{cases} \tag{2.18}$$

where $\tilde{x}_1 \in \mathbb{R}^{n-m}$ and $\tilde{x}_2 \in \mathbb{R}^m$. Moreover, A_{11} is Hurwitz stable.

Remark 2.21 Minimum-phase agents with relative degree 1 are a subclass of passifiable agents via output feedback. The advantage is that the extra structure will enable a more explicit design. This connection has already been studied in [20] although they use state feedback. There they even consider weakly minimum-phase agents. However, in our view, our design is more transparant. Moreover, they only consider unweighted graphs.

2.5.2 Discrete-Time System

Consider a general, strictly proper system Σ,

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k) + Du(k), \end{cases} \tag{2.19}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. We first define passive and passifiable systems.

Definition 2.22 The system (2.19) is called *passive* if the system is square (i.e., $m = p$) and for initial condition $x(0) = 0$, for any input u and for any $T \geq 0$ we have

$$\sum_{k=0}^{T} y^{\mathrm{T}}(k)u(k) \geq 0.$$

The system is called *passifiable via static output feedback* if the system is square and there exists a matrix H such that for initial condition $x(0) = 0$, for any input v and for any $T \geq 0$, we have

$$\sum_{k=0}^{T} y^{\mathrm{T}}(k)u(k) \geq \sum_{k=0}^{T} y^{\mathrm{T}}(k)Hy(k).$$

The system is called *passifiable via static input feedforward* if the system is square and there exists a matrix R such that for initial condition $x(0) = 0$, for any input v and for any $T \geq 0$, we have

$$\sum_{k=0}^{T} y^{\mathrm{T}}(k)u(k) \geq \sum_{k=0}^{T} u^{\mathrm{T}}(k)Ru(k).$$

The positive real lemma (see e.g. [41]) gives an easy characterization when systems are passive.

Lemma 2.23 *Assume that (A, B) is controllable and (A, C) is observable with B and C full-column and full-row rank, respectively. The system (2.9) is passive if and only if there exists a matrix $P > 0$ such that*

$$G(P) = \begin{pmatrix} A^{\mathrm{T}}PA - P & A^{\mathrm{T}}PB + C^{\mathrm{T}} \\ B^{\mathrm{T}}PA - C & B^{\mathrm{T}}PB - D - D^{\mathrm{T}} \end{pmatrix} \leq 0. \tag{2.20}$$

Similar to continuous-time case, we can consider the more general concepts of *squared-down passive*, *squared-down passifiable via static output feedback* and *squared-down passifiable via static input feedforward* for a non-square system (2.9) based on the idea of *squaring down* in [129]. Since this book is considering MAS with strictly proper systems, we define these concepts for strictly proper systems. However, for discrete-time systems this implies that the system can never be *squared-down passive* and *squared-down passifiable via static output feedback*. However, strictly proper, discrete-time agents can be squared-down passifiable via static input feedforward.

Consider a linear system Σ given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{2.21}$$