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Renormings in Banach Spaces

A Toolbox

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Dedicated to the memory of Zdeněk Frolík
who managed to unite the classical Čech topological tradition
with the activities of the new Banach space workshop in Prague

A mathematician is a person who can find analogies between theorems. A better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.

Stefan Banach

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Preface

Introduction

Banach spaces provide a framework for modern mathematical analysis, as they blend classical analysis, geometry, topology and linearity. This makes Banach space theory a wonderful, elegant, and active research area in Mathematics. Many problems in modern linear and nonlinear analysis are of infinite-dimensional nature. Renormings represent a big and important part. They deal with structural properties, geometry, differential calculus etc. In this area, there are many important open problems.

In this text, "differentiability" and "smoothness" are synonymous terms, as they are "rotundness" and "strict convexity". These and related concepts will be properly defined in Chapters 4 and 5, respectively.

Questions concerning the supply of smooth functions on Banach spaces are of crucial importance in differential calculus. Smooth functions are usually obtained from equivalent smooth norms and smooth norms are in turn often constructed by using dual rotund norms. The existence of equivalent smooth or rotund norms or nontrivial smooth functions on a particular Banach space depends on its structure and has a profound impact on its geometry. For example, a separable Banach space that admits a Fréchet smooth function with bounded support must have a separable dual (Theorems 276, 283, and 284); if this function is, moreover, infinitely differentiable, then the space must contain a copy of c_0 or ℓ_p (Theorem 516). If the space does not contain c_0 and has a C^2-smooth function with bounded support, it is superreflexive (Theorem 453). If for a compact space K, its Cantor derivative $K^{(\omega_1)}$ is empty, then $C(K)$ admits a C^∞-smooth norm (Theorem 813). However, there is K such that $K^{(\omega_1)}$ is a singleton and $C(K)$ does not admit any Gâteaux differentiable norm [Hay99] (cf., e.g., [DeGoZi93, Chapter VII], [Ziz03, Theorem VII.6.4], and Theorem 529).

This text attempts to provide a modern and concise guide to some basic renorming results and techniques of Banach spaces and their applications, and discusses some streams of contemporary research in this area. We tried to keep a reader-friendly relaxed form to enjoy the text. For understanding the relations of concepts, we provide here as many counterexamples as possible. We made an effort to write the text in an informative manner, briefly pointing out main ideas and motivations for the results and the connections between them. The experienced reader may skip some detailed explanations in the introductory parts of the text. We include them to provide for some training in techniques used in this area, specially for young students. In many instances, specially in more advanced topics, we prefer to sketch the construction and motivate the result pointing out the main idea behind, better than to provide all details of a proof. In case that it is missing at all, we refer then either to the original paper or to books where the results appear in full.

This style enables us to discuss much wider spectrum of things and their relations than it could go to the regular book form. This allows us to include observations and comments that may help the reader, suggesting ways to complete the information provided here.

The text can be considered as a commented guide to optional graduate courses in Banach spaces. It starts with a review of basic tools and continues with several rather independent sections for the instructor's choice. The course would discuss many motivating open problems, and thus provides material for a research seminar as well.

The central topic of the text consists in renorming Banach spaces. This means to endow the given space with an equivalent norm that enjoys extra properties:

> If $\|\cdot\|$ is a norm on a Banach space X, another norm $|||\cdot|||$ is **equivalent** to $\|\cdot\|$ if two positive numbers α and β exist such that $\alpha\|x\| \leq |||x||| \leq \beta\|x\|$ for all $x \in X$.

Observe that the topology on X induced by the norm does not change when such new norm is adopted, although the metric properties may be really different. The renorming theory has thus many applications in the geometry of Banach spaces and in problems in analysis on them, and it is an active, large area of research in Banach spaces.

Note that in a given finite-dimensional space, all norms are equivalent (see Proposition 3 below). On the other hand, on any given infinite-dimensional Banach space $(X, \|\cdot\|)$ there is always a norm $|||\cdot|||$ that is not equivalent to $\|\cdot\|$ (see again Proposition 3).

Usually, any good result in renorming has many corollaries and it ensures many applications, since it is often stronger than the mere required particular property. Consider in this direction the Krein–Milman Theorem 7, differentiability of convex functions, and some other properties discussed below in the text.

