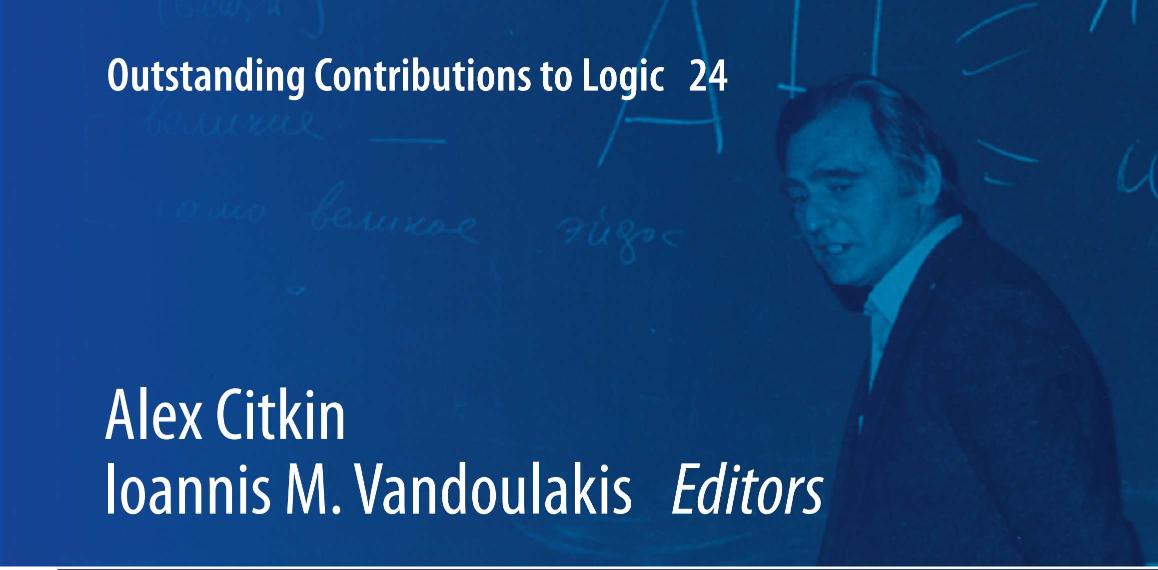

Outstanding Contributions to Logic 24

Alex Citkin
Ioannis M. Vandoulakis *Editors*

V. A. Yankov on Non-Classical Logics, History and Philosophy of Mathematics

Springer

Outstanding Contributions to Logic

Volume 24

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Preface

This volume is dedicated to Vadim Yankov (Jankov[1]), the Russian logician, historian and philosopher of mathematics and political activist who was prosecuted in the former USSR.

In 1964, he defended his dissertation *Finite implicative structures and realizability of formulas of propositional logic* under the supervision of A. A. Markov. In the 1960s, Yankov published nine papers dedicated to non-classical propositional logics, predominantly to intermediate logics. Even today, these publications—more than fifty years later—still hold their place among the most quotable papers in logic. The reason for this is very simple: not only Yankov obtained significant results in propositional logic, but he also developed a machinery that has been successfully used to obtain new results up until our days.

Yankov studied the class of all intermediate logics, as well as some particular intermediate logic. He proved that the class of all intermediate logic ExtInt is not denumerable, and that there are intermediate logics lacking the finite model property, and he had exhibited such a logic. In addition, he proved that ExtInt contains infinite strongly ascending, strongly descending and independent (relative to set inclusion) subclasses of logics, each of which is defined by a formula on just two variables. Thus, it became apparent that ExtInt as a lattice has a quite a complex structure.

In 1953, G. Rose gave a negative answer to a hypothesis that the logic of realizability, introduced by S. Kleene in an attempt to give precise intuitionistic semantics to Int, does not coincide with Int. In 1963, Yankov constructed the infinite series of realizable formulas not belonging to Int.

In his 1968 paper, Yankov studied the logic of the weak law of excluded middle, and the logic defined relative to Int by a single axiom $\neg p \vee \neg\neg p$. Nowadays, this logic is often referred to as a Yankov (or Jankov) logic. In particular, Yankov has discovered that this logic has a very special place in ExtInt: it is the largest logic,

[1] In Russian, the last name is Янков. In the translations of papers of the 1960s by the American Mathematical Society, the last name was transliterated as “Jankov,” while in the later translations, the last name is transliterated as “Yankov,” which perhaps is more correct. In this volume, the reader will see both spellings.

a positive fragment of which coincides with the positive fragment of Int, while all extensions of the Yankov logic have distinct positive fragments.

In his seminal 1969 paper, Yankov described in detail the machinery mentioned above. The reader can find more on Yankov's achievements in intermediate logics in the exposition included in this volume.

However, not only Yankov's results in studying intermediate logics are important. His papers instigate the transition from matrix to algebraic semantics. Already in his 1963 papers, he started to use what is now known as Heyting or pseudo-Boolean algebras. At the same time, H. Rasiowa and R. Sikorski's book, *The Mathematics of Methamatematics*, was published, in which the pseudo-Boolean algebras were studied. Yankov made the Russian translation of this book (published in 1972), and it greatly influenced the researchers in the former Soviet Union. Besides, Yankov was one of the pioneers who studied not only intermediate logics—extensions of Int, but also extensions of positive and minimal logics and their fragments. It would not be an overstatement to say that Yankov is one of the most influential logicians of his time.

At the end of the 1960s and in the 1970s, Yankov got more involved in the political activities. In 1968, he joined other prominent mathematicians and co-signed the famous letter of the 99 Soviet mathematicians addressed to the Ministry of Health and the General Procurator of Moscow asking for the release of imprisoned Esenin-Vol'pin. As a consequence, Yankov lost his job at the Moscow Institute of Physics and Technology (MIPT), and most of the mathematicians who signed this letter faced severe troubles.

Since 1972, he started to publish abroad, for instance, in the dissident journal *Kontinent*, founded in 1974 by writer Vladimir Maximov that was printed in Paris and focused on the politics of the Soviet Union. In issue 18, he published the article "On the possible meaning of the Russian democratic movement." In 1981–1982 he wrote a "Letter to Russian workers on the Polish events," on the history and goals of the "Solidarity" trade union. Following these events, he was arrested in August 1982, and on January 21, 1983, the Moscow City Court sentenced him to four years in prison and three years in exile for anti-Soviet agitation and propaganda. He served his term in the Gulag labor camp, called "Dubravny Camp" in Mordovia, near Moscow, and exile in Buryatia in south-central Siberia. He was released in January 1987 and rehabilitated in 1991.

Despite his hard life in the Camp and the exile, Yankov started to study philosophy and the classic Greek language. The second editor was impressed when he visited him at home in Dolgoprudnyj, near Moscow, in 1990, and Yankov started to analyze the syntax of a passage from Plato's *Parmenides* in classic Greek. When he asked him where he studied classical Greek so competently, he was stunned Yankov's unexpected answer: "In prison"!

Thus, Yankov's philosophical concerns were shaped while he was imprisoned. His first, possibly philosophical publication was printed abroad in issue 43 (1985) of the journal *Kontinent*, entitled "Ethical-philosophical treatise," where he outlines his philosophical conception of existential history. A publication on the same theme in Russia was made possible only ten years later, in the journal *Voprosy Filosofii* (1998, 6).

After Yankov's acquaintance with the second editor's Ph.D. Thesis, he agreed to become a member of the Committee of Reviewers and then started to examine the history of Greek mathematics systematically but from a specific logical point of view. He was primarily concerned about the ontological aspects of Greek mathematical theories and the relevant ontological theories in pre-Socratic philosophy. He stated a hypothesis on the rise of mathematical proof in ancient Greece, which integrated into the broader context of his inquiry of the pre-Socratic philosophy.

This volume is a minimal appreciation to a mathematician and scholar who deserves our respect and admiration.

