Mathematics Education in the Digital Era

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Patrick W. Thompson *Editors*

Quantitative Reasoning in Mathematics and Science Education

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Mathematics Education in the Digital Era

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Introduction

Quantitative Reasoning in Mathematics and Science Education in the Digital Era

The digital era is a period beginning in the mid-20th century and leading its way into the 21st century. Technology characterizes this era as it provides access to widespread information in various electronic forms; therefore, it “increases the speed and breadth of knowledge turnover within the economy and society” (Shepherd, 2004, p. 1). The digital era demands specific 21st-century skills and abilities such as critical thinking, creativity, collaboration, communication, and flexibility. These skills are central to STEM disciplines (Beswick & Fraser, 2019), with which the teachers and learners need to be equipped. Even though technology is a crucial driver for such a “skills agenda, simply assisting students to develop up-to-date technology skills is not sufficient” (Beswick & Fraser, 2019, p. 958) to promote such an agenda. This is where we believe quantitative reasoning comes to the fore as it lays the foundation for developing these skills within STEM subjects. This book focuses on quantitative reasoning as an orienting framework to analyze learning, teaching, and curriculum. Different chapters of the book delve into quantitative reasoning related to the learning and teaching diverse mathematics and science concepts, conceptual analysis of mathematical and scientific ideas, and analysis of school mathematics (K-16) curricula in different contexts.

Quantitative reasoning is “an individual’s analysis of a situation into a quantitative structure” (Thompson, 1990, p. 13) such that it entails “the mental actions of an individual conceiving a situation, constructing quantities of his or her conceived situation, and both developing and reasoning about relationships between these constructed quantities” (Moore et al., 2009, p. 3). Thompson and Carlson (2017) point out that envisioning a situation in terms of a quantitative structure is advantageous for students’ positioning “to propagate information about how to calculate values of quantities in the structure in terms of arithmetic or algebraic expressions that are implied by the structure” (p. 440). Particularly, quantitative reasoning provides “content and meaning for numerical and symbolic expression and computation”

(Smith III & Thompson, 2008, p. 41). Envisioning a situation in terms of quantities and relationships among quantities is important to establish a foundation for reasoning about covariation, which plays a crucial role in learners' development of more complex mathematical and scientific ideas in critical ways (Thompson & Carlson, 2017). These suggest that quantitative reasoning is a key in education, and the proposed book unveils its particulars. In this regard, Johnson's chapter focuses on the "relationships" as an intellectual need and uses mathematizing to describe a category of a way of thinking emerging from that need. These relationships are essential in both mathematics and science. Gonzales' chapter uses quantitative reasoning to develop an understanding of the energy budget as a system of interrelated quantities and utilizes covariational reasoning to investigate climate change with a critical lens. In addition, Brahmia and Olshon's chapter discusses physics quantitative literacy as the blending of conceptual and procedural mathematics to generate and apply models relating physics quantities to each other.

The relevant literature suggests that quantitative reasoning supports the learning of arithmetic and algebra and plays a vital role in learning concepts foundational to calculus, geometry, trigonometry, physics, and so on. The literature studies provided detailed accounts of how quantitative reasoning can play an essential role in learning and teaching different mathematical and scientific concepts. In this book, Moore et al. chapter provides an analysis of concept construction from a quantitative reasoning perspective. In addition, Paoletti et al. chapter further describes a task sequence to construct covariational relationships among quantities and distinguish nonlinear and linear relationships. Moreover, based on a 15-year research program, Carlson et al. chapter explores how to support instructors in making their precalculus teaching more engaging, meaningful, and coherent using quantitative relationships symbolically and graphically.

Quantitative reasoning also provides a propitious arena for the conceptual analysis of mathematical and scientific ideas. Thompson (2008) defined conceptual analysis of mathematical ideas as a method "to describe ways of understanding ideas that have the potential of becoming goals of instruction or of being guides for curricular development" (p. 58). Conceptual analyses are "extremely powerful" because they offer concrete examples of learning trajectories (Thompson, 2008). The book gives examples of such analyses from different areas of mathematics. For instance, Akar, Zembat, Arslan, and Belin's chapter provides such analysis of isometries and their conceptualization. Nunes and Bryant's chapter considers numbers and number systems as models of quantitative relations and investigates how action schemas used in different situations support students' understanding of quantities and numbers. Ellis et al. chapter provides examples from linear and quadratic functions by identifying a sequence of conceptual activities and examples of associated student reasoning and task design principles to guide curricular decisions.

The use of quantitative reasoning in the development of ideas in curricula has also been given prominence since 2010 in Common Core State Standards for Mathematics (CCSSM) (Johnson, 2016). However, Johnson argued that despite greater inclusion of quantity and quantitative reasoning in CCSSM, a lack of emphasis on forming and interpreting relationships between quantities that change together remains a challenge. Thompson and Carlson (2017) proposed researching the systematic analysis of different curricular approaches that support students in developing quantitative and covariational reasoning. Akar, Watanabe, and Turan's chapter exemplifies such systematic analysis based on quantitative reasoning for a Japanese textbook series and curricular resources.

Quantitative reasoning is also crucial for other disciplines, including science. Duschl and Bismack (2013) stated, "quantitative reasoning is represented as a component of model-based reasoning that bridges the divide between mathematics and science" (p. 122). Similarly, further elaborating on quantitative reasoning, Thompson (2011) offered a detailed definition of quantification as "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit" (p. 37). Thompson considered this definition as a link between mathematics and science education. One can undoubtedly establish such connections between mathematics and other disciplines, and this book contributes to such an initiative. For example, Jin et al. chapter uses the mathematization of science dwelling on quantitative reasoning to quantify phenomena and construct knowledge and as a cross-cutting theme to build curricular coherence in physical and life sciences.

Although not exhausting all quantitative reasoning work, we point to the importance of quantitative reasoning and its crucial role in mathematics and science education with this book. Thompson's introductory chapter highlights that many scholars have based their work on quantitative reasoning as a framework to investigate and think about learning and teaching, conceptual analyses, curricular efforts, and links to other disciplines for decades. However, there seems to be a void in collecting this work together and pondering quantitative reasoning from different angles. This book provides ways to cluster the work established so far and can be considered as a reference book to be used by researchers, teacher educators, curriculum developers, and pre- and in-service teachers. We hope that it finds its place in the mathematics and science education literature within the digital era.

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Contents

Quantitative Reasoning as an Educational Lens

Patrick W. Thompson

I must begin by thanking Gülseren Karagöz Akar, Ismail Özgür Zembat, and Selahattin Arslan for including me in their effort to produce this book. While I am listed as an editor, they did the heavy lifting of conceptualizing the book and working with authors. My role was more as a consultant than an editor. I am nevertheless grateful they thought to include me.

