

Monografie Matematyczne  
Instytut Matematyczny PAN  
Vol. 75

Antonio José Guirao  
Vicente Montesinos  
Václav Zizler

# Renormings in Banach Spaces

A Toolbox

 Birkhäuser



# Monografie Matematyczne

## Volume 75

Founded in 1932 by

S. Banach, B. Knaster, K. Kuratowski, S. Mazurkiewicz, W. Sierpinski, H. Steinhaus

### **Managing Editor:**

Yuri Tomilov, Polish Academy of Science, Warsaw, Poland

### **Editorial Board:**

Jean Bourgain, IAS, Princeton, USA

Joachim Cuntz, Universität Münster, Germany

Ursula Hamenstädt, Universität Bonn, Germany

Gilles Pisier, Texas A&M University, USA

Piotr Pragacz, IMPAN, Poland

Andrew Ranicki, University of Edinburgh, Edinburgh, UK

Slawomir Solecki, University of Illinois, Urbana-Champaign, USA

Przemyslaw Wojtaszczyk, IMPAN and Warsaw University, Poland

Jerzy Zabczyk, IMPAN, Poland

Henryk Zoladek, Warsaw University, Poland

Starting in the 1930s with volumes written by such distinguished mathematicians as Banach, Saks, Kuratowski, and Sierpinski, the original series grew to comprise 62 excellent monographs up to the 1980s. In cooperation with the Institute of Mathematics of the Polish Academy of Sciences (IMPAN), Birkhäuser now resumes this tradition to publish high quality research monographs in all areas of pure and applied mathematics.

Antonio José Guirao • Vicente Montesinos  
Václav Zizler

# Renormings in Banach Spaces

A Toolbox

 Birkhäuser

Antonio José Guirao  
Departamento de Matemática Aplicada  
Universitat Politècnica de València  
Valencia, Spain

Vicente Montesinos  
Departamento de Matemática Aplicada  
Universitat Politècnica de València  
Valencia, Spain

Václav Zizler  
University of Alberta  
Edmonton, AB, Canada

ISSN 0077-0507

ISSN 2297-0274 (electronic)

Monografie Matematyczne

ISBN 978-3-031-08654-0

ISBN 978-3-031-08655-7 (eBook)

<https://doi.org/10.1007/978-3-031-08655-7>

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This book is published under the imprint Birkhäuser, [www.birkhauser-science.com](http://www.birkhauser-science.com) by the registered company Springer Nature Switzerland AG

The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

**Dedicated to the memory of Zdeněk Frolík**

who managed to unite the classical Čech topological tradition  
with the activities of the new Banach space workshop in Prague

A mathematician is a person who can find analogies between theorems. A better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.

Stefan Banach

# Contents

<b>Preface</b>	<b>xvii</b>
Introduction . . . . .	xvii
Notation . . . . .	xxv
Typography . . . . .	xxvi
<b>I An Introductory Course in Renorming</b>	<b>1</b>
<b>1 Norms, normed spaces, Banach spaces</b>	<b>3</b>
1.1 Norms, normed and Banach spaces . . . . .	3
1.2 Equivalent norms . . . . .	5
1.2.1 Definition . . . . .	5
1.3 Duality . . . . .	5
1.4 A few basic examples . . . . .	10
1.5 Finite-dimensional spaces . . . . .	12
<b>2 Some basic definitions and tools</b>	<b>15</b>
2.1 A short utility-grade approach to locally convex spaces . . . . .	15
2.2 Extreme points . . . . .	18
2.3 Some basic results in Banach space theory . . . . .	23
<b>3 Equivalent norms</b>	<b>27</b>
3.1 On the definition of equivalent norms . . . . .	27
3.2 Finite-dimensionality and equivalent norms . . . . .	29
3.3 Dual equivalent norms . . . . .	30
3.3.1 Norms on the dual versus dual norms . . . . .	30
3.4 Equivalent norms and norming subspaces . . . . .	35
3.4.1 Norming and 1-norming subspaces . . . . .	35
3.4.2 Separating subspaces . . . . .	36
3.4.3 Characterizing norming subspaces . . . . .	38
3.4.4 Getting norming subspaces . . . . .	40
3.5 Some other results related to norming subspaces . . . . .	44
3.6 The interplay between norms and their unit balls . . . . .	45
3.6.1 Minkowski functionals . . . . .	45
3.6.2 Support functions . . . . .	46
3.6.3 Polarity . . . . .	47
3.6.4 Fenchel conjugation . . . . .	48
3.6.5 The infimal convolution of two convex functions . . . . .	51
3.7 Norm and ball arithmetics; duality . . . . .	54

3.7.1	Sum of two norms . . . . .	55
3.7.2	Inverse summation of two norms . . . . .	56
3.7.3	Supremum of two norms . . . . .	58
3.7.4	The infimal convolution of two norms . . . . .	59
3.7.5	A basic result . . . . .	60
3.7.6	Dealing with seminorms . . . . .	60
3.7.7	The predual norm of a $p$ -sum of $w^*$ -lower semicontinuous seminorms . . . . .	63
<b>4</b>	<b>Basic differentiability in Banach spaces</b>	<b>65</b>
4.1	Gâteaux and Fréchet differentiability . . . . .	66
4.1.1	The basic definitions . . . . .	66
4.1.2	Lipschitz functions I . . . . .	68
4.1.3	Convex functions . . . . .	70
4.1.4	The case of the norm . . . . .	76
4.1.5	The set of points of differentiability of general functions on a Banach space . . . . .	83
4.2	Strengthening Gâteaux differentiability . . . . .	86
4.3	Uniformities in differentiability . . . . .	87
4.4	Norms defined by level sets . . . . .	91
<b>5</b>	<b>Basic rotundity</b>	<b>95</b>
5.1	Strict convexity . . . . .	95
5.2	Uniform convexity . . . . .	97
5.2.1	Introduction . . . . .	97
5.2.2	A consequence of the uniform convexity . . . . .	97
5.2.3	A weak version of the uniform convexity . . . . .	98
5.3	Local uniform convexity . . . . .	99
5.3.1	Introduction . . . . .	99
5.3.2	Some consequences of the local uniform convexity . . . . .	100
5.3.3	A weak version of the local uniform rotundity property . . . . .	103
5.4	Topological properties of the unit sphere . . . . .	104
5.5	Variants of uniform convexity and local uniform convexity . . . . .	107
5.5.1	2-rotundness and weakly 2-rotundness . . . . .	107
5.5.2	Rotundness in directions . . . . .	107
5.5.3	Near uniform convexity . . . . .	109
5.5.4	Midpoint and weak midpoint local uniform convexity . . . . .	109
5.6	Using the square of the norm . . . . .	111
5.6.1	Some useful computations using $\ \cdot\ ^2$ , and some tools . . . . .	112
5.6.2	Describing rotundity . . . . .	113
5.7	Extension of rotund norms . . . . .	116