New York, USA — Alex Citkin
Hagen, Germany — Ioannis M. Vandoulakis

Contents

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Chapter 1
Short Autobiography

Vadim A. Yankov

My full name is Vadim Anatol'evich Yankov. I was born on February 1st 1935, in Taganrog, Russia. During the Second World War, I was evacuated to Sverdlovsk. In 1952, I enrolled in the Department of Philosophy of Moscow State University. Faced with the "troubles" related to the ideologization in the humanities fields in the Soviet time, in 1953 I decided to transfer to the Department of Mechanics and Mathematics. In 1956, I was expelled from the University. The reasons given were my sharp criticism of the Komsomol, participation in a complaint against the conditions at the University students' cafeteria, and publication of an independent students' newspaper. Later, I was accepted into the University's distance remote program and obtained my diploma in 1959.

Since 1958, I have been employed in the Programming Department at the Steklov Institute of Mathematics (later, the Programming Department of the Institute of Mathematics of the Siberian Branch of the USSR Academy of Sciences). I worked in the research group developing one of the first programming languages, the ALPHA, an extension of ALGOL. After my graduation in 1959, I became a post-graduate student at the Department of Mathematics of the Moscow State University. Under the supervision of Andrey Markov, I prepared my thesis "Finite implicative structures

Editors' Note. In the book series "Outstanding Contributions to Logic" it is customary to include a scientific autobiography of the person the volume is dedicated to. Unfortunately, V. A. Yankov is not in a position to write his scientific autobiography, so we included a translation of his very brief and formal autobiography written a long time ago for the human resources department, and translated from Russian by Fiona Citkin. His scientific biography can be found in the overview papers by Citkin, Indrzejczak, Denisova, and Vandoulakis, which have been included in this volume.

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and realizability of formulas of propositional logic,"[1] for which I was awarded my PhD Degree in 1964. Since 1963, I have been an Assistant Lecturer at the Moscow Institute of Physics and Technology. In 1968, I was dismissed after co-signing the letter of my colleagues addressed to the Ministry of Health and the General Procurator of Moscow asking for the release of Aleksandr Esenin-Volpin.[2]

From 1968 to 1974, I worked as a Senior Lecturer at the Moscow Aviation Institute. Due to teaching overload during this period, my scientific achievement was substantially reduced.[3] Moreover, I was fired by the Institute's administration for my dissent views and discussions concerning the Soviet intervention in Czechoslovakia in 1968. During 1974–1982 I worked at the Enterprise Resource Planning Department of the Moscow Institute for Urban Economics.

From 1982 to 1987, I was imprisoned and exiled; the official reason for my arrest was anti-Soviet propaganda found in my publications on Soviet politics in foreign political journals.[4] During my confinement, I started studying Classic Greek by comparing Thucydides' works in the original and its Russian translation.

After my release in 1987, I worked in the Institute of Thermal Metallurgical Units and Technologies "STALPROEKT" until 1991.

Since 1991, I have been an Associate Professor at the Department of Mathematics, Logic and Intellectual Systems, Faculty of Theoretical and Applied and the Department of Logical and Mathematical Foundations of Humanitarian Knowledge, Institute of Linguistics of the Russian State University for the Humanities in Moscow. During this period, my research interests shifted to philosophy, history of philosophy and history of mathematics. I started lecturing regular courses on philosophy and the history of philosophy at the Russian State University of the Humanities, delivered a series of lectures in the Seminar of Philosophy of Mathematics of the Moscow

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[1] *Editors' Note*. In Plisko's paper in this volume, the reader can find more information on Yankov's results in realizability.

[2] *Editors' Note*. The letter was signed by 99 prominent mathematicians. As a consequence, many of them had been compelled to leave their positions in academia. For more details, see Fuchs D.B. "On Soviet Mathematics of the 1950th and 1960th" in Golden Years of Soviet Mathematics, American Mathematical Society, 2007, p. 221.

[3] *Editors' Note.* In the early 1960s, Yankov published a series of papers dedicated to propositional logic, especially, intermediate propositional logics. In these papers he announced the results which were further developed and published with proofs in 1968–1969 (cf. the complete list of papers at the end of this volume). At this time, Yankov mentioned (in a letter to A. Citkin) that "the focus of my research interests has been shifted." More about Yankov's contribution to the theory of intermediate logics can be found in Citkin's expository paper in this volume.

[4] *Editors' Note*. During this period, Yankov published abroad some papers in which he criticized the Soviet regime. In November 1981 – January 1982 he published a "Letter to Russian workers about the Polish events" in which he expressed his support to the Polish workers that struggled for freedom. On the 9th of August 1982, when Yankov left his apartment to go to the office, he was arrested. In January 1983, he was sentenced to four years in prison and three years of exile. During the *Perestroika*, in January 1987, he was released, and then rehabilitated on the 30th of October 1991 from all charges against him.

State University and published papers in the history of mathematics. This activity culminated in the publication of my book *Interpretation of Early Greek Philosophy* in 2011.[5]

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[5] *Editors' Note*. The papers by Denisova and Vandoulakis in this volume have been devoted to Yankov's contribution to philosophy and history of mathematics.

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Part I
Non-Classical Logics

Chapter 2
V. Yankov's Contributions to Propositional Logic

Alex Citkin

Abstract I give an exposition of the papers by Yankov published in the 1960s in which he studied positive and some intermediate propositional logics, and where he developed a technique that has successfully been used ever since.

Keywords Yankov's formula · Characteristic formula · Intermediate logic · Implicative lattice · Weak law of excluded middle · Yankov's logic · Positive logic · Logic of realizability · Heyting algebra

2020 Mathematics Subject Classification: Primary 03B55 · Secondary 06D20 · 06D75

2.1 Introduction

V. Yankov started his scientific career in early 1960s while writing his Ph.D. thesis under A. A. Markov's supervision. Yankov defended thesis "Finite implicative lattices and realizability of the formulas of propositional logic" in 1964. In 1963, he published three short papers Jankov (1963a, b, c) and later, in Jankov (1968a, b, c, d, 1969), he provided detailed proofs together with new results. All these papers are primarily concerned with studying *super-intuitionistic* (or super-constructive, as he called them) propositional logics, that is, logics extending the intuitionistic propositional logic Int. Throughout the present paper, the formulas are propositional formulas in the signature $\to, \wedge, \vee, \mathfrak{f}$, and as usual, $\neg p$ denotes $p \to \mathfrak{f}$ and $p \leftrightarrow q$ denotes $(p \to q) \wedge (q \to p)$; the logics are the sets of formulas closed under the rules Modus Ponens and substitution.

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To put Yankov's achievements in a historical context, we need to recall that Int was introduced by Heyting (cf. Heyting 1930[1]), who defined it by a calculus denoted by IPC as an attempt to construct a propositional logic addressing Brouwer's critique of the law of excluded middle and complying with intuitionistic requirements. Soon after, Gödel (cf. Gödel 1932) observed that Int cannot be defined by any finite set of finite logical matrices and that there is a strongly descending (relative to set-inclusion) set of super-intuitionistic logics (*si-logics* for short); thus, the set of si-logics is infinite. Gödel also noted that IPC possesses the following property: for any formulas A, B, if $\mathsf{IPC} \vdash (A \vee B)$, then $\mathsf{IPC} \vdash A$, or $\mathsf{IPC} \vdash B$—the *disjunction property*, which was later proved by Gentzen.

Even though Int cannot be defined by any finite set of finite matrices, it turned out that it can be defined by an infinite set of finite matrices (cf. Jaśkowski 1936), in other words, Int enjoys the finite model property (f.m.p. for short). This led to a conjecture that every si-logic enjoys the f.m.p., which entails that every si-calculus is decidable.

At the time when Yankov started his research, there were three objectives in the area of si-logics: (a) to find a logic that has semantics suitable from the intuitionistic point of view, (b) to study the class of si-logics in more details, and (c) to construct a convenient algebraic semantics.

By the early 1960s the original conjecture that Int is the only si-logic enjoying the disjunction property and that the realizability semantics introduced by Kleene is adequate for Int were refuted: in Kreisel and Putnam (1957), it was shown that the logic of IPC endowed with axiom $(\neg p \to (q \vee r)) \to ((\neg p \to q) \vee (\neg p \to r))$ is strictly larger than Int, and in Rose (1953), a formula that is realizable but not derivable in IPC was given. Using the technique developed by Yankov, Wroński proved that in fact, there are continuum many si-logics enjoying the disjunction property (cf. Wroński 1973).