In analysis in infinite dimensions, some problems require further techniques for their solution. For example, though every finite convex function on a finite-dimensional space is necessarily continuous (see Theorem 52), note that every infinite-dimensional Banach space admits a discontinuous linear functional (see the paragraph after the proof of Theorem 49). Bounded linear functionals do not necessarily attain their maxima on closed balls (as an easy example, observe that if B_{c_0} is the closed unit ball of the space c_0 and f is the continuous linear functional on c_0 defined by $f(x) = \sum \frac{1}{2^i} x_i$ for $x = (x_i)$, then the supremum of f over B_{c_0} is 1, but it is not difficult to show that for every $x \in B_{c_0}$, $f(x) < 1$. Therefore there is no closest point to the origin in this $f^{-1}(1)$. Also, there is no extreme point in B_{c_0}. Another example of a continuous linear functional that does not attain its supremum on the closed unit ball of a Banach space is given after Theorem 12).

An infinite-dimensional Rolle's theorem does not hold true (see, e.g., [FHHMZ11, Exercise 7.25]), and so new variational principles are needed. We will meet among others Ekeland's, compact, and smooth variational principles here (see Chapter 11).

Let us see an example of how renormings can help, typically, in some of these questions. Let $(X, \|\cdot\|_1)$ be a reflexive Banach space and let $\|\cdot\|_2$ be an equivalent strictly convex norm on X (its existence is guaranteed by a theorem of S.L. Troyanski (Theorem 218 below). By a result of K.S. Lau (Theorem 260 below), there exists an interior point c_1 of $B_{\|\cdot\|_1}$ from which a point $y_1 \in S_{\|\cdot\|_1}$ at farthest $\|\cdot\|_2$-distance exists. Then it is not difficult to see that y_1 is an extreme point of $B_{\|\cdot\|_1}$. We shall see in the text that, in a similar way, J. Lindenstrauss proved that all closed bounded convex sets in reflexive spaces do have an extreme point with extra properties (Theorem 259), by using M.I. Kadets and S.L. Troyanski renorming by locally uniformly convex norms (a property better than simple strict convexity). Similarly, if x_0 is a point of Fréchet smoothness of the second norm, then x_0 is also a Fréchet smooth point of the first norm. This, via a dualization of the Kadets–Troyanski result and the Lindenstrauss method, gives Fréchet smooth points on balls in reflexive spaces (see Item 5 in Subsection 5.3.2). These results of M.I. Kadets, Troyanski, and Lindenstrauss have been one of the most important results in this part of Banach space theory. Let us note that these are in a sense the best possible results in this direction; as we shall see later, measure theory can hardly be used here. It is the (miraculous) power of the concept of differentiability that allows for refraining from the requirement of reflexivity in the cases mentioned above. This is one of the goals of the smooth variational principle (Theorem 289) we will meet in this text.

Another thing we would like to mention at the beginning is that M.I. Kadets proved that if X is a Banach space with a Schauder basis $\{e_i, f_i\}$, then X can be renormed so that if $\|x_n\| \to \|x\|$ and $f_i(x_n) \to f(x)$ for each i, then $\|x_n - x\| \to 0$ (Corollary 177). So this norm shares this property with the Hilbertian norm. He used this approach for solving the long-standing problem of M. Fréchet and S. Banach whether that all infinite-dimensional separable Banach spaces are homeomorphic (Theorem 186). We shall discuss it in this text.

Let us briefly explicit here our point of view about the actual situation of renorming:

1. The present understanding of the basics is rather satisfactory in the separable case for local uniform rotundity, C^1-Fréchet and Gâteaux smoothness.
2. The understanding of basics in local uniform rotundity and C^1-Fréchet smoothness is relatively satisfactory for the class of weakly compactly generated spaces and their relatives.

3. The understanding of Gâteaux smoothness is not yet satisfactory in nonseparable spaces.
4. The understanding of higher order smoothness is not yet satisfactory in separable or nonseparable spaces.
5. There is a huge potential for more research in applications of renormings in linear and nonlinear analysis, in particular in Convexity Theory and Approximation.

There are many techniques for providing equivalent norms in specific Banach spaces or in wide classes of them. The state of affairs in Renorming Theory up to 1993 was presented in [DeGoZi93]. Later on, some remarkable contributions appeared, and some of them are presented in [MOTV09]. Now it is almost impossible to collect all sources and references in this area. Some are provided in the bibliography at the end of this manual. Among them we may add to the previously mentioned references, for example, [BeLi00], [BoVa09], [Di75], [Fab97], [FHHMPZ01], [FHHMZ11], [Go01b], [GoLaZi14], [HHZ96], [HMVZ07], [HáJo14], [HáMoZi12], [Meg98], [PelBe79], and [Ziz03]. Only a tiny part of the theory and techniques of renorming was discussed in [FHHMZ11] and [HMVZ07]. This is why we prepared this text, where we try to upgrade the information, including also some examples, counterexamples, and application of renorming. A big portion of the results are mentioned here for the first time in book form. Emphasis will be done on basic techniques, so that the reader may use them —or convenient variations—for constructing norms with various required properties.