<b>6</b>	<b>Structural properties of spaces</b>	<b>119</b>
6.1	Bases and projectional resolutions of the identity . . . . .	119
6.1.1	Biorthogonal systems . . . . .	119
6.1.2	Markushevich bases . . . . .	119
6.1.3	Schauder bases . . . . .	121
6.1.4	Projectional resolutions of the identity I . . . . .	122
6.1.5	Markushevich bases versus projectional resolutions of the identity . . . . .	123
6.1.6	Several classes of Banach spaces defined in terms of Marku- shevich bases or projectional resolutions of the identity . .	124
6.1.7	Superreflexivity, finite representability . . . . .	125
6.1.8	Generation and strong generation . . . . .	126
6.2	Some classes of compact topological spaces . . . . .	127
6.3	Extra definitions needed . . . . .	128
<b>7</b>	<b>The use of Šmulyan’s tests</b>	<b>131</b>
7.1	Some comments on the Šmulyan tests . . . . .	131
7.2	Šmulyan’s lemma and $w^*$ -convergence . . . . .	133
<b>8</b>	<b>Asplund averaging I</b>	<b>135</b>
8.1	Asplund original approach . . . . .	135
8.2	Applications . . . . .	138
<b>9</b>	<b>Tools for renorming</b>	<b>141</b>
9.1	Two simple facts . . . . .	141
9.2	An elementary strictly convex renorming . . . . .	143
9.2.1	A strictly convex renorming of the space . . . . .	143
9.2.2	A strictly convex renorming of the dual space . . . . .	144
9.2.3	A strictly convex and Gâteaux smooth renorming of the space	145
9.3	Transfer methods by using the square of the norm . . . . .	145
9.3.1	Clarkson’s method . . . . .	146
9.4	Some “ad-hoc” norms on classical spaces . . . . .	151
9.4.1	Day’s norm I: Day’s norm on $\ell_\infty(\Gamma)$ . . . . .	151
9.4.2	Some other particular norms on $c_0(\Gamma)$ . . . . .	154
9.4.3	Phelps’ norm on $\ell_1$ . . . . .	156
9.5	Locally uniformly rotund renormings . . . . .	157
9.5.1	Kadets renorming theorem . . . . .	157
9.5.2	Some consequences of Kadets’ renorming theorem . . . . .	163
9.5.3	Kadets homeomorphism theorem for separable spaces . . .	164
9.5.4	Some statements concerning homeomorphic normed spaces	169

<b>II</b>	<b>An Intermediate Course in Renorming</b>	<b>171</b>
<b>10</b>	<b>Locally uniformly convex renorming of nonseparable spaces</b>	<b>173</b>
10.1	Weakly compactly generated spaces I: Some introductory results . . .	174
10.2	Lindenstrauss–Troyanski tools . . . . .	176
10.2.1	A basic Lindenstrauss lemma . . . . .	177
10.2.2	An application to reflexive spaces . . . . .	178
10.2.3	A glimpse of the weakly compactly generated case and Hahn–Banach extension operators . . . . .	180
10.3	Projectional resolutions of the identity II: A tool . . . . .	182
10.4	Troyanski’s renorming theorem, and extensions . . . . .	187
10.4.1	Troyanski’s original proof . . . . .	187
10.4.2	Moltó–Orihuela–Troyanski transfer method . . . . .	188
10.4.3	Godefroy–Fabian transfer results . . . . .	193
10.5	Renormings for weakly compactly generated spaces . . . . .	196
10.6	Weakly compactly generated spaces II . . . . .	198
10.7	Day’s norm II: Its locally uniformly convex behaviour on $c_0(\Gamma)$ . .	199
10.8	Asplund averaging II . . . . .	202
<b>11</b>	<b>Variational principles</b>	<b>205</b>
11.1	Lindenstrauss’ extremal structure of $\ell_1$ . . . . .	205
11.2	Lindenstrauss’ results on norm-attaining operators . . . . .	207
11.3	Norm-attaining operators and strongly exposed points . . . . .	209
11.4	Farthest points . . . . .	211
11.5	Properties $\alpha$ and $\beta$ . Uniformly strongly exposed points . . . . .	215
11.6	Asplund spaces I . . . . .	216
11.6.1	Introduction . . . . .	216
11.6.2	Some equivalences . . . . .	219
11.6.3	Some tools and proofs on rough norms . . . . .	223
11.7	The use of variational principles . . . . .	231
11.7.1	Ekeland’s and smooth variational principles . . . . .	231
11.7.2	Other geometric statements equivalent to the Ekeland variational principle . . . . .	237
11.7.3	The compact variational principle and related results . . . .	240
<b>12</b>	<b>Projectional resolutions of the identity III</b>	<b>249</b>
12.1	Introduction . . . . .	249
12.2	Four useful properties of projectional resolutions of the identity . .	250
12.3	Spaces with projectional resolutions of the identity . . . . .	252
12.4	Examples of spaces without projectional resolutions of the identity	255
12.5	Results on projectional resolutions of the identity in the 1970s and 1980s . . . . .	256

<b>13 Smooth approximation of norms by norms</b>	<b>261</b>
13.1 Smooth approximations in separable spaces . . . . .	261
13.2 Smooth approximation in nonseparable spaces . . . . .	262
<b>14 Smooth partitions of unity in nonseparable spaces</b>	<b>263</b>
<b>15 Smooth norms in dense subspaces</b>	<b>269</b>
<b>16 Miscellaneous applications</b>	<b>271</b>
16.1 Difference of two convex continuous functions . . . . .	271
16.2 Some topological issues . . . . .	271
16.3 An application of locally uniformly rotund renorming to lower semi- continuous functions . . . . .	273
<b>17 Bumps depending locally on finitely many coordinates</b>	<b>275</b>
<b>18 Summary on renorming for uniformly rotund in every direction, strictly convex, and weakly uniformly rotund spaces</b>	<b>277</b>
<b>19 Examples on <math>C^1</math>-smoothness</b>	<b>279</b>
19.1 Uniformly Gâteaux differentiable norms and related results . . . . .	279
19.2 Uniformly Kadets–Klee smooth norms . . . . .	282
<b>20 Examples on Rotundity</b>	<b>285</b>
20.1 Normal structure . . . . .	285
20.2 M.A. Smith’s renormings of $\ell_2$ . . . . .	290
20.3 M.A. Smith’s renormings of $c_0$ and $\ell_1$ . . . . .	294
20.4 A chart . . . . .	302
 <b>III Advances and Developments in Renorming, and Applications</b>	 <b>303</b>
<b>21 Nonlinear transfer techniques</b>	<b>305</b>
21.1 Deville’s master lemma and applications . . . . .	305
21.2 Nonlinear transfer . . . . .	311
<b>22 Lipschitz functions II</b>	<b>315</b>
22.1 The Rademacher theorem . . . . .	316
22.2 Preiss’ differentiation of Lipschitz functions . . . . .	321
<b>23 Spaces isomorphic to Hilbert spaces</b>	<b>323</b>
<b>24 Superreflexive spaces</b>	<b>325</b>
<b>25 The Kingdom of Tsirelson’s space</b>	<b>329</b>

<b>26</b>	<b>The <math>\mathcal{L}_\infty</math> spaces</b>	<b>335</b>
<b>27</b>	<b>Higher-order smoothness</b>	<b>337</b>
27.1	Higher-order Gâteaux smooth norms . . . . .	337
27.2	Higher-order smoothness . . . . .	338
27.3	Smooth norms in $C(K)$ spaces . . . . .	342
27.4	Survey on higher-order smooth norms on $C(K)$ spaces . . . . .	345
<b>28</b>	<b>James boundaries</b>	<b>349</b>
28.1	Introduction . . . . .	349
28.2	The boundary problem and the strong boundary problem . . . . .	350
28.3	Some techniques . . . . .	352
<b>29</b>	<b>The Radon–Nikodým property</b>	<b>357</b>
<b>30</b>	<b>Strongly subdifferentiable norms</b>	<b>361</b>
<b>31</b>	<b>The Banach–Saks property</b>	<b>363</b>
<b>32</b>	<b>Transitive norms</b>	<b>365</b>
<b>33</b>	<b>Norms with the Mazur intersection property</b>	<b>367</b>
<b>34</b>	<b>Nicely smooth Banach spaces</b>	<b>369</b>
<b>35</b>	<b>Weak Hadamard differentiability</b>	<b>373</b>
35.1	Introduction . . . . .	373
35.2	Sequential convergence in $X^*$ , boundedness, and differentiability . . . . .	373
<b>36</b>	<b>Lipschitz Asplund spaces</b>	<b>377</b>
<b>37</b>	<b>Lipschitz-free spaces</b>	<b>379</b>
37.1	Introduction . . . . .	379
37.2	Lipschitz-free spaces . . . . .	379
37.3	The extremal structure of $B_{\mathcal{F}(M)}$ . . . . .	382
37.4	Lipschitz-free spaces on Banach spaces . . . . .	382
<b>38</b>	<b>Polyhedral spaces</b>	<b>385</b>
38.1	Polyhedral spaces . . . . .	385
38.2	Tilings . . . . .	388
<b>39</b>	<b>Smooth functions on <math>c_0(\Gamma)</math></b>	<b>391</b>