1 Origins of a Theory of Quantitative Reasoning and Its Applicability

Humans have been reasoning quantitatively for thousands of years. I did not invent quantitative reasoning. I developed a *theory* of quantitative reasoning—a theory with the aim of explaining how individuals might come to reason about the world as they see it through a measurement lens (including not seeing it through a measurement lens) and implications for students' mathematical learning. My early work was motivated by wanting to understand students' difficulties with story problems—descriptions of settings designed by textbook authors that included a question about the setting. This interest was sparked in the spring of 1985 by James Greeno in his presentation of Valerie Shalin's work (Shalin, 1987; Shalin & Bee, 1985) to the mathematics education faculty at San Diego State University. Shalin designed a computer interface of notecards to represent quantities and arrows among notecards to show relationships. I realized Shalin had devised a way to represent relationships among quantities without having to rely on formulas or expressions. Shalin had not, however, explicated what she meant by quantity or quantitative relationship, nor did

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she include a theory of how relationships among quantities imply methods for evaluating them. However, I immediately saw the theoretical power of having a way to represent quantities and relationships without formulas or expressions.

In 1986 I was invited to contribute a chapter on artificial intelligence (AI) in mathematics education to an NCTM publication on learning and teaching algebra (Thompson, 1989). I wanted to include a discussion of Shalin's and Greeno's computer program, but was unsuccessful in obtaining more information about it. I therefore decided to write an AI program, *Word Problem Analyst* (WPA), inspired by Shalin's interface and discuss the aspects of quantitative reasoning as I conceived it embodied in the program. I will not recap all the insights I gained from writing WPA (and revising it over the next four years) except to say writing it, with support from the US National Science Foundation, provided a testbed for creating a scheme theory for ideas of quantity and the development of mathematical reasoning from quantitative reasoning (Thompson, 1990, 2011).

The following problem and Figs. 1, 2, 3, 4, 5, 6 and 7 illustrate the use of WPA to model someone conceptualizing a problem in terms of quantities and relationships among quantities and the algebra that can be inferred from this structure.

> MEA Export is to deliver an oil valve to Costa Rica. The valve's price is $5000. Freight charges to Costa Rica are $100. Insurance is 1.25% of Costa Rica's total cost. Costa Rica's total cost includes the costs of the valve, insurance, and freight. What is Costa Rica's total cost? (Thompson, 1990, p. 39)

Figure 1 shows a person's (say, José's) conception that there are six quantities involved in this situation: Total Cost to Costa Rica, the costs of Freight, Valve, and Insurance, the Insurance Rate, and the cost of Insurance and Freight together. At

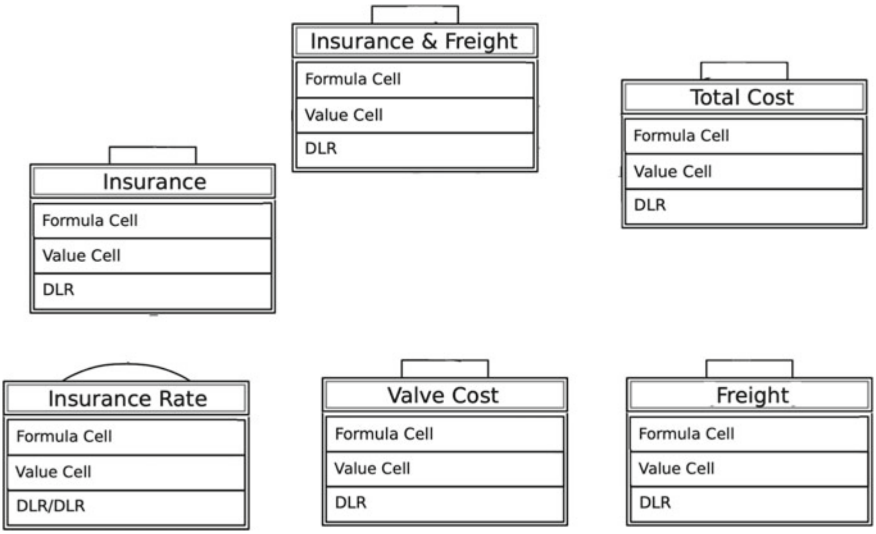

Fig. 1 José's understanding of quantities involved in the situation

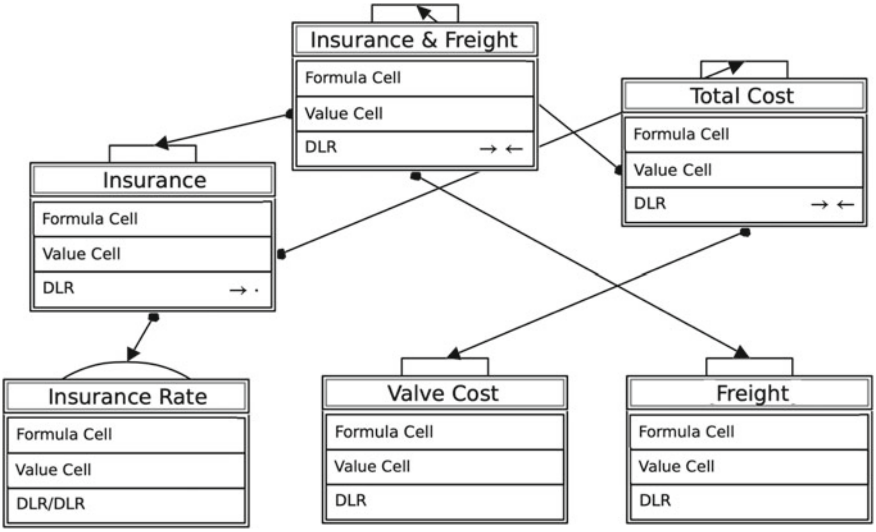

Fig. 2 José's conception of relationships among quantities in the situation

this moment José has not conceptualized any relationships among quantities. Each notecard reflects the schematic nature of a conceived quantity—a natural language description of an object's attribute, a unit in which the attribute is measured, and a potential value for the quantity's measure. Each notecard also has a "Formula Cell". This represents José's anticipation that a quantity's value might be calculated from relationships with other quantities.

Figure 2 shows the relationships José envisioned among quantities: *Total Cost* is made by an additive combination of *Insurance and Freight* and the cost of the *Valve*. *Insurance and Freight* is made by an additive combination of the cost of *Insurance* and the cost of *Freight*. The cost of *Insurance* is made by instantiating the *Insurance Rate* with the *Total Cost* to Costa Rica. Notice that at this moment, José has not thought about any calculations.