When presenting and using the various relevant concepts we shall stress the relationship between them. Quite often results —even important ones— come without a detailed proof (only a sketch of the underlying ideas is provided), and many of them actually without proof at all (in those cases we shall always refer to sources). There are four main reasons for skipping some proofs: First, if incorporated, then the length of the text would have been unpractical. Second, that nowadays almost —if not all— sources are available at a click, and what is missing is, maybe, a map to search and locate what the interested reader is looking for. Third, that we prefer to focuss instead on the underlying ideas that may enlighten the panorama, especially stressing what is missing in some other texts like, e.g., [FHHMZ11], [DeGoZi93], [HMVZ07]. And finally, that we have in mind that the most efficient streamlined proof is, sometimes, not the best for a future use. In other words, that reading the original —sometimes distant— sources is, quite often, not only an exercise in mathematical history, but a must for true understanding (read the masters!). Let us quote here N.H. Abel: “It appears to me that if one wants to make progress in mathematics one should study the masters and not the pupils” (N.H. Abel (1802–1829), quoted from an unpublished source by O. Ore in [Ore74, p. 138]). The Abstract of H.M. Edwards in [Edw81], that we reproduce here, insists on this matter: “It is as good an idea to read the masters now as it was in Abel’s time. The best mathematicians know this and do it all the time. Unfortunately, students of mathematics normally spend their early years using textbooks (which may be, but usually aren't, written by masters) and taking lecture courses which are self-contained and make little or no reference to the primary literature of the subject. The students are left to discover on their own the wisdom of Abel's advice. In this they are being cheated".

This way the text may help the reader to get an overview of the whole body of the renorming theory better than to focus on detailed explanations— trying to summarize results on individual classes of spaces—. Thus, we can often afford to discuss more than one proof of the given result. For example, we discuss several proofs of the fact that non-trivial Fréchet differentiable functions cannot live in a space where there exists a convex continuous function that is nowhere Fréchet differentiable: Kurzweil's, Ekeland's, Whitfield's, Borwein–Preiss, and Deville's methods. We show several alternative proofs of the Bishop–Phelps Theorem 14. We discuss several methods in constructing projections in nonseparable spaces: Lindenstrauss', Plichko's, Valdivia's, Orihuela's, and Troyanski's methods, just to name a few authors involved. For relatives of weakly compactly generated spaces we consider both Troyanski et al. and Godefroy et al. transfer methods. Note they both involve a very sophisticated use of the interplay of convexity and compactness. We will see in the text that each one of them is used many times in this area. In the other direction, we use in several circumstances recent results to provide streamlined proofs of some older ones. This approach also enables us to effectively summarize results concerning a given individual property.

In any case, to make this text easily readable, we shall try to provide all the required definitions, make comments on them, and give hints to significant results —especially if the material is not easily accessible— trying to be to some extent self-contained. We hope that the text may be interesting to experts in Banach spaces in general, as a second reading on renormings, and that may help in supervising PhD theses on this subject. To assist the reader, we prepared organized lists of properties of particular spaces and in Chapter 51 a list of coordinates of various counterexamples. In Chapter 52 a list of easier problems to start independent work on this subject is included.

Of course, some basic knowledge of Banach space theory is required, as contained in basic undergraduate courses in functional analysis and in the introductory chapters of [FHHMZ11]. For a more extensive background, the reader may consult, for instance, the introductory article by Johnson and Lindenstrauss [JoLi03], [FHHMZ11], [LinTz77], [Di84], and [AlKa06]. In particular, it is an asset if the reader is more or less familiar with some standard known implications between the very basic notions of rotundity, smoothness and their duality, as contained, e. g., in [DeGoZi93], [FHHMZ11, Chapters 7, 8]. A certain knowledge of convex functions and, in particular, norms, is certainly needed. Practicing on this is a must. We may suggest the reader to look, for example, to [BoVa09], [FHHMZ11], [HMVZ07], [Phe93], [Gil82], and [MoZiZi15].

A basic knowledge of set theory and a certain background in topology are a must. We may suggest to consult [BessPe75], [Du66], [Eng85], [Kam50], and [Kel55].