<b>40</b>	<b>Kottman-type results on separated sets</b>	<b>393</b>
40.1	Whitley’s results . . . . .	393
40.2	Separated and symmetrically separated sets . . . . .	394
40.3	Equilateral sets in infinite-dimensional spaces . . . . .	396
<b>41</b>	<b>Three-space properties</b>	<b>399</b>
<b>42</b>	<b>Polynomials on Banach spaces</b>	<b>401</b>
<b>43</b>	<b>Szlenk derivation and applications</b>	<b>403</b>
<b>44</b>	<b>Further miscellaneous applications</b>	<b>407</b>
44.1	Intermediate differentiation of Lipschitz functions . . . . .	411
44.2	Bates’ results, onto mappings, ranges of derivatives . . . . .	412
<b>45</b>	<b>Miscellaneous topics</b>	<b>415</b>
45.1	Phelps’ property U . . . . .	415
45.2	Pointwise uniformly rotund spaces . . . . .	417
45.3	Spaces with $w^*$ -sequentially compact dual balls . . . . .	418
45.4	Injections I . . . . .	421
45.5	Vařák spaces II . . . . .	424
45.6	Uniformly Gâteaux differentiable norms II . . . . .	426
45.7	Strongly uniformly Gâteaux differentiable norms . . . . .	428
45.8	“Pathologies” in weakly compactly generated spaces . . . . .	428
45.9	Unconditional bases . . . . .	429
45.10	Fundamental biorthogonal systems . . . . .	431
<b>46</b>	<b>Weakly compactly generated spaces and their relatives III</b>	<b>433</b>
46.1	Asplund spaces II . . . . .	435
46.2	Scattered compacta and Asplund spaces II . . . . .	436
46.3	$\sigma$ -Fréchet smooth norms . . . . .	438
46.4	The Daugavet property . . . . .	440
<b>47</b>	<b>Valdivia compacta</b>	<b>443</b>
<b>48</b>	<b>Renorming classical spaces</b>	<b>445</b>
48.1	Spaces of bounded functions . . . . .	445
48.1.1	Strict convexity . . . . .	446
48.1.2	Smoothness . . . . .	452
48.1.3	Spaces of bounded functions with countable support . . . . .	458
48.2	“Haydon’s forest” . . . . .	459
48.3	Injections II . . . . .	461
48.4	Johnson–Lindenstrauss spaces . . . . .	461
48.5	Markushevich bases II . . . . .	468
48.6	Weakly Lindelöf determined spaces I . . . . .	470

48.7	Double arrow space . . . . .	473
48.8	Space of continuous functions on ordinals . . . . .	473
48.9	Weakly Lindelöf determined spaces II . . . . .	474
48.10	Kunen compact space . . . . .	474
48.11	Strongly weakly compactly generated spaces . . . . .	475
48.12	Effros–Borel structure . . . . .	477
<b>49</b>	<b>Symmetric norms</b>	<b>479</b>
<b>50</b>	<b>Strictly convex renorming</b>	<b>481</b>
50.1	Introduction . . . . .	481
50.2	Characterization of strictly convex renormings . . . . .	482
<b>51</b>	<b>Some coordinates</b>	<b>485</b>
51.1	Differentiability . . . . .	485
51.1.1	Gâteaux differentiability (G) . . . . .	485
51.1.2	Uniform Gâteaux differentiability (UG) . . . . .	486
51.1.3	Very smoothness (VS) . . . . .	486
51.1.4	Strong uniform Gâteaux differentiability (SUG) . . . . .	486
51.1.5	Nice smoothness (NS) . . . . .	486
51.1.6	Fréchet differentiability (F) . . . . .	487
51.1.7	$C^2$ -smoothness . . . . .	487
51.1.8	$C^\infty$ -smoothness . . . . .	487
51.2	Rotundity . . . . .	487
51.2.1	Strict convexity (R) . . . . .	487
51.2.2	Local uniform convexity (LUR) . . . . .	488
51.2.3	2-rotundness (2R) . . . . .	488
51.2.4	Weak 2-rotundness (W2R) . . . . .	488
51.2.5	Average local uniform convexity (ALUR) . . . . .	488
51.2.6	Midpoint local uniform convexity (MLUR) . . . . .	489
51.2.7	Near uniform convexity (NUC) . . . . .	489
51.2.8	Uniformly rotund in every direction (URED) . . . . .	489
51.2.9	Weak uniform convexity (WUR) . . . . .	490
51.3	Some extra properties . . . . .	490
51.3.1	Kadets–Klee and sequential Kadets–Klee (KK) (SKK) . . . . .	490
51.3.2	Uniform Kadets–Klee (UKK) . . . . .	490
51.3.3	Normal structure . . . . .	490
51.3.4	Asplundness . . . . .	490
51.3.5	Mazur intersection property (MIP) . . . . .	491
51.4	Weak compact generation (WCG) and relatives . . . . .	491
51.4.1	Weak Lindelöf determinacy (WLD) . . . . .	491
51.5	Injections . . . . .	492
51.6	Markushevich bases . . . . .	493
51.7	Miscellanea . . . . .	493

<b>52 Open questions</b>	<b>495</b>
52.1 Differentiability . . . . .	495
52.1.1 Gâteaux differentiability . . . . .	495
52.1.2 Fréchet differentiability . . . . .	496
52.1.3 Higher-order smoothness . . . . .	497
52.1.4 Space $c_0$ . . . . .	498
52.1.5 Space $\ell_1$ . . . . .	498
52.1.6 Space $\ell_2$ . . . . .	498
52.1.7 $C(K)$ and related spaces . . . . .	498
52.2 Asplund and weak Asplund spaces . . . . .	499
52.3 Rotundity . . . . .	500
52.4 Weakly compactly generated spaces and relatives . . . . .	501
52.5 Auerbach and Markushevich bases . . . . .	501
52.6 Miscellanea . . . . .	501
52.7 Distortion, hereditarily indecomposable spaces . . . . .	503
52.8 Mazur intersection property . . . . .	503
52.9 Lipschitz functions and Lipschitz Asplund spaces . . . . .	504
52.9.1 Lipschitz functions . . . . .	504
52.9.2 Lipschitz-free spaces . . . . .	504
52.9.3 Lipschitz Asplund spaces . . . . .	504
<b>Bibliography</b>	<b>505</b>
<b>List of Figures</b>	<b>549</b>
<b>Indices and how to use them</b>	<b>553</b>
General Index . . . . .	553
Index of Symbols . . . . .	553
Index of Authors . . . . .	553
Renormings . . . . .	553
Impossible Renormings . . . . .	555
<b>General Index</b>	<b>557</b>
<b>Symbol Index</b>	<b>606</b>
<b>Author Index</b>	<b>610</b>
<b>Renorming Index</b>	<b>615</b>
<b>Impossible Renorming Index</b>	<b>624</b>

# Preface

## Introduction

Banach spaces provide a framework for modern mathematical analysis, as they blend classical analysis, geometry, topology and linearity. This makes Banach space theory a wonderful, elegant, and active research area in Mathematics. Many problems in modern linear and nonlinear analysis are of infinite-dimensional nature. Renormings represent a big and important part. They deal with structural properties, geometry, differential calculus etc. In this area, there are many important open problems.

In this text, “differentiability” and “smoothness” are synonymous terms, as they are “rotundness” and “strict convexity”. These and related concepts will be properly defined in Chapters 4 and 5, respectively.