In Heyting (1941), Heyting suggested an algebraic semantics, and in 1940s, McKinsey and Tarski introduced an algebraic semantics based on topology. In his Ph.D. (Rieger 1949), which is not widely known even nowadays, Rieger essentially introduced what is called a "Heyting algebra," and in Rieger (1957), he constructed an infinite set of formulas on one variable that are mutually non-equivalent in IPC. It turned out (cf. Nishimura 1960) that every formula on one variable is equivalent in IPC to one of Rieger's formulas. We need to keep in mind that the book (Rasiowa and Sikorski 1963) was published only in 1963. In 1972, this book had been translated into Russian by Yankov, and it greatly influenced the studies in the area of si-logics.

By the 1960s, it also became apparent that the structure of the lattice of the si-logics is more complex than expected: in Umezawa (1959) it has was observed that the class of si-logics contains subsets of the order type of ω^{ω}; in addition, it contains infinite subsets consisting of incomparable relative to set-inclusion logics.

Generally speaking, there are two ways of defining a logic: semantically by logical matrices or algebras, and syntactically, by calculus. In any case, it is natural to ask whether two given logical matrices, or two given calculi define the same logic. More

[1] The first part was translated in Heyting (1998).

precisely, is there an algorithm that, given two finite logical matrices decides whether their logics coincide, and is there an algorithm that given two formulas A and B decides whether calculi $\mathsf{IPC} + A$ and $\mathsf{IPC} + B$ define the same logic? The positive answer to the first problem was given in Łoś (1949). But in Kuznetsov (1963), it was established that in a general case (in the case when one of the logics can be not s.i.), the problem of equivalence of two calculi is unsolvable. Note that if every si-logic enjoys the f.m.p., then every si-calculus would be decidable and consequently, the problem of equivalence of two calculi would be decidable as well.

In Jankov (1963a), Yankov considers four calculi:

(a) $\mathsf{CPC} = \mathsf{IPC} + (\neg\neg p \to p)$—the classical propositional calculus;
(b) $\mathsf{KC} = \mathsf{IPC} + (\neg p \vee \neg\neg p)$—the calculus of the weak law of excluded middle (nowadays the logic of KC is referred to as Yankov's logic);
(c) $\mathsf{BD}_2 = \mathsf{IPC} + ((\neg\neg p \wedge (p \to q) \wedge ((q \to p) \to p)) \to q)$;
(d) $\mathsf{SmC} = \mathsf{IPC} + (\neg p \vee \neg\neg p) + ((\neg\neg p \wedge (p \to q) \wedge ((q \to p) \to p)) \to q)$—the logic of SmC is referred to as Smetanich's logic and it can be also defined by $\mathsf{IPC} + ((p \to q) \vee (q \to r) \vee (r \to s))$

and he gives a criterion for a given formula to define it relative to IPC (cf. Sect. 2.7). In Jankov (1968a), Yankov studied the logic of KC, and he proved that it is the largest si-logic having the same positive fragment as Int. Moreover, in Jankov (1968d), Yankov showed that the positive logic, which is closely related to the logic of KC, contains infinite sets of mutually non-equivalent, strongly descending, and strongly ascending chains of formulas (cf. Sect. 2.6).

Independently, a criterion that determines by a given formula A whether $\mathsf{Int} + A$ defines Cl was found in Troelstra (1965). In Jankov (1968c), Yankov gave a proof of this criterion as well as a proof of a similar criterion for Johansson's logic (cf. Sect. 2.5).

In Jankov (1963b), Yankov constructed infinite sets of realizable formulas that are not derivable in IPC and that are not derivable from each other. Moreover, he presented the seven-element Heyting algebra in which all realizable formulas are valid (cf. Sect. 2.8).

Jankov (1963c) is perhaps the best-known Yankov's paper, and it is one of the most quoted papers even today. In this paper, Yankov established a close relation between syntax and algebraic semantics: with every finite subdirectly irreducible Heyting algebra $\mathbf{A}$ he associates a formula $X_{\mathbf{A}}$—a characteristic formula of $\mathbf{A}$, such that for every formula B, the refutability of B in $\mathbf{A}$ (i.e. $\mathbf{A} \not\models B$) is equivalent to $\mathsf{IPC} + B \vdash X_{\mathbf{A}}$. Jankov (1963c) is a short paper and does not contain proofs. The proofs and further results in this direction are given in Jankov (1969), and we discuss them in Sect. 2.3. Let us point out that characteristic formulas in a slightly different form were independently discovered in de Jongh (1968).

Applying the developed machinery of characteristic formulas, Yankov proved (cf. Jankov 1968b) that there are continuum many distinct si-logics, and that among them there are logics lacking the f.m.p. Because the logic without the f.m.p. presented by Yankov was not finitely axiomatizable, it left a hope that perhaps all si-calculi enjoy the f.m.p. (this conjecture was refuted in Kuznetsov and Gerčiu 1970.)

Let us start with the basic definitions used in Yankov's papers.

2.2 Classes of Logics and Their Respective Algebraic Semantics

2.2.1 Calculi and Their Logics

Propositional formulas are formulas built in a regular way from a denumerable set of propositional variables Var and connectives.

Consider the following six propositional calculi with axioms from the following formulas:

$$p \to (q \to p); \quad (p \to (q \to r)) \to ((p \to q) \to (p \to r)); \tag{I}$$
$$(p \wedge q) \to p; \quad (p \wedge q) \to q; \quad p \to (q \to (p \wedge q)); \tag{C}$$
$$p \to (p \vee q); \quad q \to (p \vee q); \quad (p \to r) \to ((q \to r) \to ((p \vee q) \to r)); \tag{D}$$
$$\mathfrak{f} \to p. \tag{N}$$

they have inference rules Modus Ponens and substitution:

Calculus	Connectives	Axioms	Description	Logic
IPC	$\to, \wedge, \vee, \mathfrak{f}$	I,C,D,N	intuitionistic	Int
MPC	$\to, \wedge, \vee, \mathfrak{f}$	I,C,D	minimal or Johansson's	Min
PPC	$\to, \wedge, \vee$	I,C,D	positive	Pos
IPC$^-$	$\to, \wedge, \mathfrak{f}$	I,C,N	$\{\to, \wedge, \mathfrak{f}\}$ – fragment of IPC	Int$^-$
MPC$^-$	$\to, \wedge, \mathfrak{f}$	I,C	$\{\to, \wedge, \mathfrak{f}\}$ – fragment of MPC	Min$^-$
PPC$^-$	$\to, \wedge$	I,C	$\{\to, \wedge, \}$ – fragment of PPC	Pos$^-$

If $\Sigma \subseteq \{\to, \wedge, \vee, \mathfrak{f}\}$, by a Σ-formula we understand a formula containing connectives only from Σ and in virtue of the Separation Theorem (cf., e.g., Kleene 1952, Theorem 49): for every $\Sigma \in \{\{\to, \wedge, \vee\}, \{\to, \wedge, \mathfrak{f}\}, \{\to, \wedge\}\}$, if A is a C-formula $\{\to, \wedge\}$-formula, IPC $\vdash A$ if and only if PPC $\vdash A$ or IPC$^-$ $\vdash A$, or PPC$^-$ $\vdash A$.

By a C-calculus we understand one of the six calculi under consideration, and a C-logic is a logic of the C-calculus. Accordingly, C-formulas are formulas in the signature of the C-calculus. For C-formulas A and B, by $A \overset{C}{\vdash} B$ we denote that formula B is derivable in the respective C-calculus extended by axiom B; that is, $C + A \overset{C}{\vdash} B$.

The relation between PPC and MPC (or between PPC$^-$ and MPC$^-$) is a bit more complex: for any formula $\{\to, \wedge, \vee, \mathfrak{f}\}$-formula A (or any $\{\to, \wedge, \mathfrak{f}\}$-formula A), MPC $\vdash A$ (or MPC$^-$ $\vdash A$) if and only if PPC $\vdash A'$ (or PPC$^-$ $\vdash A'$), where A' is a formula obtained from A by replacing all occurrences of $\mathfrak{f}$ with a propositional variable not occurring in A (cf., e.g., Odintsov 2008, Chap. 2). In virtue of the Separation Theorem, in the previous statement, PPC or PPC$^-$ can be replaced with IPC or IPC$^-$, respectively.