There are many crucial open problems in the renorming area. Some of them are discussed, e. g., in [DeGoZi93], [FHHMZ11], [HMVZ07], [GuiMoZi16], and in [HáJo14]. A **bump function** (or just a **bump**) on a Banach space X is a function with nonempty bounded support. If ϕ is a C^k-smooth function on the real numbers such that $\phi(t) = 1$ if $t \in [-1, 1]$ and $\phi(t) = 0$ if $|t| \geq 2$, and $\|\cdot\|$ is a C^k-smooth norm —outside the origin— on X, then the composition $\phi(\|x\|)$ is a C^k-smooth bump. We will see that there are nonseparable Banach spaces that admit C^∞-smooth bumps but do not admit Gâteaux differentiable norms (Theorem 817). We will show that the sup norm of $C[0, 1]$ is nowhere Fréchet differentiable (Proposition 280) and that this space admits no Fréchet differentiable bump (Theorem 288). We will see that this is unlike finite-dimensional spaces, where Rademacher's Theorem 438 holds saying that any real-valued Lipschitz function on a finite-dimensional Banach space is Fréchet differentiable on a dense set of points. We will also meet the remarkable Preiss' Theorem 443 that on Asplund spaces —in particular, on Hilbert spaces— every real-valued Lipschitz function is Fréchet differentiable at points of a dense set.
Let us note in passing, for example, that the following problems are open.

1. Assume that a separable space X admits a C^2-smooth bump. Does it admit a C^2-smooth norm?
2. If X is an Asplund space, does X admit a C^1-smooth bump?
3. If X admits a C^1-smooth bump, does it allow for smooth approximations of continuous functions?
4. It is unknown if every norm on ℓ_2 can be approximated by real-analytic norms.
5. From the general theory, it should be noted that, amazingly, it is unknown whether every nonseparable Banach space has a infinite-dimensional separable quotient, see [GuiMoZi16, Problem 32].

This and many other open problems, famous and less famous, are discussed in this text. A list of them appears in Chapter 52. For a larger collection of open problems in the general theory of Banach spaces, we suggest to look at [GuiMoZi16].

A survey of the present situation in the area of the weakly compactly generated spaces and their relatives can be found in [FGMZ04] and [FGHZ03].

The text is divided into three parts.

(I) An introductory course in renorming that covers the basic necessary material in convexity and norms needed in this area, including the most basic renorming techniques. To make the text accessible to not-experienced readers, we explain here even basic concepts —like norming subspaces, Fenchel duality and convolution— and refer to basic results.

(II) An intermediate course in renorming covering more advanced techniques, together with applications to differentiability of convex functions and extremal structure.

(III) Advances and developments in renormings, covering contemporary research material concerning the connections to various structural properties of the spaces.

The text also shows a little bit on history and personal memories regarding when and where some of the results have been obtained, since we think that this, too, helps understanding the subject, and can provide for a moral support and encouragement: For example, the Czech and Spanish schools started from the beginning in the seventies of the last century, and now they are renowned centers of Banach spaces.

At the end of this text we provide for the reader's convenience a concise list giving the coordinates for examples and counterexamples discussed here. This information is doubled, in a telegraphic way, in several of the carefully prepared indices (General, Symbols, Authors, Renormings and Impossible Renormings), allowing for locating the information needed. We also provide a seizable list of references.

Summing up, this text is directed mainly to young research personnel in the area and can be considered as a supplement to [FHHMPZ01], [FHHMZ11], [HMVZ07], [Fab97], and [GuiMoZi16].

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The second named author wants to acknowledge the debt with the late Professor Manuel Valdivia, founder of the Spanish School of Functional Analysis. His

overwhelming contribution to Mathematics, to his many students and to the intellectual mathematical life in Valencia (Spain) deserves a very special place in this list of acknowledgement. He also wants to thank Professors Václav Zizler (Prague, Czech Republic, and Edmonton, Canada), Marián Fabian (Prague, Czech Republic), Petr Hájek (Prague, Czech Republic), and Gilles Godefroy (Paris, France) for support, cooperation, and friendship.
The third named author uses this opportunity to thank all organizers of the annual International Czech Winter and Paseky Schools on Abstract Analysis (last year was its forty eighth meeting), where many important results from renormings were presented even for the first time —for example, the solution to the scalar-plus-compact famous open problem (Theorem 474 below)—. Special appreciation goes to the founder of these Schools, the late Professor Zdeněk Frolík, of the Czech Academy of Sciences, to whom this book is devoted. The appreciation goes for thousands of hours of selfless work of the whole team of mathematicians in Prague contributing to the success of these Schools (this was needed as they wanted to keep a warm and friendly working atmosphere in these workshops; Schools are placed in Czech mountain resorts to achieve a good condition for team work). The focus has been always directed to young participants. The Schools were open even for graduate students. Special appreciation goes to Professor Kamil John, from Prague Mathematical Institute of the Czech Academy of Sciences, as it was he who brought to the Prague group the project of systematically study the linear topological structure of nonseparable Banach spaces and started to have good results in this area. The work of this group in Prague would not have been possible without a strong moral support from the Professors Czesław Bessaga and Aleksander Pełczyński from the Warsaw School. He would also like to express his appreciation to the late Professor Jiří Jelínek, who introduced him to Functional Analysis, and to the late Professor Josef Kolomý of the Charles University in Prague, who put him and Marián Fabian on the track of Renorming Theory in the 60's of the last century. Special thanks go to the previously mentioned Professors Marián Fabian (Prague, Czech Republic), Gilles Godefroy (Paris, France), Petr Hájek (Prague, Czech Republic), and Kamil John (Prague, Czech Republic), for their long-life mathematical cooperation and friendship. He also thanks Professors Franz Hering (Dortmund, Germany), John Whitfield (Thunder Bay, Canada), and again Gilles Godefroy (Paris, France) for their crucial help with emigrating Zizler family to Canada in 1983. Special thanks of the third named author go to University of Alberta in Edmonton for keeping his MathSciNet account at present. We thank many our colleagues for their criticism on the text.