Questions concerning the supply of smooth functions on Banach spaces are of crucial importance in differential calculus. Smooth functions are usually obtained from equivalent smooth norms and smooth norms are in turn often constructed by using dual rotund norms. The existence of equivalent smooth or rotund norms or nontrivial smooth functions on a particular Banach space depends on its structure and has a profound impact on its geometry. For example, a separable Banach space that admits a Fréchet smooth function with bounded support must have a separable dual (Theorems 276, 283, and 284); if this function is, moreover, infinitely differentiable, then the space must contain a copy of  $c_0$  or  $\ell_p$  (Theorem 516). If the space does not contain  $c_0$  and has a  $C^2$ -smooth function with bounded support, it is superreflexive (Theorem 453). If for a compact space  $K$ , its Cantor derivative  $K^{(\omega_1)}$  is empty, then  $C(K)$  admits a  $C^\infty$ -smooth norm (Theorem 813). However, there is  $K$  such that  $K^{(\omega_1)}$  is a singleton and  $C(K)$  does not admit any Gâteaux differentiable norm [Hay99] (cf., e.g., [DeGoZi93, Chapter VII], [Ziz03, Theorem VII.6.4], and Theorem 529).

This text attempts to provide a modern and concise guide to some basic renorming results and techniques of Banach spaces and their applications, and discusses some streams of contemporary research in this area. We tried to keep a reader-friendly relaxed form to enjoy the text. For understanding the relations of concepts, we provide here as many counterexamples as possible. We made an effort to write the text in an informative manner, briefly pointing out main ideas and motivations for the results and the connections between them. The experienced reader may skip some detailed explanations in the introductory parts of the text. We include them to provide for some training in techniques used in this area, specially for young students. In many instances, specially in more advanced topics, we prefer to sketch

the construction and motivate the result pointing out the main idea behind, better than to provide all details of a proof. In case that it is missing at all, we refer then either to the original paper or to books where the results appear in full.

This style enables us to discuss much wider spectrum of things and their relations than it could go to the regular book form. This allows us to include observations and comments that may help the reader, suggesting ways to complete the information provided here.

The text can be considered as a commented guide to optional graduate courses in Banach spaces. It starts with a review of basic tools and continues with several rather independent sections for the instructor's choice. The course would discuss many motivating open problems, and thus provides material for a research seminar as well.

The central topic of the text consists in renorming Banach spaces. This means to endow the given space with an equivalent norm that enjoys extra properties:

If  $\|\cdot\|$  is a norm on a Banach space  $X$ , another norm  $\|\|\cdot\|\|$  is **equivalent** to  $\|\cdot\|$  if two positive numbers  $\alpha$  and  $\beta$  exist such that  $\alpha\|x\| \leq \|\|\cdot\|\| \leq \beta\|x\|$  for all  $x \in X$ .

Observe that the topology on  $X$  induced by the norm does not change when such new norm is adopted, although the metric properties may be really different. The renorming theory has thus many applications in the geometry of Banach spaces and in problems in analysis on them, and it is an active, large area of research in Banach spaces.

Note that in a given finite-dimensional space, all norms are equivalent (see Proposition 3 below). On the other hand, on any given infinite-dimensional Banach space  $(X, \|\cdot\|)$  there is always a norm  $\|\|\cdot\|\|$  that is not equivalent to  $\|\cdot\|$  (see again Proposition 3).

Usually, any good result in renorming has many corollaries and it ensures many applications, since it is often stronger than the mere required particular property. Consider in this direction the Krein–Milman Theorem 7, differentiability of convex functions, and some other properties discussed below in the text.

In analysis in infinite dimensions, some problems require further techniques for their solution. For example, though every finite convex function on a finite-dimensional space is necessarily continuous (see Theorem 52), note that every infinite-dimensional Banach space admits a discontinuous linear functional (see the paragraph after the proof of Theorem 49). Bounded linear functionals do not necessarily attain their maxima on closed balls (as an easy example, observe that if  $B_{c_0}$  is the closed unit ball of the space  $c_0$  and  $f$  is the continuous linear functional on  $c_0$  defined by  $f(x) = \sum \frac{1}{2^i} x_i$  for  $x = (x_i)$ , then the supremum of  $f$  over  $B_{c_0}$  is 1, but it is not difficult to show that for every  $x \in B_{c_0}$ ,  $f(x) < 1$ . Therefore there

is no closest point to the origin in this  $f^{-1}(1)$ . Also, there is no extreme point in  $B_{c_0}$ . Another example of a continuous linear functional that does not attain its supremum on the closed unit ball of a Banach space is given after Theorem 12).

An infinite-dimensional Rolle's theorem does not hold true (see, e.g., [FHHMZ11, Exercise 7.25]), and so new variational principles are needed. We will meet among others Ekeland's, compact, and smooth variational principles here (see Chapter 11).

Let us see an example of how renormings can help, typically, in some of these questions. Let  $(X, \|\cdot\|_1)$  be a reflexive Banach space and let  $\|\cdot\|_2$  be an equivalent strictly convex norm on  $X$  (its existence is guaranteed by a theorem of S.L. Troyanski (Theorem 218 below)). By a result of K.S. Lau (Theorem 260 below), there exists an interior point  $c_1$  of  $B_{\|\cdot\|_1}$  from which a point  $y_1 \in S_{\|\cdot\|_1}$  at farthest  $\|\cdot\|_2$ -distance exists. Then it is not difficult to see that  $y_1$  is an extreme point of  $B_{\|\cdot\|_1}$ . We shall see in the text that, in a similar way, J. Lindenstrauss proved that all closed bounded convex sets in reflexive spaces do have an extreme point with extra properties (Theorem 259), by using M.I. Kadets and S.L. Troyanski renorming by locally uniformly convex norms (a property better than simple strict convexity). Similarly, if  $x_0$  is a point of Fréchet smoothness of the second norm, then  $x_0$  is also a Fréchet smooth point of the first norm. This, via a dualization of the Kadets–Troyanski result and the Lindenstrauss method, gives Fréchet smooth points on balls in reflexive spaces (see Item 5 in Subsection 5.3.2). These results of M.I. Kadets, Troyanski, and Lindenstrauss have been one of the most important results in this part of Banach space theory. Let us note that these are in a sense the best possible results in this direction; as we shall see later, measure theory can hardly be used here. It is the (miraculous) power of the concept of differentiability that allows for refraining from the requirement of reflexivity in the cases mentioned above. This is one of the goals of the smooth variational principle (Theorem 289) we will meet in this text.

Another thing we would like to mention at the beginning is that M.I. Kadets proved that if  $X$  is a Banach space with a Schauder basis  $\{e_i, f_i\}$ , then  $X$  can be renormed so that if  $\|x_n\| \rightarrow \|x\|$  and  $f_i(x_n) \rightarrow f_i(x)$  for each  $i$ , then  $\|x_n - x\| \rightarrow 0$  (Corollary 177). So this norm shares this property with the Hilbertian norm. He used this approach for solving the long-standing problem of M. Fréchet and S. Banach whether that all infinite-dimensional separable Banach spaces are homeomorphic (Theorem 186). We shall discuss it in this text.

Let us briefly explicit here our point of view about the actual situation of renorming:

1. The present understanding of the basics is rather satisfactory in the separable case for local uniform rotundity,  $C^1$ -Fréchet and Gâteaux smoothness.
2. The understanding of basics in local uniform rotundity and  $C^1$ -Fréchet smoothness is relatively satisfactory for the class of weakly compactly generated spaces and their relatives.

3. The understanding of Gâteaux smoothness is not yet satisfactory in nonseparable spaces.
4. The understanding of higher order smoothness is not yet satisfactory in separable or nonseparable spaces.
5. There is a huge potential for more research in applications of renormings in linear and nonlinear analysis, in particular in Convexity Theory and Approximation.

There are many techniques for providing equivalent norms in specific Banach spaces or in wide classes of them. The state of affairs in Renorming Theory up to 1993 was presented in [DeGoZi93]. Later on, some remarkable contributions appeared, and some of them are presented in [MOTV09]. Now it is almost impossible to collect all sources and references in this area. Some are provided in the bibliography at the end of this manual. Among them we may add to the previously mentioned references, for example, [BeLi00], [BoVa09], [Di75], [Fab97], [FHHMPZ01], [FHHMZ11], [Go01b], [GoLaZi14], [HHZ96], [HVMZ07], [HáJo14], [HáMoZi12], [Meg98], [PelBe79], and [Ziz03]. Only a tiny part of the theory and techniques of renorming was discussed in [FHHMZ11] and [HVMZ07]. This is why we prepared this text, where we try to upgrade the information, including also some examples, counterexamples, and application of renorming. A big portion of the results are mentioned here for the first time in book form. Emphasis will be done on basic techniques, so that the reader may use them—or convenient variations—for constructing norms with various required properties.

