Mathematical Problems from Applied Logic II

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Mathematical Problems from Applied Logic II

Logics for the XXIst Century

Edited by

Dov M. Gabbay
King's College London
London, UK

Sergei S. Goncharov
Sobolev Institute of Mathematics
SB Russian Academy of Sciences
Novosibirsk, Russia

Michael Zakharyaschev
Birkbeck College
London, UK

Springer

Dov M. Gabbay
Department of Computer Science
King's College London
Strand, London WC2R 2LS
UK
dg@dcs.kcl.ac.uk

Sergei S. Goncharov
Sobolev Institute of Mathematics
SB Russian Academy of Sciences
Novosibirsk 630090
Russia
gonchar@math.nsc.ru

Michael Zakharyaschev
School of Computer Science and
Information Systems
Birkbeck College
Malet Street
London WC1E 7HX
UK
michael@dcs.bbk.ac.uk

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Mathematical Problems from Applied Logic I, II
Logics for the XXIst Century

Two volumes of the *International Mathematical Series* present the most important thematic topics of logic confronting us in this century, including problems arising from successful applications areas such as computer science, AI language, etc. etc.

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Editors

Dov M. Gabbay

King's College London
London, UK

Sergei S. Goncharov

Sobolev Institute of Mathematics
SB Russian Academy of Sciences
Novosibirsk State University
Novosibirsk, Russia

Michael Zakharyaschev

Birkbeck College
London, UK

Dov M. Gabbay

Augustus De Morgan Professor of Logic
Department of Computer Science
King's College London
Strand, London WC2R 2LS
UK

dg@dcs.kcl.ac.uk
www.dcs.kcl.ac.uk/staff/dg

- Author of the books ◦ *Temporal Logics, Vols. 1,2*, OUP, 1994, 2000 ◦ *Fibred Semantics*, OUP, 1998 ◦ *Elementary Logic*, Prentice-Hall, 1998, ◦ *Fibring Logics*, OUP, 1998 ◦ *Neural-Symbolic Learning Systems* (with A. Garcez and K. Broda), Springer, 2002 ◦ *Agenda Relevance* (with J. Woods), Elsevier, 2003, etc.
- Editor-in-Chief of journals ◦ *Journal of Logic and Computation* ◦ *Logic Journal of the IGPL* ◦ *Journal of Applied Logic* (with J. Siekmann and A. Jones) ◦ *Journal of Language and Computation* (with T. Fernando, U. Reyle, and R. Kempson) ◦ *Journal of Discrete Algorithms*
- Editor of journals and series ◦ *Autonomous Agents* ◦ *Studia Logica* ◦ *Journal of Applied Non-Classical Logics* ◦ *Journal of Logic, Language & Information* ◦ *F News* ◦ *Handbook of Logic in Computer Science* (with S. Abramsky and T. Maibaum), ◦ *Handbook of Logic in AI and Logic Programming* (with C. Hogger and J.A. Robinson), ◦ *Handbook of Philosophical Logic* (with F. Guenthner) ◦ *Handbook of the History of Logic* ◦ *Handbook of the Philosophy of Science* ◦ *Studies in Logic and the Foundations of Mathematics*, etc.

Scientific interests: Logic and computation, dynamics of practical reasoning, proof theory and goal-directed theorem proving, non-classical logics and non-monotonic reasoning, labelled deductive systems, fibring logics, logical modelling of natural language.

Sergei S. Goncharov

Sobolev Institute of Mathematics
SB Russian Academy of Sciences
4, Prospekt Koptyuga
Novosibirsk 630090
Russia

gonchar@math.nsc.ru
www.math.nsc.ru

- Head of Mathematical Logic Department and Laboratory of Computability and Applied Logic, Sobolev Institute of Mathematics SB Russian Academy of Sciences
- Council Member of the *Association for Symbolic Logic*
- Professor of Logic and Dean of Mathematical Department of the Novosibirsk State University
- Vice-Chairman of *Siberian Fund of Algebra and Logic*
- Author of the books ∘ *Countable Boolean Algebras and Decidability*, Consultants Bureau, New York, 1997 ∘ *Constructive Models* (with Yu.L. Ershov), Kluwer Academic/ Plenum Publishers, 2000
- Editor-in-Chief of ∘ *Bulletin of the Novosibirsk State University. Ser. Mathematics, Mechanics, and Informatics*
- Editor of journals and series ∘ *Algebra and Logic* (Associate Ed.) ∘ *Siberian Mathematical Journal* ∘ *Siberian School of Algebra and Logic* (monograph series), Kluwer Academic / Plenum Publishers ∘ *Handbook of Recursive Mathematics* (with Yu.L. Ershov, A. Nerode, J.B. Remmel, and V.W. Marek), Vols. 1,2, Elsevier, 1998 ∘ *Computability and Models (Perspectives East and West)* (with S.B. Cooper), Kluwer Academic/ Plenum Publishers, 2003

Scientific interests: Theory of computability, computable and decidable models, abstract data types, model theory and algebra, computer science and logic programming, applied logic.