Above all, we thank our wives, Marta, Danuta, and Jarmila, respectively, for their moral support and encouragement. This book is also dedicated to them.

We would be happy if this text helped in motivating some young researchers to choose Renorming Theory as their main work field, and/or help some team of specialists prepare a new textbook on renormings. When preparing this text, we have had in mind also students that may like renormings and are staying

at universities that do not specialize in Banach spaces. For them we included a discussion of some folklore results needed in this area. At the end of the text we include many easy (and some not easy) problems for them to start with. Due to the ample bibliography included, Banach space specialists in general can use it as a reference list in this area. We hope that the books [DeGoZi93], [Fab97], [HMVZ07], [MOTV09], [FHHMZ11], and the present one, could offer a reasonable overview of the present state of renormings and their applications.
We wish readers a pleasant time using this book.

Notation

We follow the standard notation as in [LinTz77], [DuSch], or [FHHMZ11], for example. In any case, symbols will be properly defined when they will appear, and a list of them is included as a special index. Here we shall only mention some adopted conventions that maybe are not of general use: For example, if f is a function from a set A into a set B, and if $b \in B$, we shall denote the set $\{x \in A : \ f(x) = b\}$ simply by $f(x) = b$. This will be used, in particular, in many of our pictures. Thus, $f(x) = 0$ will denote the kernel $\{x \in A : \ f(x) = 0\}$ of a function f from A into a space. If nothing is said on the contrary, by the word "space" we shall have in mind an infinite-dimensional real normed space (usually denoted by X). If X^* denotes its dual space, then the action of an element $x^* \in X^*$ on an element $x \in X$ will be indistinctly denoted by $x^*(x)$ or $\langle x, x^* \rangle$. We find it convenient, when dealing with higher dual spaces, to consider in the previous evaluation notation that "even" duals are "on the left" and "odd" duals "on the right". In this way, $\langle x, x^* \rangle$ will not change order if x is considered as an element in X^{**} via the canonical linear isometric injection j from X into X^{**}; accordingly, it will be written $\langle j(x), x^* \rangle$ or, in a more simple way, $\langle x, x^* \rangle$ if there is no risk of misunderstanding. By the way, and having in mind the existence of the mapping j, X will always be considered as a linear subspace of X^{**}.

Often we say that a Banach space is, say, strictly convex if its norm is strictly convex, etc. Sometimes we say "separable Banach space" and mean "separable infinite-dimensional Banach space". Often we say "a function", meaning a "real valued function". The word "operator" refers to a linear mapping between two spaces. When we say that a space "admits a norm" we mean an *equivalent* norm.

Vectors x in a (generalized) sequence space will be denoted by $(x_\gamma)_{\gamma\in\Gamma}$ or $(x(\gamma))_{\gamma\in\Gamma}$, indistinctly (sometimes the notation will be shortened to (x_γ) if the index set is understood). The first or the second notation will be used according to notational convenience. Sequences (or generalized sequences) will be denoted by $\{x_n\}_{n=1}^{\infty}$, or by $\{x_n\}_{n\in\mathbb{N}}$ (respectively, $\{x_\gamma\}_{\gamma\in\Gamma}$ for general ordered sets Γ of indices).

Typography

To be spotted easily, definitions are included in special boxes on a gray background. Since the subject of this text is renormings, results explicitly on this matter are also included in boxes again on gray background.
To help focusing on the essential, some explanations can be avoided at a first reading; for them, a particular environment that we call “annotation”, is used,

➡ as in this example. They are preceded by a special symbol for the reader's convenience. This resource will be used for some proofs, complementary material, and detours. Skipping them should not damage much a first approach.

Historical notes will be included in a special environment headed by H

Remarks end by the symbol ®

Proofs end by the symbol □

Exercises end by the symbol ♢

Examples end by the symbol ✠

The list of open questions in Chapter 52 follows the same typographical convention than the results along the text: Those directly related to renorming are included in special boxes on a gray background. Not all the items in the list come from the text. For those that do we provide the right location.