When presenting and using the various relevant concepts we shall stress the relationship between them. Quite often results—even important ones—come without a detailed proof (only a sketch of the underlying ideas is provided), and many of them actually without proof at all (in those cases we shall always refer to sources). There are four main reasons for skipping some proofs: First, if incorporated, then the length of the text would have been unpractical. Second, that nowadays almost—if not all—sources are available at a click, and what is missing is, maybe, a map to search and locate what the interested reader is looking for. Third, that we prefer to focuss instead on the underlying ideas that may enlighten the panorama, especially stressing what is missing in some other texts like, e.g., [FHHMZ11], [DeGoZi93], [HVMZ07]. And finally, that we have in mind that the most efficient streamlined proof is, sometimes, not the best for a future use. In other words, that reading the original—sometimes distant—sources is, quite often, not only an exercise in mathematical history, but a must for true understanding (read the masters!). Let us quote here N.H. Abel: “It appears to me that if one wants to make progress in mathematics one should study the masters and not the pupils” (N.H. Abel (1802–1829), quoted from an unpublished source by O. Ore in [Ore74, p. 138]). The Abstract of H.M. Edwards in [Edw81], that we reproduce here, insists on this matter: “It is as good an idea to read the masters now as it was in Abel’s time. The best mathematicians know this and do it all the time. Unfortunately,

students of mathematics normally spend their early years using textbooks (which may be, but usually aren't, written by masters) and taking lecture courses which are self-contained and make little or no reference to the primary literature of the subject. The students are left to discover on their own the wisdom of Abel's advice. In this they are being cheated".

This way the text may help the reader to get an overview of the whole body of the renorming theory better than to focus on detailed explanations— trying to summarize results on individual classes of spaces—. Thus, we can often afford to discuss more than one proof of the given result. For example, we discuss several proofs of the fact that non-trivial Fréchet differentiable functions cannot live in a space where there exists a convex continuous function that is nowhere Fréchet differentiable: Kurzweil's, Ekeland's, Whitfield's, Borwein–Preiss, and Deville's methods. We show several alternative proofs of the Bishop–Phelps Theorem 14. We discuss several methods in constructing projections in nonseparable spaces: Lindenstrauss', Plichko's, Valdivia's, Orihuela's, and Troyanski's methods, just to name a few authors involved. For relatives of weakly compactly generated spaces we consider both Troyanski et al. and Godefroy et al. transfer methods. Note they both involve a very sophisticated use of the interplay of convexity and compactness. We will see in the text that each one of them is used many times in this area. In the other direction, we use in several circumstances recent results to provide streamlined proofs of some older ones. This approach also enables us to effectively summarize results concerning a given individual property.

In any case, to make this text easily readable, we shall try to provide all the required definitions, make comments on them, and give hints to significant results—especially if the material is not easily accessible— trying to be to some extent self-contained. We hope that the text may be interesting to experts in Banach spaces in general, as a second reading on renormings, and that may help in supervising PhD theses on this subject. To assist the reader, we prepared organized lists of properties of particular spaces and in Chapter 51 a list of coordinates of various counterexamples. In Chapter 52 a list of easier problems to start independent work on this subject is included.

Of course, some basic knowledge of Banach space theory is required, as contained in basic undergraduate courses in functional analysis and in the introductory chapters of [FHHMZ11]. For a more extensive background, the reader may consult, for instance, the introductory article by Johnson and Lindenstrauss [JoLi03], [FHHMZ11], [LinTz77], [Di84], and [AlKa06]. In particular, it is an asset if the reader is more or less familiar with some standard known implications between the very basic notions of rotundity, smoothness and their duality, as contained, e. g., in [DeGoZi93], [FHHMZ11, Chapters 7, 8]. A certain knowledge of convex functions and, in particular, norms, is certainly needed. Practicing on this is a must. We may suggest the reader to look, for example, to [BoVa09], [FHHMZ11], [HVMZ07], [Phe93], [Gil82], and [MoZiZi15].

A basic knowledge of set theory and a certain background in topology are a must. We may suggest to consult [BessPe75], [Du66], [Eng85], [Kam50], and [Kel55].

There are many crucial open problems in the renorming area. Some of them are discussed, e. g., in [DeGoZi93], [FHMMZ11], [HMMVZ07], [GuiMoZi16], and in [HáJo14]. A **bump function** (or just a **bump**) on a Banach space  $X$  is a function with nonempty bounded support. If  $\phi$  is a  $C^k$ -smooth function on the real numbers such that  $\phi(t) = 1$  if  $t \in [-1, 1]$  and  $\phi(t) = 0$  if  $|t| \geq 2$ , and  $\|\cdot\|$  is a  $C^k$ -smooth norm —outside the origin— on  $X$ , then the composition  $\phi(\|x\|)$  is a  $C^k$ -smooth bump. We will see that there are nonseparable Banach spaces that admit  $C^\infty$ -smooth bumps but do not admit Gâteaux differentiable norms (Theorem 817). We will show that the sup norm of  $C[0, 1]$  is nowhere Fréchet differentiable (Proposition 280) and that this space admits no Fréchet differentiable bump (Theorem 288). We will see that this is unlike finite-dimensional spaces, where Rademacher's Theorem 438 holds saying that any real-valued Lipschitz function on a finite-dimensional Banach space is Fréchet differentiable on a dense set of points. We will also meet the remarkable Preiss' Theorem 443 that on Asplund spaces —in particular, on Hilbert spaces— every real-valued Lipschitz function is Fréchet differentiable at points of a dense set.

Let us note in passing, for example, that the following problems are open.

1. Assume that a separable space  $X$  admits a  $C^2$ -smooth bump. Does it admit a  $C^2$ -smooth norm?
2. If  $X$  is an Asplund space, does  $X$  admit a  $C^1$ -smooth bump?
3. If  $X$  admits a  $C^1$ -smooth bump, does it allow for smooth approximations of continuous functions?
4. It is unknown if every norm on  $\ell_2$  can be approximated by real-analytic norms.
5. From the general theory, it should be noted that, amazingly, it is unknown whether every nonseparable Banach space has a infinite-dimensional separable quotient, see [GuiMoZi16, Problem 32].

This and many other open problems, famous and less famous, are discussed in this text. A list of them appears in Chapter 52. For a larger collection of open problems in the general theory of Banach spaces, we suggest to look at [GuiMoZi16].

A survey of the present situation in the area of the weakly compactly generated spaces and their relatives can be found in [FGMZ04] and [FGHZ03].

The text is divided into three parts.

- (I) An introductory course in renorming that covers the basic necessary material in convexity and norms needed in this area, including the most basic renorming techniques. To make the text accessible to not-experienced readers, we explain here even basic concepts —like norming subspaces, Fenchel duality and convolution— and refer to basic results.

- (II) An intermediate course in renorming covering more advanced techniques, together with applications to differentiability of convex functions and extremal structure.
- (III) Advances and developments in renormings, covering contemporary research material concerning the connections to various structural properties of the spaces.

The text also shows a little bit on history and personal memories regarding when and where some of the results have been obtained, since we think that this, too, helps understanding the subject, and can provide for a moral support and encouragement: For example, the Czech and Spanish schools started from the beginning in the seventies of the last century, and now they are renowned centers of Banach spaces.

At the end of this text we provide for the reader's convenience a concise list giving the coordinates for examples and counterexamples discussed here. This information is doubled, in a telegraphic way, in several of the carefully prepared indices (General, Symbols, Authors, Renormings and Impossible Renormings), allowing for locating the information needed. We also provide a sizable list of references.