Michael Zakharyaschev

School of Computer Science
and Information Systems
Birkbeck College
Malet Street
London WC1E 7HX
UK
michael@dcs.bbk.ac.uk
www.dcs.bbk.ac.uk/~michael

- Professor of Computer Science, Birkbeck University of London
- Logic coordinator of *Group of Logic, Language and Computation*
  www.dcs.kcl.ac.uk/research/groups/gllc
- Member of the Steering Committee of *Advances in Modal Logic* (a bi-annual workshop and book series in Modal Logic), www.aiml.net
- Author of the books ◦ *Modal Logic* (with A. Chagrov), Oxford Logic Guides: 35, Clarendon Press, Oxford, 1997 ◦ *Many-Dimensional Modal Logics: Theory and Applications*, (with D. Gabbay, A. Kurucz, and F. Wolter), Series in Logic and the Foundation of Mathematics, 148, Elsevier, 2003
- Editor of journals and series ◦ *Studia Logica* (Associate Ed.) ◦ *Journal of Applied Logic* ◦ *Journal of Logic and Computation* ◦ *Advances in Modal Logic*

Scientific interests: Knowledge representation and reasoning, modal and temporal logics, description logics, spatial and temporal reasoning, automated theorem proving, intuitionistic and intermediate logics.

Authors of Volume I†

Franz Baader

Technische Universität Dresden
Dresden, Germany

Lev Beklemishev

Steklov Mathematical Institute RAS
Moscow, Russia

Universiteit Utrecht
Utrecht, The Netherlands

Johan van Benthem

University of Amsterdam
Amsterdam, The Netherlands

Stanford University
Stanford, USA

S. Barry Cooper

University of Leeds
Leeds, UK

John N. Crossley

Monash University
Melbourne, Australia

† See www.springer.com ISBN 0-387-28688-8, ISSN 1571-5485.

Wilfrid A. Hodges

Queen Mary University of London
London, UK

Ralf Küsters

Institut für Informatik
und Praktische Mathematik
Christian-Albrechts-Universität zu Kiel
Kiel, Germany

Lawrence S. Moss

Indiana University
Bloomington, USA

Albert Visser

Universiteit Utrecht
Utrecht, The Netherlands

Content of Volume I†

† See www.springer.com ISBN 0-387-28688-8, ISSN 1571-5485.

Authors of Volume II

Sergei Artemov

The City University of New York
New York, USA

Moscow State University
Moscow, Russia

John Case

University of Delaware
Newark, USA

Sergei S. Goncharov

Sobolev Institute of Mathematics
SB Russian Academy of Sciences
Novosibirsk State University
Novosibirsk, Russia

Judit X. Madarász

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary

István Németi

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary

Anil Nerode

Cornell University
Ithaca, USA

Gergely Székely

Eötvös Loránd University Budapest

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary

Dimiter Vakarelov

Sofia University
Sofia, Bulgaria

Sergei Artemov

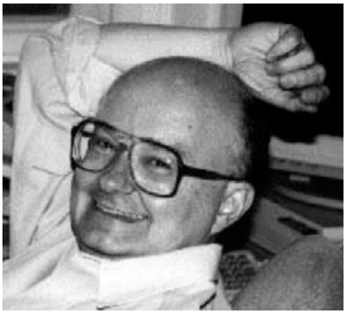
The Graduate Center of
the City University of New York
365 Fifth Avenue, New York, NY 10016
USA

Moscow State University
Vorob'evy Gory, 119899 Moscow
Russia
sartemov@gc.cuny.edu
http://web.cs.gc.cuny.edu/~sartemov/

- Distinguished Professor, the Head of the Research Laboratory for Logic and Computation, CUNY
- Editor of Annals of Pure and Applied Logic, Moscow Mathematical Journal, the monograph series *Studied in Logic and Foundations of Mathematics*

Scientific interests: Logic in computer science, mathematical logic, proof theory, knowledge representation, artificial intelligence, automated deduction,verification, optimal control, hybrid systems.

John Case



Department of Computer
and Information Sciences
University of Delaware
Newark, DE 19716
USA

case@cis.udel.edu
www.cis.udel.edu/∼case

- Professor and the former Chair of current department and former Associate Dean, Faculty of Natural Science and Mathematics, University of Buffalo (USA)
- Author (with James Royer) of *Subrecursive Programming Systems: Complexity and Succinctness*, Birkhäuser, 1994
- A Founding Editor *J. Universal Computer Science*

Scientific interests: computational learning theory, machine learning, functional and comparative genomics, lattice-connected computer models of parallel processing.

Judit X. Madarász

Department of Algebraic Logic
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, H-1053 Hungary

madarasz@renyi.hu
www.renyi.hu/~madarasz

- Author (with H. Andréka and I. Németi) of the Internet book *On the Logical Structure of Relativity Theories* [http://www.math-inst.hu/pub/algebraic-logic/olsort.html]

Scientific interests: Logic of spacetime, Gödel's rotating universe, geometry, algebraic logic.

István Németi

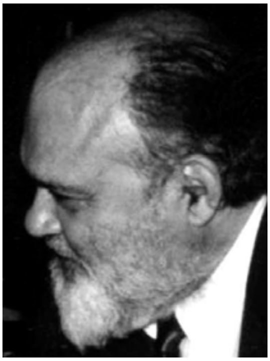
Department of Algebraic Logic
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, H-1053 Hungary

nemeti@renyi.hu
www.logicart.hu/index_hu.html
www.renyi.hu/~nemeti

- Author of the books ∘ *Cylindric Set Algebras*, Lect. Notes Math. **881**, Springer, 1981 (with L. Henkin, J. D. Monk, A. Tarski, and H. Andréka) ∘ *Decision Problems for Equational Theories of Relation Algebras*, Mem. Am. Math. Soc., 1997 (with H. Andréka and S. R. Givant) ∘ *Generalization of the Concept of Variety and Quasi-Variety to Partial Algebras through Category Theory*, Warsaw, 1983 (with H. Andréka)
- Editor of journals and series ∘ *Journal of Applied Nonclassical Logic* ∘ *Journal of Applied Logic* ∘ *Advanced Studies in Mathematics and Logic book series* ∘ *Logica Universalis*, etc.

Scientific interests: Logic as a unifying foundation of all branches of science, relativity theory, computer science and AI, universal algebra, model theory.

Anil Nerode



Goldwin Smith Professor of Mathematics
Department of Mathematics
Cornell University
545 Malott Hall
Ithaca, NY 14853-4201
USA
anil@math.cornell.edu
www.math.cornell.edu/~anil

- Chairman (1982-1987) of Department of Mathematics and Director (1987-1996) of Mathematical Sciences Institute, Cornell University • Vice President (1991-1994), American Mathematical Society • Chair, advisor, member of centers on mathematics, information technologies, engineering, etc. of numerous organizations (NAS, NSF, AMS, MAA, IBM, etc.)
- Author of the books ∘ *Automata Theory and Its Applications* (with B. Khoussainov), Birkhauser, 2001 ∘ *Logic for Applications* (with R. A. Shore), Springer, 1993, 1997 ∘ *Principles of Logic Programming. With the Cooperation of A. Sinachopoulos* (with G. Metakides), North-Holland, 1996, etc.
- Editor Emeritus ∘ *Annals of Pure and Applied Logic* • Associate Editor ∘ *Annals of Mathematics and Artificial Intelligence* • Editor of journals ∘ *Computer Modelling and Simulation* ∘ *Mathematics and Computer Modelling* ∘ *Documenta Mathematica* ∘ *Grammars* • Editor of *Handbook of Recursive Mathematics. Vols. 1, 2* (with Yu. L. Ershov, S.S. Goncharov, J.B. Remmel, and V.W. Marek) North-Holland, 1998 • Advisory Editorial Board ∘ *Discrete Mathematics and Theoretical Computer Science Series* • Editor of journals and series of past years ∘ *Proceedings of the American Mathematical Society* (1962-1965) ∘ *Advances in Mathematics* (1968-1971) ∘ *Journal of Symbolic Logic* (1968-1983) ∘ *Annals of Pure and Applied Logic* (1983-1997) ∘ *Future Generation Computing* (1984-1998) ∘ *ORCA Journal of Computing* (1988-1990), ∘ *Constraints* (1995-1999), etc.

Scientific interests: Mathematical logic, computability, complexity, automata, recursive algebra, recursive analysis, multiple agent distributed control, relaxed calculus of variations on manifolds, hybrid systems, nonmonotonic logics, modal logics for computer science, constructive logics for computer science, distributed systems, history of mathematics.

Gergely Székely

Eötvös Loránd University Budapest

Department of Algebraic Logic
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, H-1053 Hungary

turms@renyi.hu
http://www.renyi.hu/~turms

- PhD student

Scientific interests: Logic of spacetime, conceptual analysis of spacetime and relativity theories.

Dimiter Vakarelov

Department
of Mathematical Logic
and Its Applications
Sofia University
15 Tsar Osvoboditel Blvd.
1504 Sofia, Bulgaria

dvak@fmi.uni-sofia.bg

- Professor of Mathematics, Faculty of Mathematics and Computer Science, Sofia University
- Editor of ∘ *Journal of Applied Non-Classical Logic*

Scientific interests: Applied non-classical logic, modal logic, region-based theory of space.

Content of Volume II

John Case

Sergei S. Goncharov

Judit X. Madarász,
István Németi,
and Gergely Székely

Anil Nerode

Dimiter Vakarelov

On Two Models of Provability

Sergei Artemov

CUNY Graduate Center
New York, USA

Gödel's modal logic approach to analyzing provability attracted a great deal of attention and eventually led to two distinct mathematical models. The first is the modal logic GL, also known as the Provability Logic, which was shown in 1979 by Solovay to be the logic of the formal provability predicate. The second is Gödel's original modal logic of provability S4, together with its explicit counterpart, the Logic of Proofs LP, which was shown in 1995 by Artemov to provide an exact provability semantics for S4. These two models complement each other and cover a wide range of applications, from traditional proof theory to λ-calculi and formal epistemology.

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1. Introduction

In his 1933 paper [**79**], Gödel chose the language of propositional modal logic to describe the basic logical laws of provability. According to his approach, the classical logic is augmented by a new unary logical connective (modality) '$\Box$' where $\Box F$ should be interpreted as

F is provable.

> Gödel's treatment of provability as modality in [**79**] has an interesting prehistory. In his letter to Gödel [**185**] of January 12, 1931, John von Neumann actually used formal provability as a modal-like operator B and gave a shorter, modal-style derivation of Gödel's second incompleteness theorem. Von Neumann freely used such modal logic features as the transitivity axiom $B(a) \to B(B(a))$, equivalent substitution, and the fact that the modality commutes with the conjunction '$\wedge$.'

Gödel's goal was to provide an exact interpretation of intuitionistic propositional logic within a classical logic with the provability operator, hence giving classical meaning to the basic intuitionistic logical system.

According to Brouwer, the founder of intuitionism, truth in intuitionistic mathematics means the existence of a proof. An axiom system for intuitionistic logic was suggested by Heyting in 1930; its full description may be found in the fundamental monographs [**93, 106, 171**]. By IPC, we infer Heyting's intuitionistic propositional calculus. In 1931–34, Heyting and Kolmogorov gave an informal description of the intended proof-based semantics for intuitionistic logic [**91, 92, 93, 107**], which is now referred to as the *Brouwer-Heyting-Kolmogorov* (*BHK*) *semantics.* According to the *BHK*-conditions, a formula is 'true' if it has a proof. Furthermore, a proof of a compound statement is connected to proofs of its parts in the following way:

- a proof of $A \wedge B$ consists of a proof of proposition A and a proof of proposition B,

- a proof of $A \vee B$ is given by presenting either a proof of A or a proof of B,
- a proof of $A \rightarrow B$ is a construction transforming proofs of A into proofs of B,
- falsehood $\perp$ is a proposition which has no proof; $\neg A$ is shorthand for $A \rightarrow \perp$.

From a foundational point of view, it did not make much sense to understand the above 'proofs' as proofs in an intuitionistic system, which those conditions were supposed to specify. So in 1933 ([**79**]), Gödel took the first step towards developing an exact semantics for intuitionism based on **classical provability**. Gödel considered the classical modal logic S4 to be a calculus describing properties of provability in classical mathematics:

(i) *Axioms and rules of classical propositional logic*,

(ii) $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$,

(iii) $\Box F \rightarrow F$,

(iv) $\Box F \rightarrow \Box\Box F$,

(v) *Rule of necessitation*: $\dfrac{\vdash F}{\vdash \Box F}$.

Based on Brouwer's understanding of logical truth as provability, Gödel defined a translation $tr(F)$ of the propositional formula F in the intuitionistic language into the language of classical modal logic, i.e., $tr(F)$ was obtained by prefixing every subformula of F with the provability modality $\Box$. Informally speaking, when the usual procedure of determining classical truth of a formula is applied to $tr(F)$, it will test the provability (not the truth) of each of F's subformulas in agreement with Brouwer's ideas.