Part I

An Introductory Course in Renorming

Chapter 1

Norms, normed spaces, Banach spaces

In this chapter we will set the basic definitions —and some results, with their references— that will be used along the text. We include them here in order to avoid repetitions later on. For further information, the reader may access any of the texts mentioned in the references.

1.1 Norms, normed and Banach spaces

Definition 1. *A* **norm** *on a real vector space X is a function $\|\cdot\|$ from X into $\mathbb{R}$ that satisfies, for all $x, y \in X$, and $\lambda \in \mathbb{R}$,*
(i) $\|x\| \geq 0$,
(ii) $\|x\| = 0$ *if, and only if,* $x = 0$,
(iii) $\|\lambda x\| = |\lambda|.\|x\|$, *and*
(iv) $\|x + y\| \leq \|x\| + \|y\|$ *(the* **triangle inequality***).*
A real vector space X with a norm $\|\cdot\|$ is said to be a **(real) normed space***, and we speak of the normed space $(X, \|\cdot\|)$. A sequence $\{x_n\}_{n=1}^{\infty}$ in X* **$\|\cdot\|$-converges to an element** *$x \in X$ if $\|x_n - x\| \to 0$. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is* **$\|\cdot\|$-Cauchy** *if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for every $n, m \geq n_0$. A normed space $(X, \|\cdot\|)$ is a* **(real) Banach space** *if every Cauchy sequence in X is $\|\cdot\|$-convergent to some point in X (i.e., if the space is* **complete***).*

In this text, we shall deal with *real* normed spaces. Thus, the term "normed space" ("Banach space") will mean "real normed space" (respectively, "real Banach space"). Although many concepts defined below make sense in the context of normed spaces —without the hypothesis of completeness— and again several results do not depend on this requirement, Banach spaces will be the natural framework in which we shall work. Only in a few instances we shall define the concepts or formulate the results in the more general context of general (real) normed spaces (in a few instances we shall refer to the theory of locally convex spaces; a brief introduction will be presented in Section 2.1).
Thus, X, or more precisely, $(X, \|\cdot\|)$, will be a real Banach space if nothing is said to the contrary. The term "space" without other qualifications will refer to a real Banach space. If nothing is said on the contrary, the term "subspace" refers to a linear subspace.

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A norm $\|\cdot\|$ on X defines a **distance** d: Precisely, $d(x_1, x_2) := \|x_1 - x_2\|$ for $x_1, x_2 \in X$. This function d satisfies all the requirements of an abstract distance: (i) $d(x_1, x_2) \geq 0$, (ii) $d(x_1, x_2) = 0$ if, and only if, $x_1 = x_2$, (iii) $d(x_1, x_2) = d(x_2, x_1)$, and (iv) $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$, for all $x_1, x_2, x_3 \in X$. The distance defined by a norm has some extra features: For example, $d(x_1, x_2) = d(x_1 + x_3, x_2 + x_3)$, or $d(\lambda x_1, \lambda x_2) = \lambda d(x_1, x_2)$, for all $x_1, x_2, x_3 \in X$ and $\lambda \geq 0$. The space $(X, \|\cdot\|)$, with the distance defined above, is a particular instance of a metric space, and we may speak of "proximity": Two points x_1 and x_2 in X are "close" when $d(x_1, x_2)$ (i.e., $\|x_1 - x_2\|$) is small. The set of points in X whose distance to a given point $x_0 \in X$ is equal to some constant $\varepsilon > 0$ is called the **sphere centered at** x_0 **and having radius** ε. The set of points in X whose distance to a given point $x_0 \in X$ is less than or equal to some constant $\varepsilon > 0$ is, by definition, the **closed ball** $B(x_0, \varepsilon) := \{x \in X : \|x - x_0\| \leq \varepsilon\}$ **centered at** x_0 **and having radius** ε. The set $\mathrm{Int}B(x_0, \varepsilon) := \{x \in X : \|x - x_0\| < \varepsilon\}$ is called the **open ball centered at** x_0 **and having radius** ε. We usually denote by B_X (or by $B_{(X,\|\cdot\|)}$ if we want to be more precise) the set $B(0, 1)$ (i.e., $\{x \in X : \|x\| \leq 1\}$, the closed unit ball), by $\mathrm{Int}B_X$ or $\mathrm{Int}B_{(X,\|\cdot\|)}$ the open unit ball, and by $S_X := \{x \in X : \|x\| = 1\}$ the **unit sphere**. A set $S \subset X$ is said to be **bounded** if there exists some $n \in \mathbb{N}$ such that $S \subset B(0, n)$. The class $\mathcal{O}$ of all open sets defined by the metric d is a **topology** on X: A set $O \subset X$ is **open** whenever, for every $x_0 \in O$, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon)$ is contained in O (note that the empty set is considered also to be open). This topology, that comes from a norm $\|\cdot\|$ in X, shall be denoted by $\mathcal{T}_{\|\cdot\|}$, and we shall speak about the **norm topology**, or the topology **defined by the norm**. The terminology is consistent: A closed ball is a closed set, an open ball is an open set; the **topological interior** (i.e., the biggest open set contained in a given set) of the closed ball $B(x_0, \varepsilon)$ is the open ball $\mathrm{Int}B(x_0, \varepsilon)$, and the **topological closure** (i.e., the smallest closed set containing a given set) of the open ball $\mathrm{Int}B(x_0, \varepsilon)$ is the closed ball $B(x_0, \varepsilon)$.