Figure 2.1 shows the relations between the introduced logics: a double edge depicts an extension of the logic without any extension of the language (e.g., Min $\subset$ Int),

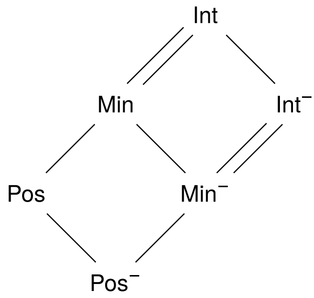

Fig. 2.1 Logics

while a single edge depicts an extension of the language but not of the class of theorems (e.g., if A is a $\{\to, \wedge, \neg\}$-formula, then $A \in \mathsf{Int}$ if and only if $A \in \mathsf{Int}^-$).

Let us observe that $((p \to \neg q) \to (q \to \neg p)) \in \mathsf{Min}^- \subseteq \mathsf{Min}$. Indeed, formula $(p \to (q \to r)) \to (q \to (p \to r))$ can be derived from the axioms (I). Hence, formula $(p \to (q \to \mathfrak{f})) \to (q \to (p \to \mathfrak{f}))$ is derivable too, that is, $(p \to \neg q) \to (q \to \neg p)$ is derivable in MPC^-.

We use $\mathbf{Ext}\mathsf{Int}$, $\mathbf{Ext}\mathsf{Min}$, $\mathbf{Ext}\mathsf{Pos}$, $\mathbf{Ext}\mathsf{Int}^-$, $\mathbf{Ext}\mathsf{Min}^-$, $\mathbf{Ext}\mathsf{Pos}^-$ to denote classes of logics extending, respectively, Int, Min, Pos, Int^-, Min^-, and Pos^-. Thus, $\mathbf{Ext}\mathsf{Int}$ is a class of all si-logics.

2.2.2 *Algebraic Semantics*

As pointed out in the Introduction, the first Yankov papers were written before the book by Rasiowa and Sikorski (1963) was published, and the terminology used by Yankov in his early papers was, as he himself admitted in Jankov (1968b), misleading. What he then called an "implicative lattice"[2] he later called a "Brouwerian algebra," and then he finally settled with the term "pseudo-Boolean algebra". We use a commonly accepted terminology, which we clarify below.

2.2.2.1 Correspondences Between Logics and Classes of Algebras.

In a meet-semilattice $\mathbf{A} = (\mathsf{A}; \wedge)$ an element c is a *complement of element* a *relative to element* b if c is the greatest element of A such that $\mathsf{a} \wedge \mathsf{c} \leq \mathsf{b}$ (e.g. Rasiowa 1974a). If a semilattice $\mathbf{A}$ for any elements a and b contains a complement of a relative to b, we say that $\mathbf{A}$ is a *semilattice with relative pseudocomplementation*, and we denote the relative pseudocomplementation by $\to$.

[2] In some translations of the Yankov paper, this term was translated as "implicative structure" (e.g. Jankov 1963a).

Proposition 2.1 *Suppose that* $\mathbf{A}$ *is a meet-semilattice and* $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathbf{A}$. *If* $\mathsf{a} \to \mathsf{b}$ *and* $\mathsf{a} \to \mathsf{c}$ *are defined in* $\mathbf{A}$, *then* $\mathsf{a} \to (\mathsf{b} \wedge \mathsf{c})$ *is defined as well and*

$$\mathsf{a} \to (\mathsf{b} \wedge \mathsf{c}) = (\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c}).$$

Proof Suppose that $\mathbf{A}$ is a meet-semilattice in which $\mathsf{a} \to \mathsf{b}$ and $\mathsf{a} \to \mathsf{c}$ are defined. We need to show that $(\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c})$ is the greatest element of $\mathbf{A}' := \{\mathsf{d} \in \mathbf{A} \mid \mathsf{a} \wedge \mathsf{d} \leq \mathsf{b} \wedge \mathsf{c}\}$.

First, we observe that $(\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c}) \in \mathbf{A}'$:

$$(\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c}) \wedge \mathsf{a} = (\mathsf{a} \wedge (\mathsf{a} \to \mathsf{b})) \wedge (\mathsf{a} \wedge (\mathsf{a} \to \mathsf{c})) \leq \mathsf{b} \wedge \mathsf{c},$$

because by the assumption, $\mathsf{a} \wedge (\mathsf{a} \to \mathsf{b}) \leq \mathsf{b}$ and $\mathsf{a} \wedge (\mathsf{a} \to \mathsf{c}) \leq \mathsf{c}$.

Next, we show that $(\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c})$ is the greatest element of $\mathbf{A}'$. Indeed, suppose that $\mathsf{d} \in \mathbf{A}'$. Then, $\mathsf{a} \wedge \mathsf{d} \leq \mathsf{b} \wedge \mathsf{c}$ and consequently,

$$\mathsf{a} \wedge \mathsf{d} \leq \mathsf{b} \text{ and } \mathsf{a} \wedge \mathsf{d} \leq \mathsf{c}.$$

Hence, by the definition of relative pseudocomplementation,

$$\mathsf{d} \leq \mathsf{a} \to \mathsf{b} \text{ and } \mathsf{d} \leq \mathsf{a} \to \mathsf{c},$$

which means that $\mathsf{d} \leq (\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c})$.

By an *implicative semilattice* we understand an algebra $(\mathsf{A}; \to, \wedge, \mathbf{1})$, where $(\mathsf{A}; \wedge)$ is a meet-semilattice with the greatest element $\mathbf{1}$ and $\to$ is a relative pseudocomplementation and accordingly, an algebra $(\mathsf{A}; \to, \wedge, \vee, \mathbf{1})$ is an *implicative lattice* if $(\mathsf{A}; \wedge\vee, \mathbf{1})$ is a lattice and $(\mathsf{A}; \to, \wedge, \mathbf{1})$ is an implicative semilattice (cf. Rasiowa 1974a). In implicative lattices, $\mathbf{0}$ denotes a constant (0-ary operation) that is the smallest element.

The logics described in the previous section have the following algebraic semantics:

Logic	Signature	Algebraic semantic	Denotation
Pos^-	$\{\to, \wedge, \mathbf{1}\}$	implicative semilattices	**BS**
Pos	$\{\to, \wedge, \vee, \mathbf{1}\}$	implicative lattices	**BA**
Min^-	$\{\to, \wedge, \mathfrak{f}, \mathbf{1}\}$	implicative semilattices with constant	**JS**
Min	$\{\to, \wedge, \vee, \mathfrak{f}, \mathbf{1}\}$	implicative semilattices with constant	**JA**
Int^-	$\{\to, \wedge, \mathbf{0}, \mathbf{1}\}$	bounded implicative semilattices	**HS**
Int	$\{\to, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$	bounded implicative lattices	**HA**

As usual, in **JS** and **JA**, we let $\neg\mathsf{a} = \mathsf{a} \to \mathfrak{f}$, while in **HS** and **HA**, $\neg\mathsf{a} = \mathsf{a} \to \mathbf{0}$. Also, we use the following denotations: $\mathcal{L} := \{\mathsf{Pos}^-, \mathsf{Pos}, \mathsf{Min}^-, \mathsf{Min}, \mathsf{Int}^-, \mathsf{Int}\}$ and $\mathcal{A} := \{\mathbf{BS}, \mathbf{BA}, \mathbf{JS}, \mathbf{JA}, \mathbf{HS}, \mathbf{HA}\}$. For each $\boldsymbol{L} \in \mathcal{L}$, $\mathrm{Mod}(\boldsymbol{L})$ denotes the respective class of algebras. By a C-algebra we shell understand an algebra in the signature

$\Sigma \cup \{\mathbf{1}\}$, and we assume that Σ is always a signature of one of the six classes of logics under consideration.

Every class from $\mathcal{A}$ forms a variety. Moreover, **HS** and **HA** are subvarieties of, respectively, **JS** and **JA** defined by the identity $\mathfrak{f} \to x = \mathbf{1}$.