Figure 3 shows that José has now attended to the information given in the problem statement. *Freight* has a value of $100, *Valve Cost* has a value of $5000, and *Insurance Rate* has a value of $1.25/100 of insurance per dollar of cost. Notice that, at this moment, José cannot make any inferences about values of other quantities.

Figure 4 shows José's decision to let *C* stand for the value of *Total Cost* to Costa Rica. Figure 5 shows an immediate consequence of letting *C* stand for the value of *Total Cost*—since *Total Cost* is made by an additive combination of *Insurance & Freight* and *Valve Cost*, and *Valve Cost* has a value of 5000, the value of *Insurance and Freight* must be *C*—5000$.

Figure 6 shows the next propagation. Since *Insurance* is made by instantiating *Insurance Rate* with the value of *Total Cost*, the value of *Insurance* will be $C * 0.0125$ dollars. Figure 7 reflects José's openness to deriving a formula for a quantity for which he already knows a value. *Insurance & Freight* is made by an additive combination

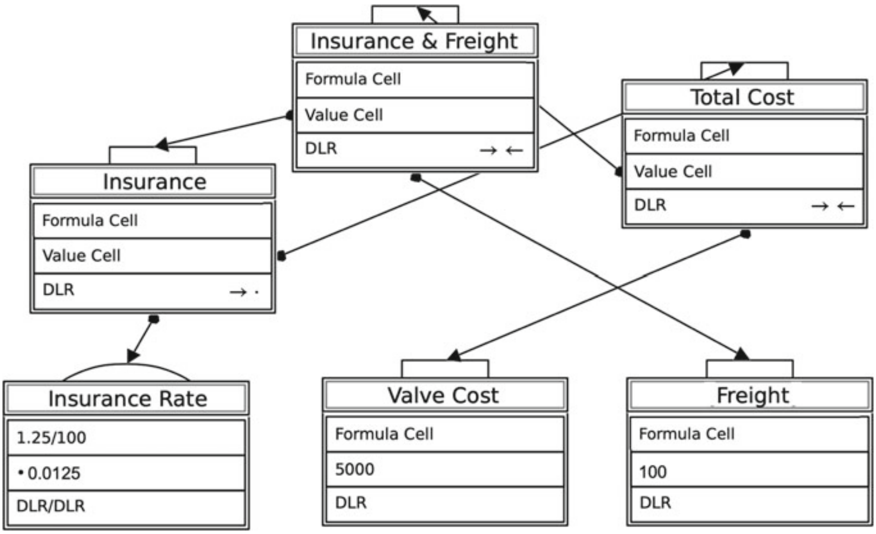

Fig. 3 Adding information given in the problem to José's conception of the situation

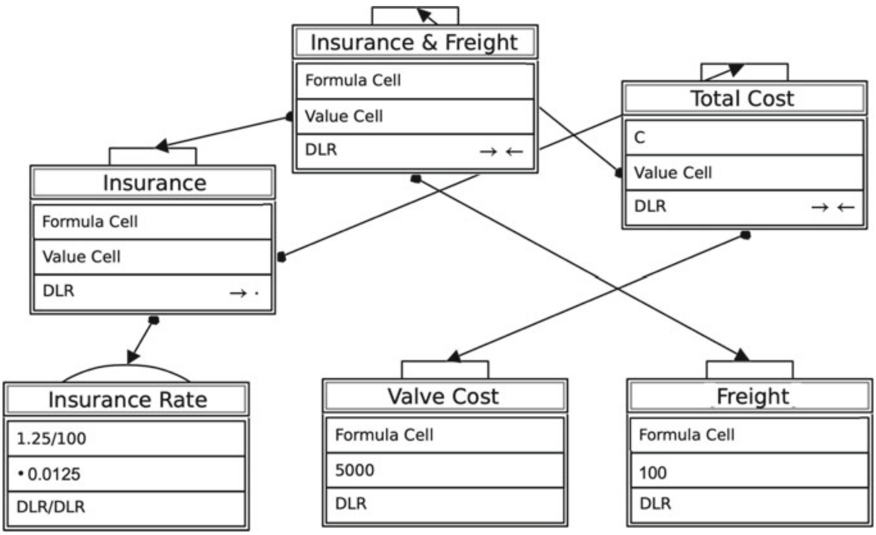

Fig. 4 Using "*C*" to stand for the value of *total cost*

of *Insurance* and *Freight*, and since its value is $C - 5000$ and *Freight's* value is 100, José infers that a formula to compute *Freight's* value is $C - 5000 - 0.0125C$. But the

