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Erdal Karapınar · Ravi P. Agarwal

Fixed Point Theory in Generalized Metric Spaces

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Erdal Karapınar: To my wife Senem Pinar and our children Can and Ulaş Ege.

Ravi P. Agarwal: To my wife Sadhna.

Preface

The fixed point theory is one of the major research areas in nonlinear functional analysis, as well as topology and applied mathematics. Indeed, a fixed point problem is very simple to state, and hence, a vast of real-world problems can be expressed in the framework of the fixed point. Naturally, the solution of a fixed point problem turns into the solution of a real-world problem. In other words, fixed point theory has wide application potential in almost all quantitative sciences due to its nature. Consequently, both the theoretical advances and possible applications have been studied densely.

In this book, we collect the basic metric fixed point results in the setting of metric spaces, as well as, in the context of the generalized metric spaces, such as b-metric spaces and partial metric spaces. The book consists of two parts: In the first part, we consider the fundamental properties of the standard metric space together with the basic fixed point theorems in the context of metric spaces. In the second part, we consider some generalization of the notion of the metric and collect some crucial fixed point theorems in this new structures. In particular, we focus on the fixed point theorems in the context of "partial metric spaces" and "b-metric spaces".

The book contains six chapters. The first three chapters present some preliminaries and historical notes on metric spaces and on mappings. In the second chapter, we recollect the basic notions such as metric, norm, and auxiliary functions. The third chapter is devoted to collect significant metric fixed point theorems. Chapter 4 aims to collect remarkable generalization of the metric space. Fixed point theorems in the context of b-metric spaces and partial metric spaces are discussed in Chaps. 5–6, respectively. In general, we aim to bring the historically important fixed point theorems in the metric spaces with giving the proofs of a few of them. Later, we indicate how these results can be generalized in the context of new abstract spaces, such as b-metric spaces and partial metric spaces. We avoid the proof of all theorems and corollaries due to analogy.

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Kingsville, USA — Ravi P. Agarwal
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Contents

Acronyms

$\mathbb{Z}$	Integers
$\mathbb{N}$	Natural numbers
$\mathbb{N}_0$	Nonnegative integers
$\mathbb{R}$	Real numbers
$\mathbb{R}_0^+$	Nonnegative real numbers
X	A nonempty set
(X, d)	Metric space
$(X, \|.\|)$	Normed space
T	A self-mapping on X
T^n	nth iteration of T
$Fix(T)$	The set of all fixed point of T in X
$Fix(T^n)$	The set of all fixed point of T^n in X
ϕ	Comparison function defined on $\mathbb{R}_0^+$
ψ	c-Comparison function defined on $\mathbb{R}_0^+$
Φ	The class of comparison function
Ψ	The class of c-comparison function
Ψ_b	The class of b-comparison function

Part I
Fixed Point Theorems in the Framework of Metric Spaces

In this first part, we aim to recollect and discuss the classical fixed point theorems as well as the recently published interesting results in the setting of (standard) metric spaces.

1 Introduction

Fixed point theory can be described as a framework for researching and investigating the existence of the solution of the equation $f(p) = p$ for a certain self-mapping f that is defined on a non-empty set X. As is expected, here, p is called the fixed point of the mapping f. On the other side, we may re-consider the fixed point equation $f(p) = p$ as $T(p) = f(p) - p = 0$ and, accordingly, finding the zeros of the mapping T and finding the fixed point of f becomes an equivalent statement. This equivalence, not only enriches the fixed point theory but also, opens the doors to a wide range of potential applications in the setting of almost all quantitative sciences. For example, let us consider one of the classical open problems of number theory, finding perfect numbers: Let p be a self-mapping on a natural number such that $p(n)$ is the sum of all divisors of n for $n > 1$. Thus, any fixed points of the function p give a perfect number. In particular, 6 is the smallest perfect numbers, and $2^{74207280} \times (2^{74207281} - 1)$, with 44, 677, 235 digits, is the biggest known perfect number.

The natural and the basic questions of the fixed point theory of can be listed as follows:

$(Q1)$ For a self-mapping f on a non-empty set X, which conditions are necessary and sufficient so that the mapping f possesses a fixed point in this structure?

$(Q2)$ What are the necessary and sufficient properties that a corresponding function f should be equipped with these conditions to guarantee that the fixed point equation $f(p) = p$ has a solution?

The fixed point theorem is split into three main research branches:

1. Metric fixed point theorem,
2. Topological fixed point theorem,
3. Discrete fixed point theorem.

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The origin of the metric fixed point theory draws back to Liouville [207] who proposed method of successive approximations for a solution of certain differential equations in 1837. Later, Picard [228] in 1890 developed this method systematically. Indeed, both of them implicitly used a fixed point approach to solve the differential equations that they dealt with. It is also known that the roots of the metric fixed point theory reach to Cauchy [226]. Roughly speaking, Cauchy proved the existence and uniqueness of an initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ for a continuous differentiable function f, first times. Lipschitz [226] refined the proof of Cauchy by using “Liptschitz constant”. Peano [226] observed that differentiability conditions on f is superfluous. On the other hand, metric fixed point theory formally appeared in the distinguish result of Banach [56] in 1922. Indeed, the renowned result of Banach [56], known also as Banach Contraction Mapping Principle, is an abstraction of the successive approximation that is mentioned above. Banach contraction mapping principle can be easily stated as “every contraction in a complete metric space has a unique fixed point”. As a historical note, we write down that this version was published by Caccioppoli [88] in 1931. hyper@itemfalse

On the other side, the origin of the topological metric fixed point theory can be sent back to the outstanding result of Poincaré [229]. The variants of Poincaré [229] were proved by Bohl [69], Brouwer [83] in 1912 and Hadamard [123] in 1910. Brouwer’s notable result is the most famous one, which states, “a continuous map on a closed unit ball in $\mathbb{R}^n$ possesses a fixed point.” An extension of this result is Schauder’s fixed point theorem [260] of 1930, which states that “a continuous map on a convex compact subspace of a Banach space possesses a fixed point.”

In 1955, Tarski [271] initiated the discrete fixed point theory by reporting a lattice-theoretical fixed point theorem. Discrete fixed point theory has been used in the economy, particularly in (Nash) equilibrium theory.

In 1974, Ćirić [97] published an interesting paper in which the investigated operators have a fixed point that is not necessarily unique. It has been called nonunique fixed points. Regarding the crucial role of the fixed point theorems in solving differential equations, we recall that not every equation has a unique solution. Naturally, not every operator can have a unique fixed point. Certain differential equations have periodic solutions; hence, the corresponding operators have periodic fixed points. Due to these reasons, many authors have been interested in the investigation of the existence of nonunique fixed points, and they have published fascinating results in this direction, see e.g. [4, 97, 121, 205, 206, 223, 281].