Clearly, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is $\mathcal{T}_{\|\cdot\|}$-convergent to $x \in X$ if, and only if, $\|x - x_n\| \to 0$. The topological concepts in the topology $\mathcal{T}_{\|\cdot\|}$ can be described by using sequences: For example, the complement of an open subset of X is said to be **closed**; it turns out that A is closed in $(X, \mathcal{T}_{\|\cdot\|})$ if, and only if, $x \in A$ whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in A that $\mathcal{T}_{\|\cdot\|}$-converges to x.

Note that $\|\cdot\|$, as a function from X into $\mathbb{R}$, is $\mathcal{T}_{\|\cdot\|}$-continuous. Indeed, a consequence of the triangle inequality (iv) in Definition 1 is that

$$\big|\|x\| - \|y\|\big| \leq \|x - y\|, \text{ for all } x, y \in X.$$

Therefore, if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X that $\mathcal{T}_{\|\cdot\|}$-converges to $x \in X$, then $\big|\|x\| - \|x_n\|\big| \leq \|x - x_n\| \to 0$.

To simplify some expressions, topological concepts in the $\mathcal{T}_{\|\cdot\|}$-topology will sometimes be written by using the prefix $\|\cdot\|$- instead of the more complicated $\mathcal{T}_{\|\cdot\|}$-, as referring to "$\|\cdot\|$-convergence" instead of "$\mathcal{T}_{\|\cdot\|}$-convergence". Both ways will be used indistinctly.

1.2 Equivalent norms

1.2.1 Definition

The concept of equivalent norms has been already presented in the Introduction. Let us repeat it here since it is the central object of our study. A wider treatment of the basic concept of equivalence of norms will be done in Chapter 3. At this stage we need just its definition:

Definition 2. *If* $(X, \|\cdot\|)$ *is a normed space, another norm* $|||\cdot|||$ *on* X *is said to be* **equivalent** *(to* $\|\cdot\|$*) whenever there exist two positive constants* $c \leq C$ *such that*

$$c\|x\| \leq |||x||| \leq C\|x\|, \text{ for all } x \in X. \tag{1.1}$$

To be able to construct equivalent norms on a given Banach space that enjoy some desired properties —essentially rotundity and/or differentiability to some degrees, see Chapters 4 and 5 for definitions— is the main purpose of this text.

1.3 Linear operators and linear functionals. Duality, weak topologies

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces, a linear mapping (also called an **operator**) $T : X \to Y$ is continuous if, and only if, it is **bounded**, i.e., if TB_X is a bounded set. It is equivalent to say that there exists $C \geq 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$. The minimum constant C with this property is said to be the **norm of** T, denoted by $\|T\|$. In other words, $\|T\| = \sup\{\|Tx\|_Y : \ x \in B_X\}$. The set $\mathcal{B}(X, Y)$ of all continuous linear operators from X into Y, with the algebraic operations sum and product by a scalar, is a vector space. When endowed with the norm $\|\cdot\|$ defined above, $\mathcal{B}(X, Y)$ becomes a normed space. If $(Y, \|\cdot\|_Y)$ is complete, then it is easy to see that so it is $(\mathcal{B}(X, Y), \|\cdot\|)$:

➡ First, notice that a Cauchy sequence is always bounded. Second, if $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{B}(X, Y), \|\cdot\|)$, and $x \in X$, then $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, hence convergent (say to $f(x)$). The so-defined mapping $f : X \to Y$ is clearly linear, and the boundedness of $\{f_n\}_{n\in\mathbb{N}}$ readily gives that f is bounded (i.e., $f(B_X)$ is a bounded set in Y), hence continuous.

The converse also holds: If $(\mathcal{B}(X, Y), \|\cdot\|)$ is complete, and $X \neq \{0\}$, then $(Y, \|\cdot\|_Y)$ is complete, too:

➡ Let $x_0 \in S_X$ and $f_0 \in S_{X^*}$ be such that $f_0(x_0) = 1$ (the existence of f_0 is guaranteed by the Hahn–Banach extension theorem, see below in this section). Let $\{y_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. For $n \in \mathbb{N}$, let us define a mapping $\phi_n \in \mathcal{B}(X,Y)$ as $\phi_n(x) := f_0(x)y_n$, for $x \in X$. Note that $\phi_n(x_0) = y_n$ for all n. Since $\|\phi_n - \phi_m\| \le \|y_n - y_m\|$ for $n, m \in \mathbb{N}$, the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is $\|\cdot\|$-Cauchy. Thus, it converges to some function $\phi \in \mathcal{B}(X,Y)$. In particular, $y_n = \phi_n(x_0) \to \phi(x_0)$.

A particular instance of the previous situation is when $(Y, \|\cdot\|_Y)$ is just $(\mathbb{R}, |\cdot|)$. The space $X^* := \mathcal{B}(X,\mathbb{R})$, called the **dual space** of X. Elements in X^* are called **continuous linear forms**. We shall use the symbol $\|\cdot\|^*$ for the defined norm on X^*, referred to as the **dual norm** (dual to $\|\cdot\|$). If there is no risk of misunderstanding, sometimes we shall use $\|\cdot\|$ instead. By the previous observation, $(X^*, \|\cdot\|^*)$ is always a Banach space, even in the case that X is not complete. This shows, in particular, that every finite-dimensional normed space is complete.

➪ Indeed, notice first that every linear mapping between two finite-dimensional normed spaces is continuous —after all, its action can be described by matrix multiplication—. If $(X, \|\cdot\|)$ is finite-dimensional, and $\{e_i\}_{i=1}^n$ is an algebraic basis, the set $\{e_i^*\}_{i=1}^n$, where $e_i^*(e_j) = \delta_{i,j}$, $i, j = 1, 2, \ldots, n$, is an algebraic basis of X^*. The mapping $T : (X, \|\cdot\|) \to (\mathcal{B}(X^*,\mathbb{R}), \|\cdot\|)$ given by $Tx(f) = f(x)$ for all $f \in X^*$ is clearly a linear isometry. It is onto: Given $\phi \in \mathcal{B}(X^*,\mathbb{R})$, put $\phi(e_i^*) = a_i$ for $i = 1, 2, \ldots, n$. Then clearly $Tx = \phi$, where $x := \sum_{i=1}^n a_i e_i$. The result follows from the completeness of $(\mathcal{B}(X^*,\mathbb{R}), \|\cdot\|)$.

Notice that $\|x^*\|^* = \sup\{\langle x, x^*\rangle : \ x \in B_{(X,\|\cdot\|)}\}$. We shall sometimes shorten this expression by writing, instead, $\sup\langle B_{(X,\|\cdot\|)}, x^*\rangle$. The same applies to similar formulas as long as it will not induce any misunderstanding.

If X and Y are normed spaces, and $T : X \to Y$ is a linear operator, we may naturally define an operator $T^* : Y^* \to X^*$ (called its **adjoint**) by the formula $\langle x, T^*x^*\rangle = \langle Tx, x^*\rangle$ for $x \in X$ and $x^* \in X^*$. It is easy to see that T^* is continuous in case that T is continuous, and that $\|T^*\| = \|T\|$.

We say that X and X^* form a **dual pair** (denoted by $\langle X, X^*\rangle$), or that they are **in duality**. Technically speaking, this means that there exists a **bilinear** (i.e., linear in each variable) **form** b from $X \times X^*$ into $\mathbb{R}$ such that, if $b(x, x^*) = 0$ for all $x^* \in X^*$, then $x = 0$ and, conversely, if $b(x, x^*) = 0$ for all $x \in X$, then $x^* = 0$. In our case, this bilinear form b is just $b(x, x^*) := x^*(x)$. Due to the intrinsic symmetry involved, we shall often use $\langle x, x^*\rangle$ instead of $x^*(x)$. Note that the definition of the norm $\|\cdot\|^*$ on X^* shows that

$$|\langle x, x^*\rangle| \le \|x\|.\|x^*\|^*, \quad \text{for all } x \in X \text{ and } x^* \in X^*. \tag{1.2}$$

The reason for the existence of nonzero continuous linear forms on any normed space $X \neq \{0\}$ is the fundamental Hahn–Banach extension theorem (see, e.g., [Kothe69, §17.3] and [FHHMZ11, Chapter II, Theorem 2.1]), that says that *if X is a vector space endowed with a* **sublinear function** p (i.e., a mapping $p : X \to \mathbb{R}$ such that $p(x_1 + x_2) \le p(x_1) + p(x_2)$ and $p(\lambda x) = \lambda p(x)$ for $x_1, x_2, x \in X$ and