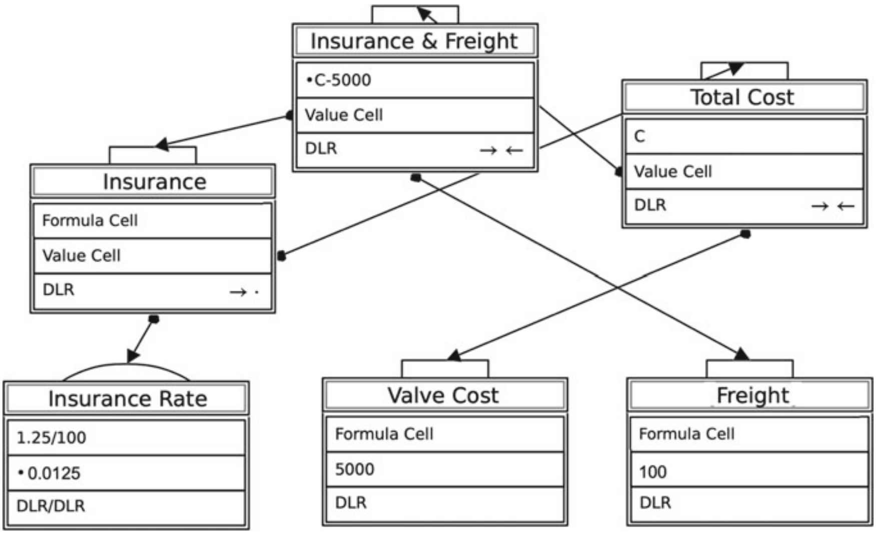

Fig. 5 Inferring a formula to compute the value of *insurance and freight*

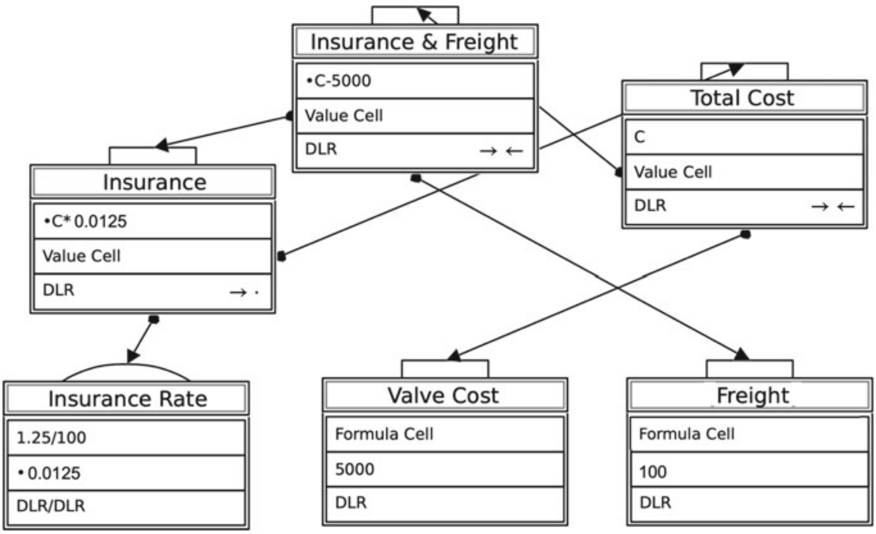

Fig. 6 Inferring a formula to calculate the value of *insurance*

value this formula must yield is the value of *Freight*, which is 100.[1] In other words, by reasoning quantitatively, José ended with the equation $C - 5000 - 0.125C = 100$.

[1] The brackets in the *Freight* notecard indicate that José ignored the fact he already knows a value of *Freight* in order to infer a formula to compute *Freight's* value.

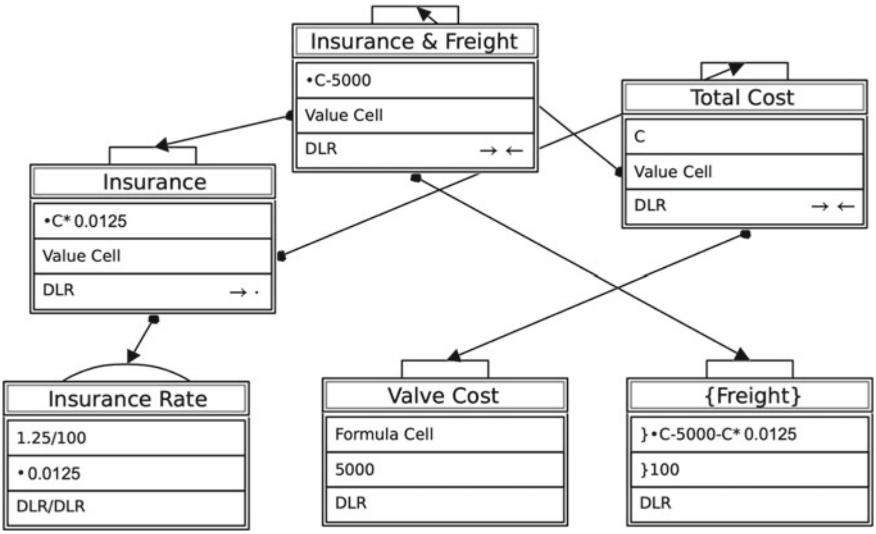

Fig. 7 Inferring a formula for the value of *freight* even though it has a known value of 100

José's conceptualization of the Costa Rica situation is not unique. It can be conceptualized in many ways. Indeed, in Thompson (1990) I illustrate how even simple problems can have very different underlying conceptualizations in terms of quantities and relationships composing it yet yield the same arithmetic or algebra.

There are three significant differences between Shalin's model and the theory I developed. First, Shalin's model did not have an underlying theory of quantity or quantification, except for the arithmetic or units developed by Schwartz (1988). An arithmetic of units, such as $\text{cm} \cdot \text{cm} = \text{cm}^2$, or $(\text{ft/s})/\text{s} = \text{ft/s}^2$,conflates arithmetic operations and quantitative operations. It is not a theory of quantitative reasoning. Rather, units are treated as if they are numbers or variables. An arithmetic of units is implied by quantitative reasoning, but it is not a theory of it. Second, the theory addressed how one propagates information throughout a quantitative structure when knowing only partial information about the context. The theory of propagation is the foundation of the model's hypotheses about students' transitions from quantity-based arithmetic to quantity-based algebra (and beyond). Third, Shalin did not make a distinction between quantitative operations and arithmetic operations, which resulted in confounding type of quantity with an arithmetic operation to calculate its value, such as describing a quantity as a difference simply because, in a particular situation, subtraction is used to calculate its value (see Greeno, 1987, p. 77).

Finally, the WPA model of José's conception of the Costa Rica situation presumed he had mature schemes for the quantities and quantitative operations depicted therein. WPA was meant to model implications of reasoning quantitatively for algebraic reasoning. It did not address ways learners *construct* quantities and quantitative operations. The theory I expressed in Thompson (1990) provided a foundation for later studies that brought coherence to understanding the development of students' schemes for quantitative comparisons, variation and covariation, ratio and rate, geometric and exponential growth, uses of notation, function, probability and statistics, and many ideas specific to calculus.

2 Chapters in This Book

I am surprised and gratified that many people found this early work, and later expansions of it, useful in their research. The chapters in this book show creative uses of quantitative reasoning as a lens for making sense of students' reasoning, for design of instruction, for curriculum design and evaluation, for teacher professional development, and for design of assessments. Johnson's use of Harel's notion of intellectual need as a motive for why students might *seek* relationships between quantities whose values vary is novel and powerful. Moore et al.'s focus on students' creation of abstract quantitative structures addresses the question of how students might generalize their quantitative reasoning in specific contexts to broader areas of application. Karagöz Akar, Watanabe and Turan created a novel way of examining mathematics textbooks by the criterion of ways they support or inhibit students' quantitative reasoning. Paoletti extends a framework for thinking about students' variational and covariational reasoning by filling a gap in it, while Ellis et al. build a learning progression based in variational and covariational reasoning to address students' development over early grades of schemes for function. Karagöz Akar, Zembat, Arslan and Belin leverage quantitative reasoning to address the issue of students' difficulties in conceiving motions in the plane as functions mapping $\mathbb{R}^2$ to $\mathbb{R}^2$. Carlson et al. leverage quantitative reasoning to address the question of how to support teachers in transitioning from speaking to students as if to themselves to engaging students in reflective discourse aimed at students' construction of coherent systems of mathematical meanings. I am especially gratified to see three chapters by science educators leveraging a theory originally aimed to support learning and teaching mathematics to address issues within science education. Jin et al. apply quantitative reasoning as a theme to enhance curricular coherence across grade levels and across a broad array of scientific concepts. González uses quantitative reasoning, especially distinctions between ratio as a quantity and rate as a quantity, to examine students' meanings for ideas central to understanding climate change. White Brahmia and Olsho turn the lens around. Instead of using quantitative reasoning as a lens on students' reasoning in physics, they use physics as a context to assess students' quantitative reasoning. Nunes and Bryant take an approach to quantitative reasoning more in line with Schwartz (1988), in which numbers represent quantities and arithmetic operations imply operations on quantities.

I suspect one reason quantitative reasoning has found such broad applicability is its fundamental stance that quantities are in a mind, not in the world. This stance forces anyone adopting it to examine ways *learners* understand situations presented to them. It forces us to ask, "What is this situation *to the learner*?" As Carlson et al. (this volume) document, adopting this stance is nontrivial for instructors who are accustomed to apply criteria of coherence only to their own understandings, not to ways their students might understand the situations presented to them or might understand their instructor's actions and utterances regarding a situation.

Another possible reason quantitative reasoning has been found broad applicability is that using it forces one to employ a level of qualitative precision that is uncommon in mathematics instruction, yet beneficial for students' learning. Distinctions among object, attribute, and measure are often unaddressed by mathematics teachers—as witnessed by the common proclivity among teachers and students to write statements like "D = distance". Carlson et al. (this volume) document difficulties precalculus instructors create for themselves and their students by their lack of precision about contextual meanings of numbers, variables, and expressions.

3 Conceptualizing Units and Conceptualizing Quantification: Aspects of Quantitative Reasoning Needing Greater Attention

Early on in developing this theory of quantitative reasoning I proposed that a quantity is a scheme—someone's conception of an object and an attribute of it the person has conceived as measurable in an appropriate unit. I also spoke repeatedly of the synergy among a person's conceptions of object, attribute, and measurability—they each mature as the person gains clarity on the others. In Thompson (2011) I gave a brief recount of 8th-graders' construction of "explosiveness of a grain silo" as a quantity. They engaged in extended discussions of just what was it that was explosive: The silo? The grain in the silo? Dust in the silo? Dust in the air within the silo? They also had to settle on a mechanism for explosions, eventually settling on oxidation at the surface of grain dust particles. This led them eventually to a unit of grain silo explosiveness: cm^2 of "dust surface area" per cm^3 of "dust volume" per ft^3 of "silo volume" in which the dust is dispersed.

I offered the example of grain silo explosiveness to illustrate the messiness of quantitative reasoning that often is unaddressed in studies employing a quantitative reasoning lens. But we need not go to uncommon quantities like "grain silo explosiveness" to see the interdependence among conceptualizations of object, attribute, and unit. In Thompson (2000) I spoke of ways students often understand area and volume as one-dimensional quantities. Area is one-dimensional when one conceives the unit as having one dimension—a square region of a particular size. Then all areas are just counts of that one-dimensional unit. Similarly, volume is one-dimensional when one conceives the unit as having one dimension—a cubic object of a particular size. Then all volumes are just counts of that one-dimensional unit. Brady and Lehrer (2020) clarified that a unit of area is conceived as two-dimensional when one conceives it as generated by two segments, one being swept along the other. This is the imagistic equivalent of understanding the interior of a rectangle being formed by the cross product of two perpendicular lines viewed as sets of points. Karagöz

Akar et al.'s chapter on isometries makes a similar point with respect to conceptualizing the Cartesian plane as $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ You obtain a two-dimensional object by the quantitative operation of multiplicative combination of two one-dimensional objects. Area and volume are just two instances of quantities teachers and researchers take as unproblematic in conceiving their unit when in fact students often conceive them in ways that are problematic for their comprehension of situations involving them.

In the following paragraphs, I offer two additional examples to illustrate the messiness of quantification and how attention to units can be helpful to students in understanding mathematical or scientific ideas. The first is conceptualizing interest rate as actually being a rate of change of one quantity with respect to another. The second is the quantification of kinetic energy.

3.1 Quantification of Interest Rate as a Rate of Change

To specify a quantity as a rate of change, we must state two quantities whose values covary. They vary with respect to each other. The "rate of change" attribute of two quantities covarying is captured by a statement of the amount one varies in relation to variations in the other.

Here are three definitions of interest rate by commonly accepted authorities:

1. "The cost of borrowing money from a lender is represented as a percentage of the principal loan amount, called the interest rate." U.S Federal Housing Administration https://www.fha.com/define/interest-rate
2. "The amount earned on a savings, checking, or money market account, or on an investment, as a certificate of deposit or bond, typically expressed as an annual percentage of the account balance or investment sum." Dictionary.com https://www.dictionary.com/browse/interest-rate
3. "The percentage usually on an annual basis that is paid by the borrower to the lender for a loan of money." Meriam-Webster.com https://www.merriam-webster.com/dictionary/rateofinterest.

I find it peculiar that, despite purporting to define interest rate, none of these statements actually defines a rate of change of one quantity with respect to another.

Imagine a bank advertisement as follows:

> We pay 3% interest per year on your deposit.

What quantities are involved in this practice of charging or paying interest? What are their units? What is the rate of change of one quantity with respect to another that is the "rate"?

The quantities are interest paid (dollars of interest), dollars on balance (basis of the percentage), and an amount of time (number of years balance is on deposit). Regarding the rate—what is it? Is it a rate of change of balance with respect to time? The rate of change of interest earned with respect to time?

The crux of the matter is to understand that "3%" has a unit: dollars of interest per dollar on balance. The unit of "3% interest per year" is ($interest/$balance)/year. The bank will pay interest at the rate of 0.03 dollars interest per dollar of balance per year. There is yet one open question: What constitutes the balance upon which interest is computed? Is it the current balance at the time of computing interest, or is it the initial balance at the time of opening the account?

The difference between simple interest and compound interest is much easier to understand when we answer these questions explicitly. "We pay 3% interest per year on your deposit, compounded quarterly" means that at the end of each quarter they will add to your balance the amount earned at the rate of (($0.03 interest per $1.00 balance at beginning of compounding period) per year) earned in 1/4 year. You earn interest over a quarter year at 1/4 the rate you would earn over a year. This is like speeding up at a rate of 10 (km/h)/h for 1/4 h. Your speed increases at a rate per 1/4 hour that is 1/4 the rate for an hour, or at a rate of (2.5 km/h) per 1/4 h.

The idea of the unit of an interest rate is related to students' difficulties distinguishing between linear and geometric growth. Graphs given in Fig. 8 show two ways to understand the phrase "… increases at a rate of 20% per month." Fig. 8a shows 20% of the original amount (e.g., $2) added to the current value (e.g., $10) to get the next value (e.g., $12). The same amount is added at the end of each month. Figure 8b shows 20% of the *current* month's value added to get the next month's value. Since the current value increases each month, the amount added at the end of each month increases.

The phrase "… increases at a rate of 20% per month" is ambiguous regarding which interpretation the speaker intends a listener to make. Being clear about the quantities and their units is clarifying. The first would be "… increases at a rate of

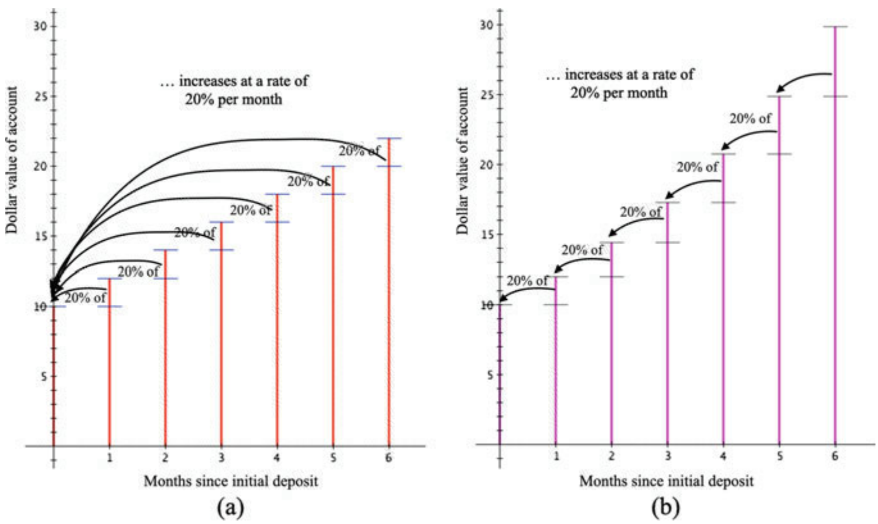

Fig. 8 Two ways to interpret the phrase, "… increases at a rate of 20% per month"

(\$0.20 interest per dollar of initial balance) per month", whereas the second would be, "… increases at a rate of (\$0.20 interest per dollar of current balance) per month".

3.2 Quantification of Kinetic Energy

A characteristic of physical quantities is how deeply their conceptualizations are interconnected. Energy is commonly defined as "the capacity to do work" (*Encyclopedia* Britannica, 2022). The idea of work is tied to the idea of applying a force to move an object some distance, while force is the idea of accelerating an object (having mass) from one velocity to another velocity. The meaning of *kinetic* energy is the work required to bring an object having mass m from velocity v to velocity 0.[2]

Jin et al. (this volume) speak of students' understanding of kinetic energy in terms of implications they draw from a formula for quantifying its measure, namely $E = \frac{1}{2}mv^2$, for how an object's kinetic energy changes when its velocity changes. Some students think doubling an object's velocity doubles its kinetic energy. Other students think doubling its velocity quadruples its kinetic energy. The issue Jin et al. addressed is students' abilities to reason about the implications of a quantification expressed in a formula. I address a more foundational issue—the quantitative reasoning involved in *quantifying* kinetic energy to *end* with the formula $E = \frac{1}{2}mv^2$. My aim here is to illustrate how conceptualizations of object, attribute, and quantifications are intertwined.

To quantify kinetic energy, we must identify an object and its attribute as a starting point of quantification—to determine a method by which to measure it and the unit in which it will be measured. In the case of kinetic energy, the "object" is anything having mass. One attribute is its motion—it is moving (at least momentarily) at a constant velocity. Another attribute is the effort (work) required to stop its motion. Work, as a quantity, is a force applied over a distance. The object's velocity, however, is not constant. Its velocity decreases as work is applied to it.

A slight twist which makes envisioning kinetic energy easier is to realize the energy required to bring an object from velocity v to velocity 0 is the same as the energy required to bring it from velocity 0 to velocity v.

Breaking down these components, and envisioning the object's velocity changing in little bits as it accelerates from 0 to v, we get

- a force of measure F is created by accelerating a mass of measure m at a rate of measure a,
- a small bit of acceleration is created by changing an object's velocity by a variation of measure dv during a variation of time of measure dt,
- a small variation in distance ds is made by going at velocity v for a small variation in time dt,

[2] I have limited these descriptions to mechanical quantities to avoid dealing with the complexities of their electro and thermal equivalents.

- a small variation of work is created by applying a force of measure F over a small variation in distance of measure ds, and
- a small variation in an object's kinetic energy of measure dE is created by a small variation in work of measure Fds that varies its velocity.

Symbolically, taking F as a measure of force, E as a measure of kinetic energy, and dE, dv, dt, and ds as infinitesimal variations in kinetic energy, velocity, time, and distance, respectively:

$$F = ma, a = \frac{dv}{dt}, ds = vdt, dE = Fds$$

$$--------------------$$

$$\begin{aligned} dE &= Fds \\ &= mads \\ &= m\left(\frac{dv}{dt}\right)vdt \\ &= mvdv \end{aligned}$$

So, a small variation in an object's kinetic energy is its momentum times a small variation in its velocity. This says an object's momentum at any velocity is its *rate of change of kinetic energy with respect to velocity.*

Recalling that the work required to decelerate an object from v to 0 is the same as the work required to accelerate it from 0 to v, an object's kinetic energy is the (hyper) sum of all infinitesimal variations in its kinetic energy as velocity varies from 0 to v. Symbolically[3]:

$$\begin{aligned} E(v) &= \int_0^v mudu \\ &= \frac{1}{2}mu^2\Big|_{u=0}^{u=v} \\ &= \frac{1}{2}mv^2 \end{aligned}$$

As I said earlier, a full, robust understanding of this quantification of kinetic energy requires understanding constituent quantities' units (units of mass, time, distance) and the units of quantities created from them (acceleration, force, momentum, work, kinetic energy)—but not in the sense of an arithmetic of units. Rather, I mean one

[3] I acknowledge that this derivation relies on students' understanding of integrals as a (hyper) sum of infinitesimal variations and on their understanding of the relationship between a rate of change function and its accumulation functions. However, they could approximate any object's kinetic energy to an acceptable accuracy with Desmos using the finite sum $E_{\text{approx}}(v) = \sum_{i=1}^{v/\Delta v} m(i\Delta v)\Delta v$, where Δv is a small increment in velocity. See Thompson et al. (2019, Ch 5) for a full development of these ideas.

must understand units in the sense of Bridgman's (1922) dimensional analysis, which attends to the *creation* of quantities from other quantities while attending to the nature of their attributes. Bridgman wrote, for example, $[F] = [m][a]$ to convey that the quantity force is formed by the quantitative operation of multiplicative combination—by accelerating an object having mass. He wrote $F = ma$ to represent how you calculate a measure of force, ending with a number with a unit that is consistent with the quantity's dimension. If you measure a mass in kg and acceleration in ((meters per second) per second), the unit of force is kg ((m/s)/s), meaning a mass measured in kg is accelerated at a rate measured in ((meters per second) per second).

How might students *know* to multiply m and a to calculate a measure of force? Hopefully, from schemes they constructed through experimentation,[4] that force is proportional to both mass and acceleration. If we increase by a factor of j the mass being accelerated at a rate a, the force of accelerating it increases by a factor of j; if we increase the acceleration of an object by a factor of k, meaning its velocity increases k times as rapidly with respect to time, the force of accelerating it increases by a factor of k. Let $F(j, k)$ represent a measure of the force of accelerating an object of j mass units at a rate of k acceleration units. Then $F(j, k) = F(j \cdot 1, k \cdot 1) = j \cdot kF(1, 1)$. This says the measure of force that accelerates a mass of measure j mass units at a rate of measure k acceleration units is $j \cdot k$ times as large as the force of accelerating a mass of measure 1 mass unit at a rate of change of velocity with respect to time of 1 acceleration unit.

Lastly, there is another question we should hope students ask with respect to quantification of kinetic energy. Since kinetic energy is equivalent to an amount of work, they hopefully ask whether $\frac{1}{2}mv^2$, our quantification of kinetic energy, actually quantifies an amount of work. If it does, then the derived unit of $\frac{1}{2}mv^2$ must, in line with Bridgman, accord with a force applied over a distance. Its unit must be of dimension $[F][d]$. Here is where arithmetic of units is useful.

The standard unit of force in the kg-meter-second system is the Newton (N), or 1 kg accelerated at 1 (m/s)/s. Keeping track of units, and using m as a measure of mass and v as a measure of velocity in the kg-meter-second system, we get

$$\begin{aligned}\frac{1}{2}mv^2 &\to \mathrm{kg\,m^2/s^2}\\ &\to \big(\mathrm{kg}\big(\mathrm{m/s^2}\big)\mathrm{m}\\ &\to (\mathrm{kg\,(m/s)/s})\,\mathrm{m}\\ &\to N\,\mathrm{m}\\ &\to [F][d]\end{aligned}$$

The unit of $\frac{1}{2}mv^2$ in the kg-meter-second system is the Newton-meter, which is of dimension $[F][d]$, so it is a unit of work.

[4] Of course, the experimentation that affords students an opportunity to construct such schemes must be crafted carefully so their abstractions are from their own activity.

4 Connections with Chapters in This Book

The examples of conceptualizing and quantifying force and kinetic energy tie together themes developed in several chapters of this volume: Brahmia and Oshlo's focus on quantification as a central aspect of scientific reasoning, Johnson's focus on mathematizing *a la* Freudenthal via an intellectual need for relationships, Jin et al.'s focus on mathematizing as a bridge between mathematics and science, Paoletti et al.'s and Ellis et al.'s focus on variation and covariation as foundational ways of thinking for students to develop understandings of functions, Moore et al.'s focus on abstracted quantitative structures as a target for students' quantitative reasoning, Gonzalez' proposal of quantitative reasoning and quantification as a central theme in climate science.

Moreover, if we consider these quantifications of force and kinetic energy as conceptual analyses of understandings we hope students construct—as a teacher's key developmental understandings of force and kinetic energy—then Carlson et al.'s analysis comes into play. As they explain, teachers must reflect upon their own quantitative understandings to become conscious of the intricacies entailed in their goals of instruction and must decenter to consider how one might support students in developing these understandings via conventions of speaking with meaning and emergent symbolization.

The example of work as a quantity relates to Moore et al.'s construct of abstract quantitative structure in a profound way. Understanding work dimensionally, as $[F][d]$, is to understand the quantitative structure of work and to understand that units will be involved, but the exact units need not be specified—they just need to be coherent with the quantities of force and distance. The example of kinetic energy also is related to Karagöz Akar, Watanabe and Turan's use of quantitative reasoning as a lens to examine mathematics textbooks' coherence. Does a textbook support teachers to engage students in reflective discourse aimed at their conceptualization of quantities, their quantification, and situations involving them that textbook authors purport to address?

The representation of kinetic energy as a function of velocity, $E(v) = \frac{1}{2}mv^2$, relates to Johnson's stance regarding intellectual need for relationships, Ellis et al.'s conceptual analysis of functions, and Paoletti's analysis of covariational reasoning. For a student (or instructor) to even consider writing "$E(v)$" requires they (1) seek a relationship between velocity and kinetic energy that remains invariant as velocity varies, (2) envision velocity varying smoothly from 0 to v regardless of the amount of time this acceleration takes, and (3) understand the notation "$E(v)$" through a scheme that entails an image of velocity and kinetic energy varying simultaneously and varying in a way that each value of velocity determines a value of kinetic energy (see Yoon & Thompson, 2020).

I can imagine mathematics educators questioning the examples of quantifying force and kinetic energy as being largely relevant to science education and less relevant to mathematics education. I disagree. Anyone who has taught arithmetic, algebra, precalculus or calculus in the United States has seen their students arrive at solutions to applied problems with little meaning or inappropriate meanings for numbers or variables in their answer. This is a serious problem. The solution to the problem of meaning, however, must be systemic. To take quantitative reasoning seriously in mathematics and science education requires attention to having students conceptualize quantities *and methods and meanings of their measures* throughout their schooling. This can range from asking students what quantity their arithmetic has evaluated, to asking them what an appropriate unit for the area of a rectangle of height 3 jibs and width of 4 jabs would be, to how one might convert measures of fuel efficiency from miles per gallon to kilometers per liter, to asking them for a useful unit of effort to complete a job (e.g., person-hour), and so on.

Moore et al.'s construct of *abstract quantitative structure* might be behind experts' utterances like "speed times time equals distance". They of course do not mean speed in *any* unit times time in *any* unit equals distance in *any* unit. Rather, they presume, without saying, this is true for a coherent system of units for speed, time, and distance. This brings to mind Carlson et al.'s explanation of the necessity for instructors to examine their own understandings and presumptions in order to consider how their expressions of them might be interpreted by students who will interpret teacher's utterances and actions through schemes quite unlike the teacher's.

5 Conclusion

I once again praise the authors' work expressed in this volume and my colleagues' who brought this collective work to our attention. I hope my call to give greater attention to the details of students' and teachers' conceptualizations of object, attribute, and measure is useful for those employing quantitative reasoning as a lens in mathematics and science education. I suspect doing this will give greater insight into difficulties students experience in learning mathematics and science and difficulties teachers experience in promoting such learning.