Remark 2.1 Let us observe that **BS** is a variety of all Brouwerian semilattices, and it was studied in detail in (cf. Köhler 1981); **BA** is a variety of all Brouwerian algebras (cf. Galatos et al. 2007); **JA** is a variety of all Johansson's algebras (j-algebras; cf. Odintsov 2008); and **HA** is a variety of all Heyting or pseudo-Boolean algebras (cf. Rasiowa and Sikorski 1963).

Let us recall the following properties of C-algebras.

Proposition 2.2 *The following holds:*

(a) every Brouwerian algebra forms a distributive lattice;
(b) every finite distributive lattice forms a Brouwerian algebra, and because it always contains the least element, it forms a Heyting algebra as well;
(c) every finite **BS***-algebra forms a Brouwerian algebra.*

(a) and (b) were observed in Rasiowa and Sikorski (1963) and Birkhoff (1948). (c) follows from the observation that in any finite **BS**-algebra **A**, for any two elements $\mathsf{a}, \mathsf{b} \in \mathbf{A}$, $\mathsf{a} \vee \mathsf{b}$ can be defined as a meet of $\{\mathsf{c} \in \mathbf{A} \mid \mathsf{a} \leq \mathsf{c}, \mathsf{b} \leq \mathsf{c}\}$.

As usual, given a formula A and a C-algebra, a map $\nu : Var \longrightarrow \mathbf{A}$ is called a *valuation* in **A**, and ν allows us to calculate a value of A in **A** by treating the connectives as operations of **A**. If $\nu(A) = \mathbf{1}$ for all valuations, we say that A is *valid* in **A**, in symbols, $\mathbf{A} \models A$. If for some valuation ν, $\nu(A) \neq \mathbf{1}$, we say that A is *refuted* in **A**, in symbols, $\mathbf{A} \not\models A$, in which case ν is called a *refuting valuation*. For a class of algebras $\mathbb{K}$, $\mathbb{K} \models A$ means that A is valid in every member of $\mathbb{K}$. Given a class of C-algebras $\mathbb{K}$, $\mathbb{K}_{fin}$ is a subclass of all finite members of $\mathbb{K}$.

For every logic $\boldsymbol{L} \in \mathcal{L}$, a respective class from $\mathcal{A}$ is denoted by $\mathrm{Mod}(\boldsymbol{L})$. A class of models $\mathbb{M}$ of logic $\boldsymbol{L}$ forms an *adequate algebraic semantics* of $\boldsymbol{L}$ if for each formula A, $A \in \boldsymbol{L}$ if and only if A is valid in all algebras from $\mathbb{M}$.

Proposition 2.3 *For every* $\boldsymbol{L} \in \mathcal{L}$ *class* $\mathrm{Mod}(\boldsymbol{L})$ *forms an adequate algebraic semantics. Moreover, each logic* $\boldsymbol{L} \in \mathcal{L}$ *enjoys the f.m.p.; that is,* $A \in \boldsymbol{L}$ *if and only if* $\mathrm{Mod}(\boldsymbol{L})_{fin} \models A$.

Proof The proofs of adequacy can be found in Rasiowa (1974a). The f.m.p. for Int follows from Jaśkowski (1936). The f.m.p. for Int^-, Pos, Pos^- follows from the f.m.p. for Int and the Separation Theorem.

As we mentioned earlier, for any formula A, $A \in \mathsf{Min}$ (or $A \in \mathsf{Min}^-$) if and only if $A^f \in \mathsf{Int}$ (or $A \in Int^-$), where A^f is a formula obtained from A by replacing every occurrence of $\mathfrak{f}$ with a new variable p. Because Int (and Int^-) enjoys the f.m.p., if $A \notin \mathsf{Min}$ (or $A \notin \mathsf{Min}^-$), there is a finite Heyting algebra **A** refuting A^f (finite **HS**-algebra refuting A^f). If ν is a refuting valuation, we can convert **A** into a **JA**-algebra (or into a **JS**-algebra) by regarding A as a Brouwerian algebra (or a Brouwerian semilattice) with $\mathfrak{f}$ being $\nu(A)$. It is clear that A is refuted in such a **JA**-algebra (**JS**-algebra).

2.2.2.2 Meet-Irreducible Elements

Let $\mathbf{A} = (\mathsf{A}; \wedge)$ be a meet-semilattice and $\mathsf{a} \in \mathbf{A}$. Element a is called *meet-irreducible*, if for every pair of elements b, c, $\mathsf{a} = \mathsf{b} \wedge \mathsf{c}$ entails that $\mathsf{a} = \mathsf{b}$ or $\mathsf{a} = \mathsf{c}$. And a is called *meet-prime* if $\mathsf{a} \leq \mathsf{b} \wedge \mathsf{c}$ entails that $\mathsf{a} = \mathsf{b}$ or $\mathsf{a} = \mathsf{c}$. For formulas where $\wedge$ is a conjunction, instead of meet-irreducible or meet-prime we say *conjunctively-irreducible* or *conjunctively-prime*.

If $\mathbf{A}$ is a semilattice, then elements a, b of $\mathbf{A}$ are *comparable* if $\mathsf{a} \leq \mathsf{b}$ or $\mathsf{b} \leq \mathsf{a}$, otherwise these elements are *incomparable*. A set of mutually incomparable elements is called an *antichain*. It is not hard to see that a meet of any finite set of elements is equal to a meet of a finite subset of mutually incomparable elements.

It is clear that every meet-prime element is meet-irreducible. In the distributive lattices, the converse holds as well.

The meet-irreducible elements play a role similar to that of prime numbers: every positive natural number is a product of primes. As usual, if a is an element of a semilattice, the representation $\mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n$ of a as a meet of finitely many meet-prime elements a_i, $i \in [1, n]$ is called a *finite decomposition* of a. This finite decomposition is *irredundant* if no factor can be omitted.

It is not hard to see that because the factors in a finite decomposition are meet-irreducible, the decomposition is irredundant if and only if the elements of its factors are mutually incomparable.

Proposition 2.4 *In any semilattice, if element* a *has a finite decomposition,* a *has a unique (up to an order of factors) irredundant finite decomposition. Thus, in finite semilattices, every element has a unique irredundant finite decomposition.*

Proof Indeed, if element a has two finite irredundant decompositions $\mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n$ and $\mathsf{a} = \mathsf{a}'_1 \wedge \cdots \wedge \mathsf{a}'_m$, then $\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n = \mathsf{a}'_1 \wedge \cdots \wedge \mathsf{a}'_m$ and

$$(\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n) \to (\mathsf{a}'_1 \wedge \cdots \wedge \mathsf{a}'_m) = \mathbf{1}.$$

Hence, for each $j \in [1, m]$,

$$(\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n) \to \mathsf{a}'_j = \mathbf{1}; \text{ that is, } (\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n) \leq \mathsf{a}'_j.$$

Because a'_j is meet-prime, $\mathsf{a}'_\mathsf{j} \in \{\mathsf{a}_1, \ldots, \mathsf{a}_\mathsf{n}\}$ and thus, $\{\mathsf{a}'_1, \ldots, \mathsf{a}'_m\} \subseteq \{\mathsf{a}_1, \ldots, \mathsf{a}_n\}$. By the same reason, $\{\mathsf{a}_1, \ldots, \mathsf{a}_n\} \subseteq \{\mathsf{a}'_1, \ldots, \mathsf{a}'_m\}$ and therefore,
$\{\mathsf{a}_1, \ldots, \mathsf{a}_n\} = \{\mathsf{a}'_1, \ldots, \mathsf{a}'_m\}$.

Proposition 2.5 (Jankov 1969). *If a meet-semilattice* $\mathbf{A}$ *has a top element and all its elements have a finite irredundant decomposition, then* $\mathbf{A}$ *forms a Brouwerian semilattice.*

Proof We need to define on semilattice $\mathbf{A}$ a relative pseudocomplement $\to$. Because every element of $\mathbf{A}$ has a finite irredundant decomposition, for any two elements $\mathsf{a}, \mathsf{b} \in \mathbf{A}$ one can consider their finite irredundant decompositions $\mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n$

and $\mathsf{b} = \mathsf{b}_1 \wedge \cdots \wedge \mathsf{b}_m$. Now, we can define $\mathsf{a} \to \mathsf{c}$, where c is a meet-prime element, and then extend this definition by letting

$$\mathsf{a} \to (\mathsf{b}_1 \wedge \cdots \wedge \mathsf{b}_m) = (\mathsf{a} \to \mathsf{b}_1) \wedge \cdots \wedge (\mathsf{a} \to \mathsf{b}_m). \tag{2.1}$$

Proposition 2.1 ensures the correctness of such an extension.