Summing up, this text is directed mainly to young research personnel in the area and can be considered as a supplement to [FHHMPZ01], [FHHMZ11], [HVMZ07], [Fab97], and [GuiMoZi16].

We thank our Institutions: The Mathematical Institute of the Czech Academy of Sciences and the Faculty of Mathematics and Physics at Charles University in Prague (Prague, Czech Republic), the University of Alberta (Edmonton, Alberta, Canada), the Real Academia de Ciencias Exactas, Físicas y Naturales of Spain (Madrid, Spain), the Universitat Politècnica de València and its Instituto de Matemática Pura y Aplicada (Valencia, Spain), for support and providing excellent working conditions. To Canadian, German, Czech and Spaniard grants, in particular Grants MTM2017-83262-C2-1-P (Spain), MTM2017-83262-C2-2-P (Spain) and PID2021-122126NB-C33 (Spain), for supporting this research over the years, and to our colleagues and friends for contributions and discussions on the subject of this work. We thank the staff of Springer Verlag, and especially Ms. Dorothy Mazlum (Birkhauser) and Mr. Marc Strauss (Springer), for their interest in this text.

The first named author wants to express his gratitude to the Murcia School of Functional Analysis (Spain), in particular to Professors Gabriel Vera, José Orihuela, Stanimir Troyanski, and the late Bernardo Cascales. His thanks also extend to Professors Petr Hájek from Prague (Czech Republic), and Robert Deville, from Bordeaux (France).

The second named author wants to acknowledge the debt with the late Professor Manuel Valdivia, founder of the Spanish School of Functional Analysis. His

overwhelming contribution to Mathematics, to his many students and to the intellectual mathematical life in Valencia (Spain) deserves a very special place in this list of acknowledgement. He also wants to thank Professors Václav Zizler (Prague, Czech Republic, and Edmonton, Canada), Marián Fabian (Prague, Czech Republic), Petr Hájek (Prague, Czech Republic), and Gilles Godefroy (Paris, France) for support, cooperation, and friendship.

The third named author uses this opportunity to thank all organizers of the annual International Czech Winter and Paseky Schools on Abstract Analysis (last year was its forty eighth meeting), where many important results from renormings were presented even for the first time—for example, the solution to the scalar-plus-compact famous open problem (Theorem 474 below)—. Special appreciation goes to the founder of these Schools, the late Professor Zdeněk Frolík, of the Czech Academy of Sciences, to whom this book is devoted. The appreciation goes for thousands of hours of selfless work of the whole team of mathematicians in Prague contributing to the success of these Schools (this was needed as they wanted to keep a warm and friendly working atmosphere in these workshops; Schools are placed in Czech mountain resorts to achieve a good condition for team work). The focus has been always directed to young participants. The Schools were open even for graduate students. Special appreciation goes to Professor Kamil John, from Prague Mathematical Institute of the Czech Academy of Sciences, as it was he who brought to the Prague group the project of systematically study the linear topological structure of nonseparable Banach spaces and started to have good results in this area. The work of this group in Prague would not have been possible without a strong moral support from the Professors Czesław Bessaga and Aleksander Pełczyński from the Warsaw School. He would also like to express his appreciation to the late Professor Jiří Jelínek, who introduced him to Functional Analysis, and to the late Professor Josef Kolomý of the Charles University in Prague, who put him and Marián Fabian on the track of Renorming Theory in the 60's of the last century. Special thanks go to the previously mentioned Professors Marián Fabian (Prague, Czech Republic), Gilles Godefroy (Paris, France), Petr Hájek (Prague, Czech Republic), and Kamil John (Prague, Czech Republic), for their long-life mathematical cooperation and friendship. He also thanks Professors Franz Hering (Dortmund, Germany), John Whitfield (Thunder Bay, Canada), and again Gilles Godefroy (Paris, France) for their crucial help with emigrating Zizler family to Canada in 1983. Special thanks of the third named author go to University of Alberta in Edmonton for keeping his MathSciNet account at present. We thank many our colleagues for their criticism on the text.

Above all, we thank our wives, Marta, Danuta, and Jarmila, respectively, for their moral support and encouragement. This book is also dedicated to them.

We would be happy if this text helped in motivating some young researchers to choose Renorming Theory as their main work field, and/or help some team of specialists prepare a new textbook on renormings. When preparing this text, we have had in mind also students that may like renormings and are staying

at universities that do not specialize in Banach spaces. For them we included a discussion of some folklore results needed in this area. At the end of the text we include many easy (and some not easy) problems for them to start with. Due to the ample bibliography included, Banach space specialists in general can use it as a reference list in this area. We hope that the books [DeGoZi93], [Fab97], [HMOV07], [MOTV09], [FHHMZ11], and the present one, could offer a reasonable overview of the present state of renormings and their applications. We wish readers a pleasant time using this book.

## Notation

We follow the standard notation as in [LinTz77], [DuSch], or [FHHMZ11], for example. In any case, symbols will be properly defined when they will appear, and a list of them is included as a special index. Here we shall only mention some adopted conventions that maybe are not of general use: For example, if  $f$  is a function from a set  $A$  into a set  $B$ , and if  $b \in B$ , we shall denote the set  $\{x \in A : f(x) = b\}$  simply by  $f(x) = b$ . This will be used, in particular, in many of our pictures. Thus,  $f(x) = 0$  will denote the kernel  $\{x \in A : f(x) = 0\}$  of a function  $f$  from  $A$  into a space. If nothing is said on the contrary, by the word “space” we shall have in mind an infinite-dimensional real normed space (usually denoted by  $X$ ). If  $X^*$  denotes its dual space, then the action of an element  $x^* \in X^*$  on an element  $x \in X$  will be indistinctly denoted by  $x^*(x)$  or  $\langle x, x^* \rangle$ . We find it convenient, when dealing with higher dual spaces, to consider in the previous evaluation notation that “even” duals are “on the left” and “odd” duals “on the right”. In this way,  $\langle x, x^* \rangle$  will not change order if  $x$  is considered as an element in  $X^{**}$  via the canonical linear isometric injection  $j$  from  $X$  into  $X^{**}$ ; accordingly, it will be written  $\langle j(x), x^* \rangle$  or, in a more simple way,  $\langle x, x^* \rangle$  if there is no risk of misunderstanding. By the way, and having in mind the existence of the mapping  $j$ ,  $X$  will always be considered as a linear subspace of  $X^{**}$ .

Often we say that a Banach space is, say, strictly convex if its norm is strictly convex, etc. Sometimes we say “separable Banach space” and mean “separable infinite-dimensional Banach space”. Often we say “a function”, meaning a “real valued function”. The word “operator” refers to a linear mapping between two spaces. When we say that a space “admits a norm” we mean an *equivalent* norm.

Vectors  $x$  in a (generalized) sequence space will be denoted by  $(x_\gamma)_{\gamma \in \Gamma}$  or  $(x(\gamma))_{\gamma \in \Gamma}$ , indistinctly (sometimes the notation will be shortened to  $(x_\gamma)$  if the index set is understood). The first or the second notation will be used according to notational convenience. Sequences (or generalized sequences) will be denoted by  $\{x_n\}_{n=1}^\infty$ , or by  $\{x_n\}_{n \in \mathbb{N}}$  (respectively,  $\{x_\gamma\}_{\gamma \in \Gamma}$  for general ordered sets  $\Gamma$  of indices).

## Typography

To be spotted easily, definitions are included in special boxes on a gray background. Since the subject of this text is renormings, results explicitly on this matter are also included in boxes again on gray background.

To help focusing on the essential, some explanations can be avoided at a first reading; for them, a particular environment that we call “annotation”, is used,

☞ as in this example. They are preceded by a special symbol for the reader’s convenience. This resource will be used for some proofs, complementary material, and detours. Skipping them should not damage much a first approach.

Historical notes will be included in a special environment headed by	H
Remarks end by the symbol	®
Proofs end by the symbol	□
Exercises end by the symbol	◇
Examples end by the symbol	✂

The list of open questions in Chapter 52 follows the same typographical convention than the results along the text: Those directly related to renorming are included in special boxes on a gray background. Not all the items in the list come from the text. For those that do we provide the right location.

Part I

An Introductory Course in  
Renorming



# Chapter 1

## Norms, normed spaces, Banach spaces

In this chapter we will set the basic definitions —and some results, with their references— that will be used along the text. We include them here in order to avoid repetitions later on. For further information, the reader may access any of the texts mentioned in the references.

### 1.1 Norms, normed and Banach spaces

**Definition 1.** A **norm** on a real vector space  $X$  is a function  $\|\cdot\|$  from  $X$  into  $\mathbb{R}$  that satisfies, for all  $x, y \in X$ , and  $\lambda \in \mathbb{R}$ ,

- (i)  $\|x\| \geq 0$ ,
- (ii)  $\|x\| = 0$  if, and only if,  $x = 0$ ,
- (iii)  $\|\lambda x\| = |\lambda| \cdot \|x\|$ , and
- (iv)  $\|x + y\| \leq \|x\| + \|y\|$  (the **triangle inequality**).

A real vector space  $X$  with a norm  $\|\cdot\|$  is said to be a **(real) normed space**, and we speak of the normed space  $(X, \|\cdot\|)$ . A sequence  $\{x_n\}_{n=1}^{\infty}$  in  $X$   $\|\cdot\|$ -converges to an element  $x \in X$  if  $\|x_n - x\| \rightarrow 0$ . A sequence  $\{x_n\}_{n=1}^{\infty}$  in  $X$  is  $\|\cdot\|$ -Cauchy if for every  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $\|x_n - x_m\| < \varepsilon$  for every  $n, m \geq n_0$ . A normed space  $(X, \|\cdot\|)$  is a **(real) Banach space** if every Cauchy sequence in  $X$  is  $\|\cdot\|$ -convergent to some point in  $X$  (i.e., if the space is **complete**).

In this text, we shall deal with *real* normed spaces. Thus, the term “normed space” (“Banach space”) will mean “real normed space” (respectively, “real Banach space”). Although many concepts defined below make sense in the context of normed spaces —without the hypothesis of completeness— and again several results do not depend on this requirement, Banach spaces will be the natural framework in which we shall work. Only in a few instances we shall define the concepts or formulate the results in the more general context of general (real) normed spaces (in a few instances we shall refer to the theory of locally convex spaces; a brief introduction will be presented in Section 2.1).

Thus,  $X$ , or more precisely,  $(X, \|\cdot\|)$ , will be a real Banach space if nothing is said to the contrary. The term “space” without other qualifications will refer to a real Banach space. If nothing is said on the contrary, the term “subspace” refers to a linear subspace.

A norm  $\|\cdot\|$  on  $X$  defines a **distance**  $d$ : Precisely,  $d(x_1, x_2) := \|x_1 - x_2\|$  for  $x_1, x_2 \in X$ . This function  $d$  satisfies all the requirements of an abstract distance: (i)  $d(x_1, x_2) \geq 0$ , (ii)  $d(x_1, x_2) = 0$  if, and only if,  $x_1 = x_2$ , (iii)  $d(x_1, x_2) = d(x_2, x_1)$ , and (iv)  $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$ , for all  $x_1, x_2, x_3 \in X$ . The distance defined by a norm has some extra features: For example,  $d(x_1, x_2) = d(x_1 + x_3, x_2 + x_3)$ , or  $d(\lambda x_1, \lambda x_2) = \lambda d(x_1, x_2)$ , for all  $x_1, x_2, x_3 \in X$  and  $\lambda \geq 0$ . The space  $(X, \|\cdot\|)$ , with the distance defined above, is a particular instance of a metric space, and we may speak of “proximity”: Two points  $x_1$  and  $x_2$  in  $X$  are “close” when  $d(x_1, x_2)$  (i.e.,  $\|x_1 - x_2\|$ ) is small. The set of points in  $X$  whose distance to a given point  $x_0 \in X$  is equal to some constant  $\varepsilon > 0$  is called the **sphere centered at  $x_0$  and having radius  $\varepsilon$** . The set of points in  $X$  whose distance to a given point  $x_0 \in X$  is less than or equal to some constant  $\varepsilon > 0$  is, by definition, the **closed ball  $B(x_0, \varepsilon) := \{x \in X : \|x - x_0\| \leq \varepsilon\}$  centered at  $x_0$  and having radius  $\varepsilon$** . The set  $\text{Int}B(x_0, \varepsilon) := \{x \in X : \|x - x_0\| < \varepsilon\}$  is called the **open ball centered at  $x_0$  and having radius  $\varepsilon$** . We usually denote by  $B_X$  (or by  $B_{(X, \|\cdot\|)}$  if we want to be more precise) the set  $B(0, 1)$  (i.e.,  $\{x \in X : \|x\| \leq 1\}$ , the closed unit ball), by  $\text{Int}B_X$  or  $\text{Int}B_{(X, \|\cdot\|)}$  the open unit ball, and by  $S_X := \{x \in X : \|x\| = 1\}$  the **unit sphere**. A set  $S \subset X$  is said to be **bounded** if there exists some  $n \in \mathbb{N}$  such that  $S \subset B(0, n)$ . The class  $\mathcal{O}$  of all open sets defined by the metric  $d$  is a **topology** on  $X$ : A set  $O \subset X$  is **open** whenever, for every  $x_0 \in O$ , there exists  $\varepsilon > 0$  such that  $B(x_0, \varepsilon)$  is contained in  $O$  (note that the empty set is considered also to be open). This topology, that comes from a norm  $\|\cdot\|$  in  $X$ , shall be denoted by  $\mathcal{T}_{\|\cdot\|}$ , and we shall speak about the **norm topology**, or the topology **defined by the norm**. The terminology is consistent: A closed ball is a closed set, an open ball is an open set; the **topological interior** (i.e., the biggest open set contained in a given set) of the closed ball  $B(x_0, \varepsilon)$  is the open ball  $\text{Int}B(x_0, \varepsilon)$ , and the **topological closure** (i.e., the smallest closed set containing a given set) of the open ball  $\text{Int}B(x_0, \varepsilon)$  is the closed ball  $B(x_0, \varepsilon)$ .

Clearly, a sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $X$  is  $\mathcal{T}_{\|\cdot\|}$ -convergent to  $x \in X$  if, and only if,  $\|x - x_n\| \rightarrow 0$ . The topological concepts in the topology  $\mathcal{T}_{\|\cdot\|}$  can be described by using sequences: For example, the complement of an open subset of  $X$  is said to be **closed**; it turns out that  $A$  is closed in  $(X, \mathcal{T}_{\|\cdot\|})$  if, and only if,  $x \in A$  whenever  $\{x_n\}_{n \in \mathbb{N}}$  is a sequence in  $A$  that  $\mathcal{T}_{\|\cdot\|}$ -converges to  $x$ .

Note that  $\|\cdot\|$ , as a function from  $X$  into  $\mathbb{R}$ , is  $\mathcal{T}_{\|\cdot\|}$ -continuous. Indeed, a consequence of the triangle inequality (iv) in Definition 1 is that

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|, \text{ for all } x, y \in X.$$

Therefore, if  $\{x_n\}_{n \in \mathbb{N}}$  is a sequence in  $X$  that  $\mathcal{T}_{\|\cdot\|}$ -converges to  $x \in X$ , then  $\left| \|x\| - \|x_n\| \right| \leq \|x - x_n\| \rightarrow 0$ .

To simplify some expressions, topological concepts in the  $\mathcal{T}_{\|\cdot\|}$ -topology will sometimes be written by using the prefix  $\|\cdot\|$ - instead of the more complicated  $\mathcal{T}_{\|\cdot\|}$ -, as referring to “ $\|\cdot\|$ -convergence” instead of “ $\mathcal{T}_{\|\cdot\|}$ -convergence”. Both ways will be used indistinctly.

## 1.2 Equivalent norms

### 1.2.1 Definition

The concept of equivalent norms has been already presented in the Introduction. Let us repeat it here since it is the central object of our study. A wider treatment of the basic concept of equivalence of norms will be done in Chapter 3. At this stage we need just its definition:

**Definition 2.** *If  $(X, \|\cdot\|)$  is a normed space, another norm  $\|\|\cdot\|\|$  on  $X$  is said to be **equivalent** (to  $\|\cdot\|$ ) whenever there exist two positive constants  $c \leq C$  such that*

$$c\|x\| \leq \|\|x\|\| \leq C\|x\|, \text{ for all } x \in X. \quad (1.1)$$

To be able to construct equivalent norms on a given Banach space that enjoy some desired properties —essentially rotundity and/or differentiability to some degrees, see Chapters 4 and 5 for definitions— is the main purpose of this text.

## 1.3 Linear operators and linear functionals. Duality, weak topologies

If  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are normed spaces, a linear mapping (also called an **operator**)  $T : X \rightarrow Y$  is continuous if, and only if, it is **bounded**, i.e., if  $TB_X$  is a bounded set. It is equivalent to say that there exists  $C \geq 0$  such that  $\|Tx\|_Y \leq C\|x\|_X$  for all  $x \in X$ . The minimum constant  $C$  with this property is said to be the **norm of  $T$** , denoted by  $\|T\|$ . In other words,  $\|T\| = \sup\{\|Tx\|_Y : x \in B_X\}$ . The set  $\mathcal{B}(X, Y)$  of all continuous linear operators from  $X$  into  $Y$ , with the algebraic operations sum and product by a scalar, is a vector space. When endowed with the norm  $\|\cdot\|$  defined above,  $\mathcal{B}(X, Y)$  becomes a normed space. If  $(Y, \|\cdot\|_Y)$  is complete, then it is easy to see that so it is  $(\mathcal{B}(X, Y), \|\cdot\|)$ :

☛ First, notice that a Cauchy sequence is always bounded. Second, if  $\{f_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(\mathcal{B}(X, Y), \|\cdot\|)$ , and  $x \in X$ , then  $\{f_n(x)\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $Y$ , hence convergent (say to  $f(x)$ ). The so-defined mapping  $f : X \rightarrow Y$  is clearly linear, and the boundedness of  $\{f_n\}_{n \in \mathbb{N}}$  readily gives that  $f$  is bounded (i.e.,  $f(B_X)$  is a bounded set in  $Y$ ), hence continuous.

The converse also holds: If  $(\mathcal{B}(X, Y), \|\cdot\|)$  is complete, and  $X \neq \{0\}$ , then  $(Y, \|\cdot\|_Y)$  is complete, too:

☛ Let  $x_0 \in S_X$  and  $f_0 \in S_{X^*}$  be such that  $f_0(x_0) = 1$  (the existence of  $f_0$  is guaranteed by the Hahn–Banach extension theorem, see below in this section). Let  $\{y_n\}_{n \in \mathbb{N}}$  be a Cauchy sequence in  $Y$ . For  $n \in \mathbb{N}$ , let us define

a mapping  $\phi_n \in \mathcal{B}(X, Y)$  as  $\phi_n(x) := f_0(x)y_n$ , for  $x \in X$ . Note that  $\phi_n(x_0) = y_n$  for all  $n$ . Since  $\|\phi_n - \phi_m\| \leq \|y_n - y_m\|$  for  $n, m \in \mathbb{N}$ , the sequence  $\{\phi_n\}_{n \in \mathbb{N}}$  is  $\|\cdot\|$ -Cauchy. Thus, it converges to some function  $\phi \in \mathcal{B}(X, Y)$ . In particular,  $y_n = \phi_n(x_0) \rightarrow \phi(x_0)$ .

A particular instance of the previous situation is when  $(Y, \|\cdot\|_Y)$  is just  $(\mathbb{R}, |\cdot|)$ . The space  $X^* := \mathcal{B}(X, \mathbb{R})$ , called the **dual space** of  $X$ . Elements in  $X^*$  are called **continuous linear forms**. We shall use the symbol  $\|\cdot\|^*$  for the defined norm on  $X^*$ , referred to as the **dual norm** (dual to  $\|\cdot\|$ ). If there is no risk of misunderstanding, sometimes we shall use  $\|\cdot\|$  instead. By the previous observation,  $(X^*, \|\cdot\|^*)$  is always a Banach space, even in the case that  $X$  is not complete. This shows, in particular, that every finite-dimensional normed space is complete.

☛ Indeed, notice first that every linear mapping between two finite-dimensional normed spaces is continuous —after all, its action can be described by matrix multiplication—. If  $(X, \|\cdot\|)$  is finite-dimensional, and  $\{e_i\}_{i=1}^n$  is an algebraic basis, the set  $\{e_i^*\}_{i=1}^n$ , where  $e_i^*(e_j) = \delta_{i,j}$ ,  $i, j = 1, 2, \dots, n$ , is an algebraic basis of  $X^*$ . The mapping  $T : (X, \|\cdot\|) \rightarrow (\mathcal{B}(X^*, \mathbb{R}), \|\cdot\|)$  given by  $Tx(f) = f(x)$  for all  $f \in X^*$  is clearly a linear isometry. It is onto: Given  $\phi \in \mathcal{B}(X^*, \mathbb{R})$ , put  $\phi(e_i^*) = a_i$  for  $i = 1, 2, \dots, n$ . Then clearly  $Tx = \phi$ , where  $x := \sum_{i=1}^n a_i e_i$ . The result follows from the completeness of  $(\mathcal{B}(X^*, \mathbb{R}), \|\cdot\|)$ .

Notice that  $\|x^*\|^* = \sup\{\langle x, x^* \rangle : x \in B_{(X, \|\cdot\|)}\}$ . We shall sometimes shorten this expression by writing, instead,  $\sup\langle B_{(X, \|\cdot\|)}, x^* \rangle$ . The same applies to similar formulas as long as it will not induce any misunderstanding.

If  $X$  and  $Y$  are normed spaces, and  $T : X \rightarrow Y$  is a linear operator, we may naturally define an operator  $T^* : Y^* \rightarrow X^*$  (called its **adjoint**) by the formula  $\langle x, T^*x^* \rangle = \langle Tx, x^* \rangle$  for  $x \in X$  and  $x^* \in X^*$ . It is easy to see that  $T^*$  is continuous in case that  $T$  is continuous, and that  $\|T^*\| = \|T\|$ .

We say that  $X$  and  $X^*$  form a **dual pair** (denoted by  $\langle X, X^* \rangle$ ), or that they are **in duality**. Technically speaking, this means that there exists a **bilinear** (i.e., linear in each variable) **form**  $b$  from  $X \times X^*$  into  $\mathbb{R}$  such that, if  $b(x, x^*) = 0$  for all  $x^* \in X^*$ , then  $x = 0$  and, conversely, if  $b(x, x^*) = 0$  for all  $x \in X$ , then  $x^* = 0$ . In our case, this bilinear form  $b$  is just  $b(x, x^*) := x^*(x)$ . Due to the intrinsic symmetry involved, we shall often use  $\langle x, x^* \rangle$  instead of  $x^*(x)$ . Note that the definition of the norm  $\|\cdot\|^*$  on  $X^*$  shows that

$$|\langle x, x^* \rangle| \leq \|x\| \|x^*\|^*, \quad \text{for all } x \in X \text{ and } x^* \in X^*. \quad (1.2)$$

The reason for the existence of nonzero continuous linear forms on any normed space  $X \neq \{0\}$  is the fundamental Hahn–Banach extension theorem (see, e.g., [Kothe69, §17.3] and [FHHMZ11, Chapter II, Theorem 2.1]), that says that *if  $X$  is a vector space endowed with a sublinear function  $p$*  (i.e., a mapping  $p : X \rightarrow \mathbb{R}$  such that  $p(x_1 + x_2) \leq p(x_1) + p(x_2)$  and  $p(\lambda x) = \lambda p(x)$  for  $x_1, x_2, x \in X$  and