> Even earlier, in 1928, Orlov published the paper [**147**] in Russian, in which he considered an informal modal-like operator of provability, introduced modal postulates (ii)–(v), and described the translation $tr(F)$ from propositional formulas to modal formulas. On the other hand, Orlov chose to base his modal system on a type of relevance logic; his system fell short of S4.

From Gödel's results in [**79**], and the McKinsey-Tarski work on topological semantics for modal logic [**130**], it follows that the translation $tr(F)$ provides a proper embedding of the intuitionistic logic IPC into $\mathsf{S4}$, i.e., an embedding of IPC into classical logic extended by the provability operator.

Theorem 1.1 (Gödel, McKinsey, Tarski). IPC *proves* $F \Leftrightarrow$ $\mathsf{S4}$ *proves* $tr(F)$.

Still, Gödel's original goal of defining IPC in terms of classical provability was not reached, since the connection of $\mathsf{S4}$ to the usual mathematical notion of provability was not established. Moreover, Gödel noticed that the straightforward idea of interpreting modality $\Box F$ as *F is provable in a given formal system T* contradicted Gödel's second incompleteness theorem (cf. [**48, 51, 70, 89, 165**] for basic information concerning proof and provability predicates, as well as Gödel's incompleteness theorems).

> Indeed, $\Box(\Box F \rightarrow F)$ can be derived in $\mathsf{S4}$ by the rule of necessitation from the axiom $\Box F \rightarrow F$. On the other hand, interpreting modality $\Box$ as the predicate $\mathsf{Provable}_T(\cdot)$ of formal provability in theory T and F as contradiction, i.e., $0 = 1$, converts this formula into the false statement that the consistency of T is internally provable in T:
>
> $$\mathsf{Provable}_T\left(\ulcorner Consis(T) \urcorner\right) .$$
>
> To see this, it suffices to notice that the following formulas are provably equivalent in T:
>
> $$\mathsf{Provable}_T(\ulcorner 0=1 \urcorner) \rightarrow (0=1) ,$$
> $$\neg \mathsf{Provable}_T(\ulcorner 0=1 \urcorner) ,$$
> $$Consis(T) .$$
>
> Here $\ulcorner \varphi \urcorner$ stands for the Gödel number of φ. Below we will omit Gödel number notation whenever it is safe, for example, we will write $\mathsf{Provable}(\varphi)$ and $\mathsf{Proof}(t, \varphi)$ instead of $\mathsf{Provable}(\ulcorner \varphi \urcorner)$ and $\mathsf{Proof}(t, \ulcorner \varphi \urcorner)$.

The situation after Gödel's paper [**79**] can be described by the following figure where '$\hookrightarrow$' denotes a proper embedding:

$$\mathsf{IPC} \hookrightarrow \mathsf{S4} \hookrightarrow ? \hookrightarrow \textit{CLASSICAL PROOFS}\,.$$

In a public lecture in Vienna in 1938 [**80**], Gödel suggested using the format of explicit proofs *t is a proof of F* for interpreting his provability calculus S4, though he did not give a complete set of principles of the resulting logic of proofs. Unfortunately, Gödel's work [**80**] remained unpublished until 1995, when the Gödelian logic of proofs had already been axiomatized and supplied with completeness theorems connecting it to both S4 and classical proofs.

The provability semantics of S4 was discussed in [**48, 51, 56, 81, 108, 117, 121, 133, 138, 140, 141, 145, 157, 158**] and other papers and books. These works constitute a remarkable contribution to this area, however, they neither found the Gödelian logic of proofs nor provided S4 with a provability interpretation capable of modeling the *BHK* semantics for intuitionistic logic. Comprehensive surveys of work on provability semantics for S4 may be found in [**12, 17, 21**].

The Logic of Proofs LP was first reported in 1994 at a seminar in Amsterdam and at a conference in Münster. Complete proofs of the main theorems of the realizability of S4 in LP, and about the completeness of LP with respect to the standard provability semantics, were published in the technical report [**10**] in 1995. The foundational picture now is

$$\mathsf{IPC} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP} \hookrightarrow \textit{CLASSICAL PROOFS}\,.$$

The correspondence between intuitionistic and modal logics induced by Gödel's translation $tr(F)$ has been studied by Blok, Dummett, Esakia, Flagg, Friedman, Grzegorczyk, Kuznetsov, Lemmon, Maksimova, McKinsey, Muravitsky, Rybakov, Shavrukov, Tarski, and many others. A detailed survey of modal companions of intermediate (or superintuitionistic) logics is given in [**60**]; a brief one is in [**61**], Sections 9.6 and 9.8.