Suppose $\mathsf{c} \in \mathbf{A}$ is meet-prime and $\mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n$ is a finite irredundant decomposition of a. Then we let

$$\mathsf{a} \to \mathsf{c} = \begin{cases} \mathbf{1}, & \text{if } \mathsf{a}_i \leq \mathsf{c} \text{ for some } i \in [1, n]; \\ \mathsf{c}, & \text{otherwise.} \end{cases}$$

Let us show that $\mathsf{a} \to \mathsf{c}$ is a pseudocomplement of a relative to c, that is, we need to show that $\mathsf{a} \to \mathsf{c}$ is the greatest element of $\mathbf{A}' := \{\mathsf{d} \in \mathbf{A} \mid \mathsf{a} \wedge \mathsf{d} \leq \mathsf{b}\}$.

Indeed, if $\mathsf{a}_i \leq \mathsf{c}$ for some $i \in [1, n]$, then

$$\mathbf{1} \wedge \mathsf{a} = \mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n \leq \mathsf{a}_i \leq \mathsf{c},$$

and obviously, $\mathbf{1}$ is the greatest of $\mathbf{A}'$.

Suppose now that $\mathsf{a}_i \not\leq \mathsf{c}$ for all $i \in [1, n]$. In this case, $\mathsf{a} \to \mathsf{c} = \mathsf{c}$, it is clear that $\mathsf{a} \wedge \mathsf{c} \leq \mathsf{c}$ (i.e., $\mathsf{a} \in \mathbf{A}'$), and we only need to verify that $\mathsf{d} \leq \mathsf{c}$ for every $\mathsf{d} \in \mathbf{A}'$.

Indeed, suppose that $\mathsf{a} \wedge \mathsf{d} \leq \mathsf{c}$; that is, $\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n \wedge \mathsf{d} \leq \mathsf{c}$. Then, $\mathsf{d} \leq \mathsf{c}$ because c is meet prime and $\mathsf{a}_i \not\leq \mathsf{c}$ for all $i \in [1, n]$.

Immediately from Propositions 2.5 and 2.2(c), we obtain the following statement.

Corollary 2.1 *Every finite meet-semilattice* $\mathbf{A}$ *with a top element in which every element has an irredundant finite decomposition forms a Brouwerian algebra. And because* $\mathbf{A}$ *is finite and has a bottom element,* $\mathbf{A}$ *is a Heyting algebra.*

2.2.3 Lattices Ded_C *and* $\mathsf{Lind}_{(C,k)}$

On the set of all C-formulas, relation $\overset{C}{\vdash}$ is a quasiorder and hence, the relation

$$A \overset{C}{\approx} B \quad \overset{\text{def}}{\iff} \quad A \overset{C}{\vdash} B \text{ and } B \overset{C}{\vdash} A$$

is an equivalence relation. Moreover, the set of all C-formulas forms a semilattice relative to connecting formulas with $\wedge$. It is not hard to see that equivalence $\overset{C}{\approx}$ is a congruence and therefore, we can consider a quotient semilattice which is denoted by Ded_C.

For each $k > 0$, we consider the set of all formulas on variables $p_1, \ldots, p_k$. This set formulas a semilattice relative to connecting two given formulas with $\wedge$. It is not hard to see that relation

$$A \overset{C}{\sim} B \quad \overset{\text{def}}{\Longleftrightarrow} \quad \overset{C}{\vdash} A \leftrightarrow B$$

is a congruence, and by $\mathsf{Lind}_{(C,k)}$ we denote a quotient semilattice.

Theorem 2.1 (Jankov 1969) *For any C and $k > 0$, semilattices $\mathsf{Lind}_{(C,k)}$ and Ded_C are distributive lattices.*

Proof For $C \in \{\mathsf{PPC}, \mathsf{MPC}, \mathsf{IPC}\}$, it was observed in Rasiowa and Sikorski (1963). If $C \in \{\mathsf{PPC}^-, \mathsf{MPC}^-, \mathsf{IPC}^-\}$, by the Diego theorem (cf., e.g., Köhler 1981), lattice $\mathsf{Lind}_{(C,k)}$ is a finite implicative semilattice and, hence, a distributive lattice.

To convert Ded_C into a lattice we need to define a meet. Given two formulas A and B, we let

$$A \vee' B \quad = \quad (A \to p) \wedge ((B' \to p) \to p),$$

where formula B' is obtained from B by replacing the variables in such a way that formulas A and B have no variables in common, and p is a variable not occurring in formulas A and B'. If $C \in \{\mathsf{PPC}, \mathsf{MPC}, \mathsf{IPC}\}$, one can take

$$A \vee' B \quad = \quad A \vee B'.$$

A proof that Ded_C is indeed a distributive lattice can be found in Jankov (1969).

Meet-prime and meet-irreducible elements in $\mathsf{Lind}_{(C,k)}$ and Ded_C are called *conjunctively prime* and *conjunctively irreducible*, and because these lattices are distributive, every conjunctively irreducible formula is conjunctively prime and vice versa.

2.2.3.1 Congruences, Filters, Homomorphisms

Let us observe that every C-algebra $\mathbf{A}$ has a $\{\to, \wedge, \mathbf{1}\}$-reduct that is a Brouwerian semilattice, and therefore, any congruence on $\mathbf{A}$ is at the same time a congruence on its $\{\to, \wedge, \mathbf{1}\}$-reduct. It is remarkable that the converse is true too: every congruence on a $\{\to, \wedge, \mathbf{1}\}$-reduct can be lifted to the algebra.

Any congruence on a C-algebra $\mathbf{A}$ is uniquely defined by the set $\mathbf{1}/\theta := \{\mathsf{a} \in \mathbf{A} \mid (\mathsf{a}, \mathbf{1}) \in \theta\}$: indeed, it is not hard to see that $(\mathsf{b}, \mathsf{c}) \in \theta$ if and only if $(\mathsf{b} \leftrightarrow \mathsf{c}, \mathbf{1}) \in \theta$ (cf. Rasiowa 1974a). A set $\mathbf{1}/\theta$ forms a filter of $\mathbf{A}$: a subset $\mathsf{F} \subseteq \mathbf{A}$ is a *filter* if $\mathbf{1} \in \mathsf{F}$ and $\mathsf{a}, \mathsf{a} \to \mathsf{b} \in \mathsf{F}$ yields $\mathsf{b} \in \mathsf{F}$. The set of all filters of C-algebra $\mathbf{A}$ is denoted by $\mathsf{Flt}(\mathbf{A})$. It is not hard to see that a meet of an arbitrary system of filters is a filter and hence, $\mathsf{Flt}(\mathbf{A})$ forms a complete lattice. A set-join of two filters does not need to be a filter, but a join of any ascending chain of filters is a filter.

As we saw, every congruence is defined by a filter. The converse is true too: any filter F of a C-algebra $\mathbf{A}$ defines a congruence

$$(\mathsf{a}, \mathsf{b}) \in \theta_{\mathsf{F}} \iff (\mathsf{a} \leftrightarrow \mathsf{b}) \in \mathsf{F}.$$

Moreover, the map $\mathsf{F} \longrightarrow \theta_{\mathsf{F}}$ is an isomorphism between complete lattices of filters and complete lattice of congruences (cf. Rasiowa 1974a). It is clear that any nontrivial C-algebra has at least two filters: $\{\mathbf{1}\}$ and the set of all elements of the algebra. The filter $\{\mathbf{1}\}$ is called *trivial*, and the filters that do not contain all the elements of the algebra are called *proper*. In what follows, by $\mathbf{A}/\mathsf{F}$ and a/F we understand $\mathbf{A}/\theta_{\mathsf{F}}$ and $\mathsf{c}/\theta_{\mathsf{F}}$.