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An Intellectual Need for Relationships: Engendering Students' Quantitative and Covariational Reasoning

Heather Lynn Johnson

People encounter situations involving change and variation as citizens of the world. For instance, sea levels are rising as the oceans continue to absorb heat from the atmosphere. One may read about this phenomenon in newspaper articles or encounter graphs representing rising sea levels over time. By engaging in quantitative and covariational reasoning (Carlson et al., 2002; Thompson, 1994, 2011; Thompson & Carlson, 2017), people can interpret and make meaning of such situations (e.g., González, 2021). Not only are these forms of mathematical reasoning productive for being informed citizens, but they also underlie key mathematical concepts such as rate and function (Thompson & Carlson, 2017). Hence, it is crucial for students to develop and engage in such reasoning, and for opportunities to occur throughout their schooling, across K-12 and university mathematics courses. Yet, from a student's point of view, what may serve as a catalyst, so students can actualize potential opportunities? Drawing on Harel's construct of "intellectual need" (1998, 2008b, 2013), I offer an intellectual need for relationships, which is a need to explain how elements work together, as in a system. I argue that this need can engender students' quantitative and covariational reasoning.

To illustrate, consider a situation involving Sam, who is walking from home to the corner store. There are a number of attributes that students may separate from the situation; two include Sam's distance from home and Sam's distance from the store. Engaging in quantitative reasoning (Thompson, 1994, 2011), a student can conceive of the possibility of measuring those attributes, even if they do not find particular amounts of measure. For instance, a student may have a sense of a length of a stretchable cord extending from Sam's current location to home or the store. As Sam is walking, each distance changes, increasing or decreasing depending on Sam's route. Engaging in covariational reasoning (Carlson et al., 2002; Thompson & Carlson,

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2017), a student can conceive of relationships between the changing distances. For instance, with a direct route, Sam's distance from home increases while the distance from the store decreases. By forming and interpreting relationships between attributes, students can mathematize (Freudenthal, 1973) such situations in terms of quantities and covariation.

Results of researchers' investigations of students' quantitative and covariational reasoning represent both challenge and promise. Even accomplished university students have demonstrated difficulty (e.g., Carlson et al., 2002; Moore et al., 2019a, 2019b), while middle and secondary students have shown promising evidence (e.g., Ellis et al., 2020; Johnson, 2012). I argue that students' intellectual need for such reasoning may account, in part, for differences in these findings. For example, consider a task in which students are to sketch a Cartesian graph relating Sam's distance from home and Sam's distance from the store. Some students may find such a task problematic; they may wonder how to measure and relate the different distances as they sketch their graph. In contrast, other students may think the task is an exercise in finding a resulting graph that is an instance of some familiar graph. If students are focused on getting end results, they may miss opportunities to engage in quantitative and covariational reasoning.

Harel (1998, 2008b, 2013) put forth the construct of intellectual need, rooted in Piaget's constructivist theory. To illustrate, say a student encounters a situation that is problematic for them, and as a result of engaging with that situation, they develop some new mathematical knowledge. The "problematic-ness" of that situation, from the student's point of view, is the student's intellectual need. For example, one student may intend for Sam's graph to represent a relationship between distances. Another student may intend to represent Sam's physical motion on the walk. While both students find the situation problematic, the first student's goal is more compatible with quantitative and covariational reasoning.

Harel (2008a) has posited two different forms of mathematical knowledge that can emerge from students' intellectual needs: ways of understanding (products of mental action) and ways of thinking (characteristics of mental action). For example, a conception of function can be a product of mental action, and a correspondence approach can be a characteristic of mental action. Through broad categories, Harel has illuminated three ways of thinking (2008a) and five forms of intellectual need (2013), leaving room for the possibility that more categories can emerge. I argue for an expansion of the ways of thinking and forms of intellectual need put forward by Harel.

I organize this chapter into six sections. First, I discuss theoretical underpinnings of quantitative and covariational reasoning. Second, I offer Freudenthal's term, "mathematizing" (Freudenthal, 1973), to represent an additional category of a way of thinking that can emerge from students' intellectual need. Third, I explain what I mean by an intellectual need for relationships, and how that need may engender students' quantitative and covariational reasoning. Fourth, I put forward four facets of such a need. Fifth, I address task design considerations for each facet, using a digital Ferris wheel task to illustrate. Sixth, I discuss implications for theory and practice.

1 Theorizing Quantitative and Covariational Reasoning

Thompson rooted the theory of quantitative reasoning (1994, 2011) in Piaget's constructivist theory, which assumes that individuals develop new understandings by reorganizing their existing conceptions. From this lens, the distances I identified in the situation of Sam walking from home to the store would not be "out there" for a student to observe. Rather, they would be a person's conception of the situation. In the theory of quantitative reasoning, Thompson explains how individuals may conceive of situations in terms of attributes that are possible to measure, such as the distances in Sam's situation. Engaging in quantitative reasoning involves conceptions of quantities, a quantification process, and quantitative operations. A student's quantitative reasoning can entail some or all of these elements.

Quantities are a foundational element of the theory. Per Thompson (1994), a quantity is an individual's conception of an attribute in a situation as being possible to measure. This means that quantities are human creations; through their conceptions, individuals transform attributes into quantities. For example, in Sam's situation, a student can transform attributes into quantities by separating those attributes (e.g., distance) from the physical motion described in the situation (e.g., Sam's walking). Essential to Thompson's theory is a distinction between conceiving of the possibility of measurement and the act of determining particular amounts of measure. This means that students can think of measuring Sam's distance from the store without finding certain amounts of distance.

With quantification, Thompson (2011) explained a three-part process by which an individual can formalize this "possible to measure-ness." First, they would conceive of an attribute that could be measured, such as Sam's distance. Second, they would conceive of a unit of measure for the attribute. This might be a standard unit, such as a meter or foot, or a nonstandard unit, such as one of Sam's steps. Third, they would conceive of a proportional relationship between the unit and the attribute's measure. That is, they could iterate one of the units, such as a step length, to measure Sam's distance from the store. As with quantity, an essential aspect of quantification was that an individual did not need to actually measure Sam's distance from the store with the indicated unit, just think of the possibility of doing so.

Thompson (1994, 2011) put forward quantitative operations to describe mental activity in which individuals could employ a quantitative lens on situations and conceive of new kinds of quantities. Thompson identified a "difference" as one such quantity that students could create via additive comparison. For example, at any instant in Sam's walk, a student might compare Sam's distance to the store and Sam's distance from home to create a new quantity, the difference between the distances. As with quantity and quantification, students could engage in quantitative operations without determining particular amounts of difference.

With Fig. 1, I express interconnections between quantity, quantification, and quantitative operations. Because both quantification and quantitative operations extend from quantities, I have placed unidirectional arrows between quantity and those elements. Conceiving of an attribute as being possible to measure is the first part