If $\mathbf{A}$ is a C-algebra and $\mathsf{B} \subseteq \mathbf{A}$ is a subset of elements, there is the least filter $[\mathsf{B})$ of $\mathbf{A}$ containing B: $[\mathsf{B}) = \bigcap\{\mathsf{F} \in \mathsf{Flt}(A) \mid \mathsf{B} \subseteq \mathsf{F}\}$, and we write $[\mathsf{a})$ instead of $[\{\mathsf{a}\})$. The reader can easily verify that for any element a of a C-algebra $\mathbf{A}$, $[\mathsf{a}) = \{\mathsf{b} \in \mathbf{A} \mid \mathsf{a} \leq \mathsf{b}\}$.

Immediately from the definitions of a filter and a homomorphism, the following holds.

Proposition 2.6 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are* C*-algebras and* $\varphi : \mathbf{A} \longrightarrow \mathbf{B}$ *is a homomorphism of* $\mathbf{A}$ *onto* $\mathbf{B}$*. Then*

(a) If F *is a filter of* $\mathbf{A}$*, then* $\varphi(\mathsf{F})$ *is a filter of* $\mathbf{B}$*;*
(b) If F *is a filter of* $\mathbf{B}$*, then* $\varphi^{-1}(\mathsf{F})$ *is a filter of* $\mathbf{A}$*.*

A nontrivial algebra $\mathbf{A}$ is called *subdirectly irreducible* (s.i. for short) if the meet of all nontrivial filters is a nontrivial filter; or, in terms of congruences, the meet of all congruences that are distinct from the identity is distinct from the identity congruence.

Because every element a of a C-algebra $\mathbf{A}$ defines a filter $[\mathsf{a})$, the meet of all nontrivial filters of $\mathbf{A}$ coincides with $\bigcap\{[\mathsf{a}), \mathsf{a} \in \mathbf{A} \mid \mathsf{a} \neq \mathbf{1}\}$ and consequently, $\mathbf{A}$ is s.i. if and only if the set $\{\mathsf{a} \in \mathbf{A} \mid \mathsf{a} \neq \mathbf{1}\}$ contains the greatest element which is referred to as a *pretop* element or an *opremum* and is denoted by $\mathsf{m}_{\mathbf{A}}$.

Let us observe that immediately from the definition of a pretop element, if $\mathsf{m}_{\mathbf{A}}$ is a pretop element of a C-algebra $\mathbf{A}$ and F is a filter of $\mathbf{A}$, then, $\mathsf{m}_{\mathbf{A}} \in \mathsf{F}$ if and only if F is nontrivial. In terms of homomorphism, this can be stated in the following way.

Proposition 2.7 *Suppose that* $\mathbf{A}$ *is an s.i.* C*-algebra and* $\varphi : \mathbf{A} \longrightarrow \mathbf{B}$ *is a homomorphism of* $\mathbf{A}$ *into* C*-algebra* $\mathbf{B}$*. Then* φ *is an isomorphism if and only if* $\varphi(\mathsf{m}_{\mathbf{A}}) \neq \mathbf{1}_{\mathbf{B}}$*.*

The following simple proposition was observed in Jankov (1969) and it is very important in what follows.

Proposition 2.8 *Let* $\mathbf{A}$ *be a nontrivial* C*-algebra,* $\mathsf{a}, \mathsf{b} \in \mathbf{A}$ *and* $\mathsf{a} \not\leq \mathsf{b}$*. Then, there is a maximal (relative to* $\subseteq$*) filter* F *of* $\mathbf{A}$ *such that* $\mathsf{a} \in \mathsf{F}$ *and* $\mathsf{b} \notin \mathsf{F}$*. Furthermore,* $\mathbf{A}/\mathsf{F}$ *is an s.i.* C*-algebra with* b/F *being the pretop element.*

Proof First, let us observe that the condition $\mathsf{a} \not\leq \mathsf{b}$ is equivalent to $\mathsf{b} \notin [\mathsf{a})$. Thus, $\mathcal{F} := \{\mathsf{F} \in \mathsf{Flt}(\mathbf{A}) \mid \mathsf{a} \in \mathsf{F}, \mathsf{b} \notin \mathsf{F}\} \neq \varnothing$.

Next, we recall that the joins of ascending chains of filters are filters and therefore, $\mathcal{F}$ enjoys the ascending chain condition. Thus, by the Zorn Lemma, $\mathcal{F}$ contains a maximal element.

Let F be a maximal element of $\mathcal{F}$. We need to show that b/F is a pretop element of $\mathbf{A}/\mathsf{F}$.

Because $\mathsf{b} \notin \mathsf{F}$ (cf. the definition of $\mathcal{F}$), we know that $\mathsf{b}/\mathsf{F} \neq \mathbf{1}_{\mathbf{A}/\mathsf{F}}$.

Let $\varphi : \mathbf{A} \longrightarrow \mathbf{A}/\mathsf{F}$ be a natural homomorphism. By Proposition 2.6, for every filter F' of $\mathbf{A}/\mathsf{F}$, the preimage $\varphi^{-1}(\mathsf{F}')$ is a filter of $\mathbf{A}$. Because $\mathbf{1}_{\mathbf{A}/\mathsf{F}} \in \mathsf{F}'$,

$$\mathsf{F} = \varphi^{-1}(\mathbf{1}_{\mathbf{A}/\mathsf{F}}) \subseteq \varphi^{-1}(\mathsf{F}').$$

Hence, if $\mathsf{F}' \supsetneq \mathbf{1}_{\mathbf{A}}/\mathsf{F}$, then $\mathsf{b} \in \varphi^{-1}(\mathsf{F}')$ (because F is a maximal filter not containing b), and consequently, $\mathsf{b}/\mathsf{F} \in \mathsf{F}'$. Thus, b/F is in every nontrivial filter of $\mathbf{A}/\mathsf{F}$, which means that $\mathbf{A}/\mathsf{F}$ is s.i. and that b/F is a pretop element of $\mathbf{A}/\mathsf{F}$.

Corollary 2.2 *Suppose that $A \to B$ is a C-formula refuted in a C-algebra $\mathbf{A}$. Then there is an s.i. homomorphic image $\mathbf{B}$ of algebra $\mathbf{A}$ and a valuation ν in $\mathbf{B}$ such that*

$$\nu(A) = \mathbf{1}_{\mathbf{B}} \quad \textit{and} \quad \nu(B) = \mathsf{m}_{\mathbf{B}}.$$

Proof Suppose that ξ is a refuting valuation in $\mathbf{A}$; that is, $\xi(A \to B) \neq \mathbf{1}_{\mathbf{A}}$. Let $\xi(A) = \mathsf{a}$ and $\xi(B) = \mathsf{b}$. Then, $\mathsf{a} \not\leq \mathsf{b}$ and by Proposition 2.8, there is a filter F of $\mathbf{A}$ such that $\mathsf{a} \in \mathsf{F}$, $\mathsf{b} \notin \mathsf{F}$ and $\mathbf{A}/\mathsf{F}$ is subdirectly irreducible with b/F being a pretop element of $\mathbf{A}/\mathsf{F}$. Thus, one can take a natural homomorphism $\eta : \mathbf{A} \longrightarrow \mathbf{A}/\mathsf{F}$ and let $\nu = \eta \circ \xi$.

$$\begin{tikzcd} p_i \arrow[d, "\xi"'] \arrow[dr, "\nu"] & \\ \mathsf{a}_i \arrow[r, "\eta"'] & \mathsf{a}_i/\mathsf{F} \end{tikzcd}$$

It is not hard to see that ν is a desired refuting valuation.

Suppose that $\boldsymbol{L}$ is an extension of one of the logics from $\mathcal{L}$ and A is a formula in the signature of $\boldsymbol{L}$. We say that a C-algebra $\mathbf{A}$ in the signature of $\boldsymbol{L}$ *separates A from $\boldsymbol{L}$* if all formulas from $\boldsymbol{L}$ are valid in $\mathbf{A}$ (i.e., $\mathbf{A} \in \mathrm{Mod}(\boldsymbol{L})$), while formula A is not valid in $\mathbf{A}$, that is, if $\mathbf{A} \models \boldsymbol{L}$ and $\mathbf{A} \not\models A$.

Corollary 2.3 *Suppose that $\boldsymbol{L}$ is a C-logic and A is a C-formula. If a C-algebra $\mathbf{A}$ separates formula A from $\boldsymbol{L}$, then there is an s.i. homomorphic image $\mathbf{B}$ of $\mathbf{A}$ and a valuation ν in $\mathbf{B}$ such that $\nu(A) = \mathsf{m}_{\mathbf{B}}$.*

Proof If formula A is invalid in $\mathbf{A}$, then there is a refuting valuation ξ in $\mathbf{A}$ such that $\xi(A) = \mathsf{a} < \mathbf{1}$. By Proposition 2.8, there is a maximal filter F of $\mathbf{A}$ such that $\mathsf{a} \notin \mathsf{F}$. Then, $\mathbf{B} := \mathbf{A}/\mathsf{F}$ is an s.i. algebra, and $\nu = \eta \circ \xi$, where ν is a natural homomorphism, is a desired refuting valuation.

Let us note that because $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$, the finiteness of $\mathbf{A}$ yields the finiteness of $\mathbf{B}$.

Remark 2.2 In Jankov (1969), Corollary 2.3 (the Descent Theorem) is proved only for finite algebras. Yankov, being a disciple of Markov and sharing the constructivist view on mathematics, avoided using the Zorn Lemma which is necessary for proving Proposition 2.8 for infinite algebras.

2.3 Yankov's Characteristic Formulas

One of the biggest achievements of Yankov, apart from the particular results about si-logics, is the machinery that he had developed and used to establish these results. This machinery rests on the notion of a characteristic formula that he introduced in Jankov (1963c) and studied in detail in Jankov (1969).

2.3.1 Formulas and Homomorphisms

With each finite C-algebra $\mathbf{A}$ in the signature Σ we associate a formula $D_{\mathbf{A}}$ on variables $\{p_{\mathsf{a}}, \mathsf{a} \in \mathbf{A}\}$ in the following way: let $\Sigma_2 \subseteq \Sigma$ be a subset of all binary operation and $\Sigma_0 \subseteq \Sigma$ be a subset of nullary operations (constants); then

$$D_{\mathbf{A}} = \bigwedge_{\circ \in \Sigma_2} (p_{\mathsf{a}} \circ p_{\mathsf{b}} \leftrightarrow p_{\mathsf{a} \circ \mathsf{b}}) \wedge \bigwedge_{\mathfrak{c} \in \Sigma_0} (\mathfrak{c} \leftrightarrow p_{\mathfrak{c}}).$$

Example 2.1 Let $\mathbf{3} = (\{\mathsf{a}, \mathsf{b}, \mathbf{1}\}; \to, \wedge, \mathbf{1})$ be a Brouwerian semilattice, $\mathsf{a} \leq \mathsf{b} \leq \mathbf{1}$, and the operations are defined by the Cayley tables:

$\to$	a	b	**1**
a	**1**	**1**	**1**
b	a	**1**	**1**
1	a	b	**1**

$\wedge$	a	b	**1**
a	a	a	a
b	a	b	b
1	a	b	**1**

Then, in the Cayley tables, we replace the elements with the respective variables:

$\to$	p_{a}	p_{b}	$p_{\mathbf{1}}$
p_{a}	$p_{\mathbf{1}}$	$p_{\mathbf{1}}$	$p_{\mathbf{1}}$
p_{b}	p_{a}	$p_{\mathbf{1}}$	$p_{\mathbf{1}}$
$p_{\mathbf{1}}$	p_{a}	p_{b}	$p_{\mathbf{1}}$

$\wedge$	p_{a}	p_{b}	$p_{\mathbf{1}}$
p_{a}	p_{a}	p_{a}	p_{a}
p_{b}	p_{a}	p_{b}	p_{b}
$p_{\mathbf{1}}$	p_{a}	p_{b}	$p_{\mathbf{1}}$

and we express the above tables in the form of a formula:

$$\begin{aligned} D_3 = {} & (p_a \to p_a) \leftrightarrow p_1 \ \wedge (p_a \to p_b) \leftrightarrow p_1 \ \wedge (p_a \to p_1) \leftrightarrow p_1 \ \wedge \\ & (p_b \to p_a) \leftrightarrow p_a \ \wedge (p_b \to p_b) \leftrightarrow p_1 \ \wedge (p_b \to p_1) \leftrightarrow p_1 \wedge \\ & (p_1 \to p_a) \leftrightarrow p_a \ \wedge (p_1 \to p_b) \leftrightarrow p_b \ \wedge (p_1 \to p_1) \leftrightarrow p_1 \wedge \\ & (p_a \wedge p_a) \leftrightarrow p_a \quad \wedge (p_a \wedge p_b) \leftrightarrow p_a \quad \wedge (p_a \wedge p_1) \leftrightarrow p_a \quad \wedge \\ & (p_b \wedge p_a) \leftrightarrow p_a \quad \wedge (p_b \wedge p_b) \leftrightarrow p_b \quad \wedge (p_b \wedge p_1) \leftrightarrow p_b \quad \wedge \\ & (p_1 \wedge p_a) \leftrightarrow p_a \quad \wedge (p_1 \wedge p_b) \leftrightarrow p_b \quad \wedge (p_1 \wedge p_1) \leftrightarrow p_1 \quad \wedge \\ & \mathbf{1} \leftrightarrow p_1. \end{aligned}$$

Let us note that formula D_3 is equivalent in Pos^- to a much simpler formula,

$$D' = ((p_b \to p_a) \to p_b) \wedge p_1.$$

The importance of formula $D_{\mathbf{A}}$ rests on the following observation.

Proposition 2.9 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are* C*-algebras. If for valuation* ν *in* $\mathbf{B}$, $\nu(D_{\mathbf{A}}) = \mathbf{1}_{\mathbf{B}}$, *then the map*

$$\eta : \mathsf{a} \mapsto \nu(p_a)$$

is a homomorphism.

Proof Indeed, for any $\mathsf{a}, \mathsf{b} \in \mathbf{A}$ and any operation $\circ$, formula $p_a \circ p_b \leftrightarrow p_{a \circ b}$ is a conjunct of $D_{\mathbf{A}}$ and hence, $\nu(p_a \circ p_b) = \nu(p_{a\circ b})$, because $\nu(D_{\mathbf{A}}) = \mathbf{1}_{\mathbf{b}}$. Thus,

$$\eta(\mathsf{a} \circ \mathsf{b}) = \nu(p_{a\circ b}) = \nu(p_a \circ p_b) = \nu(p_a) \circ \nu(p_b) = \eta(p_a) \circ \eta(p_b).$$

It is not hard to see that η preserves the operations and therefore, η is a homomorphism.

Let us note that using any set of generators of a finite C-algebra $\mathbf{A}$, one can construct a formula having properties similar to $D_{\mathbf{A}}$. Suppose that elements $\mathsf{g}_1, \dots, \mathsf{g}_n$ generate algebra $\mathbf{A}$. Then, each element $\mathsf{a} \in \mathbf{A}$ can be expressed via generators, that is, there is a formula $B_a(p_{g_1}, \dots, p_{g_n})$ such that $\mathsf{a} = B_a(\mathsf{g}_1, \dots, \mathsf{g}_n)$. If we substitute in $D_{\mathbf{A}}$ each variable p_a with formula B_a, we obtain a new formula $D'_{\mathbf{A}}(p_{g_1}, \dots, p_{g_n})$, and this formula will posses the same property as formula $D_{\mathbf{A}}$. Because $D'_{\mathbf{A}}$ depends on the selection of formulas B_a, we use the notation $D_{\mathbf{A}}[B_{a_1}, \dots, B_{a_m}]$, provided that $\mathsf{a}_1, \dots, \mathsf{a}_m$ are all elements of $\mathbf{A}$.

Proposition 2.10 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are* C*-algebras. If* ν *is a valuation in* $\mathbf{B}$ *and* $\nu(D_{\mathbf{A}}[B_{a_1}, \dots, B_{a_m}]) = \mathbf{1}_{\mathbf{B}}$, *then the map*

$$\eta : \mathsf{a} \mapsto \nu(B_a)$$

is a homomorphism.

Example 2.2 Let $\mathbf{3}$ be a three-element Heyting algebra with elements $\mathbf{0}, \mathsf{a}, \mathbf{1}$. It is clear that $\mathbf{A}$ is generated by element a: