Lecture Notes in Electrical Engineering 897

Santanu Saha Ray
H. Jafari
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Suchandan Kayal *Editors*

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Preface

This edited book, "**Lecture Notes in Electrical Engineering**", is an outcome of the **International Conference on Applied Analysis, Computation and Mathematical Modelling in Engineering (AACMME-2021)**. The contents of this book are intended to present an overall idea about the recent advances in latest developments and researches in the field of Mathematical Science and its applications.

This book focuses on the comparative study of some wavelet based numerical methods to solve initial value problems. It also investigates the enhancement of natural convection heat transfer using hybrid nanofluids over a moving vehicle plate. This book addresses the provoked flow pattern due to an impulsive motion of porous wavy wall with no slip suction velocity under the influence of magnetic field.

In this book, a linear stability analysis is applied to study the onset of bio-convection in a suspension of negatively geotactic swimmers saturated with a non-Darcy porous fluid layer under the effect of high frequency and Small-amplitude vertical vibrations. This book also studies the impact of two temperatures on a generalized thermoelastic plate with thermal loading.

Baffle spacing has a decisive effect on heat transfer and pumping power. The development of baffle spacing significantly dominates the turbulence created inside the shell and tube heat exchanger and heat transfer. This book studies the impact of baffle spacing in both global and local thermohydraulic characteristics.

In this book, Kudryashov and modified Kudryashov techniques have been implemented to acquire new exact solutions of the time fractional (2+1)-dimensional CBS equation. This book also explores the impact of double dispersion effects on the nonlinear convective flow of power-law fluid along an inclined plate.

This book emphasizes the Soret and viscous dissipation effects on mixed convective flow of an incompressible micropolar fluid over a vertical frustum of a cone embedded in a non-Darcy porous medium subject to convective boundary condition. It proposes convergence and comparison theorems for three-step alternating iteration method for rectangular linear system. This book also studied thermal hydraulic performance of helical baffle shell and tube heat exchanger using RSM Method.

This book investigates a newly proposed dual-mode Kawahara equation. It finds out the soliton and periodic solutions of the Kawahara equation. In this book, the

Lie transformation method has been used to find out the group invariant solutions of (2+1)-dimensional modified Calogero-Bogoyavlenskii-Schiff (mCBS) equation.

This book addresses the estimation and classification of two logistic distributions with a common scale and different location parameters. Bayes estimates are computed using Metropolis-Hastings method using gamma and normal prior distributions. The Bayes estimates are compared with some of the existing estimates with respect to the bias and mean squared error. Utilizing these estimates some classification rules are proposed to classify a single observation into one of the two logistic populations under the same model.

The book considers the problem of testing of hypothesis for the quantile when independent random samples are drawn from two normal populations with a common mean and order restricted variances. Several test procedures are proposed and are evaluated through their sizes and powers using a simulation procedure.

In this book, various geometrical parameters of the planted roof are studied to optimize the dimensional parameters by means of independent and dependent variables using an exact mathematical model. Using experiment, the factors influencing the performance of the planted roof activity are identified to optimize the performance of the heat flow through planted roof.

This book deals with the modal analysis of a Jeffcott functionally graded (FG) rotor system, consisting of an FG shaft mounted on linear bearings at the ends. The material gradation is applied following the exponential gradation law, whereas the thermal gradients across the radius of the FG shaft are achieved through the exponential temperature distribution method 3D finite element modelling and the modal analysis of the FG rotor system are carried out using ANSYS software. The influence of the material gradation and temperature gradients on the rotor-bearing system's natural and whirl frequencies are studied.

This book presents five-point finite difference method to solve the two-dimensional Laplace and Poisson equations on regular and irregular regions. Dirichlet and Robin boundary conditions are considered for solving the system of equations in each iteration. The obtained numerical results are compared with analytical solutions.

This book also focuses on the selection of the best ultra-sound machine using ELECTRE method based on the user's criteria. This study considers six criteria to select best one from five alternatives.

This book examines the processes included for initiation along with expansion of a crack on the web of the rail weldment in order to anticipate the direction of fracture crack and secondary, the intervals of weld inspections. The finite element study for the expected cracking is performed to measure the brief history of stress intensity factors. Computational simulations and experimental findings made by RDSO on three-dimensional growth of fatigue crack are compared.

This book deals with a higher-order wave equation with delay term and variable exponents. Under suitable conditions, they prove the nonexistence of solutions in a finite time. There is no research related to higher-order wave equations with delay term and variable exponents.

In this book, the existence result of a solution to continuous nonlinear, initial value problem is studied. A special type of problem representing the time evolution of particle number density due to the coagulation, multi-fragmentation events among the particles present in a system has been considered. The proof of the main theorem is based on the contraction mapping principle. Initially the local existence of nonnegative solutions for these compactly supported kernels has been also proved in this book. The study is completed by examining the mass conservation law of the existing solution.

This book also introduces a new sequence of Szasz—Kantorovich type operators based on Boas - Buck type polynomials which include Brenke type polynomials, Sheffer polynomials and Appell polynomials. The error is estimated in the approximation by these operators in terms of the Lipschitz type maximal function, Peetre's K-functional and Ditzian–Totik modulus of smoothness. The order of convergence is also studied of these operators for unbounded functions by using the weighted modulus of continuity. This study also covers quantitative-Voronovoskaya-type theorem and Gruss Voronovskaya-type theorem.

A study on the numerical modeling and simulation of heat distribution inside the skin tissue for cancer treatment with external exponential heating is also presented in this book. The two-dimensional Pennes bio-heat model for thermal therapy based on Fourier's law of heat conduction is considered in this study. The mathematical model's numerical solution is obtained using Crank Nicolson finite difference approximation and radial basis function approximation for time and space. The effects of thermophysical properties of the skin on the temperature profile in the tissue are also explained.

Overall, the chapters create new avenues and present intriguing information to comprehend the difficulties and provide answers for various challenges, which would assist readers grasp and implement for the new development and mathematically analyse physical problems.

The editors would like to express their appreciation to Springer, the Springer Editor, for publishing these chapters in "Lecture Notes in Electrical Engineering." We are also grateful to the anonymous reviewers who provided worthwhile review reports that resulted in significant modifications and enhancements to these chapters.

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Pretoria, South Africa — H. Jafari
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Comparative Study of Some Wavelet-Based Numerical Methods to Solve Initial Value Problems

Kshama Sagar Sahu and Mahendra Kumar Jena

Abstract Ordinary differential equations, in particular initial value problems, play a vital role in applied mathematics. There are many methods available to solve these initial value problems. The operational matrix method based on wavelet is a recent one. In this paper, we briefly review some operational matrix methods. The operational matrix method from the Haar wavelet, the frame, and the Legendre wavelet is considered. We give a comparison of the solution by providing several numerical examples.

Keywords Frame · Haar wavelet · Legendre wavelet · Operational matrices

1 Introduction

Many mathematical models in real-life problems involve ordinary differential equations (ODEs). The solution of these ODE plays a vital role in solving real-life problems. Sometimes, it is not easy to find an analytical solution to the ODE. In such cases, we depend upon the numerical solution. Several methods exist to solve ODE numerically, but the operational matrix method based on wavelet is a recent trend. Many researchers use the operational matrix in solving differential equations. The method using an operational matrix to solve ODE is called as operational matrix method. In this method, the given ODE is converted to an algebraic equation. Solving the algebraic equation, we get the solution of the given ODE.

Wavelet theory is a wide field in science and engineering. It constitutes a family of functions constructed from dilation and translation of a single function called the mother wavelet [14]. Wavelets are used in signal processing, image compression, and many more. In 1997, Chen and Hsiao [2] introduce the operational matrix of

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integration from Haar wavelet. Many researchers used this operational matrix to find the solution of differential equations numerically. This method got the popularity as it is simple and easy. Later, so many operational matrices based on wavelets have been introduced. Now, this operational matrix method is not limited to solve ODE only. It is widely used to solve fractional differential equations [8, 16], partial differential equations [18], integral equations [1], integro-differential equations [17].

Some well-known operational matrix methods are the Haar wavelet operational matrix (HWOM) method, Legendre's wavelet operational matrix (LWOM) method, and the frame operational matrix (FOM) method. All these methods have been derived using an operational matrix of integration.

Frames first appear in 1952 [3, 7]. These are considered some kinds of alternatives to wavelets. They are more useful when compactly supported and obtained from a single prototype function by dilation and translation. Like wavelets, a function in $L^2(\mathbb{R})$ can also be expressed as a linear combination of frame elements [3]. In this paper, we consider the frame constructed from the linear cardinal B-spline. First, we find out the operational matrices, and then with the help of these operational matrices, we find the approximate solutions of initial value problems (IVPs).

The remaining part of the paper is organized as follows. In Sect. 2, we review the Haar wavelet operational matrix method. In Sect. 3, we present the frame operational matrix method. Legendre wavelet operational matrix method is outlined in Sect. 4. Some numerical examples are given in Sect. 5. A conclusion is given in Sect. 6.

2 Haar Wavelet Operational Matrix Method

In this section, we first find out the operational matrices from the Haar wavelet for the different resolutions. These operational matrices are then used to solve the IVPs. The given IVP is transferred to an algebraic equation which involves the operational matrices. The algebraic equation is then solved, and as a result, we get an approximate solution of the IVP.

Definition 1 (*Haar Wavelet*) Let $m = 2^j$, $j = 0, 1, \ldots J$, $k = 0, 1, \ldots m - 1$, and $i = m + k + 1$. Here, i and j denote wavelet number and the level of wavelet respectively, whereas k is the translation parameter. The maximum level of resolution is J. The minimum value of $i = 2$, and the maximum value is $2M$. The Haar wavelet family for $t \in [A, B]$ is given as [10, 11]

$$h_i(t) = \begin{cases} 1 & t \in [\xi_1(i), \xi_2(i)] \\ -1 & t \in [\xi_2(i), \xi_3(i)] \\ 0 & \text{otherwise} \end{cases},$$

where

$$\xi_1(i) = A + 2k\mu\Delta x,$$
$$\xi_2(i) = A + (2k+1)\mu\Delta x,$$
$$\xi_3(i) = A + 2(k+1)\mu\Delta x,$$

$$\mu = \frac{M}{m}, \quad \text{and} \quad \Delta x = \frac{(B-A)}{2M}.$$

The scaling function is $h_1(t) = 1$ for $t \in [A, B]$ and 0 elsewhere. The Haar wavelet are orthogonal to each other:

$$\int_A^B h_i(t)\,h_l(t) = 2^{-j}\delta_{ij} = \begin{cases} 2^{-j}, & i = l = 2^j + k \\ 0, & i \neq l. \end{cases} \tag{1}$$

Haar wavelets form a good basis for this orthogonal property. Any function $y(t)$ which is square-integrable in the interval $[A, B]$ can be expanded into a Haar wavelet expansion

$$y(t) = \sum_{i=1}^{2M} a_i h_i(t),$$

where $a_i = 2^j \int_A^B y(t)\,h_i(t)\,\mathrm{d}t$.

2.1 Operational Matrix of Integration

Let us define Haar wavelet matrix [2, 10, 11, 13] of order $2M \times 2M$ by

$$H_{2M\times 2M} = (h_i(t_l))_{i=1,l=1}^{2M\times 2M} = \begin{bmatrix} h_1(t_1) & h_1(t_2) & \cdots & h_1(t_{2M}) \\ h_2(t_1) & h_2(t_2) & \cdots & h_2(t_{2M}) \\ \vdots & \vdots & \vdots & \vdots \\ h_{2M}(t_1) & h_{2M}(t_2) & \cdots & h_{2M}(t_{2M}) \end{bmatrix}.$$

In general,

$$H_{2M\times 2M} = \left[h_{2M}\left(\tfrac{1}{4M}\right)\ h_{2M}\left(\tfrac{3}{4M}\right)\ \cdots\ h_{2M}\left(\tfrac{4M-1}{4M}\right) \right].$$

The operational matrices are defined as follows:

$$(PH)_{il} = \int_0^{t_l} h_i\,(t)\,dt, \tag{2}$$

$$(QH)_{il} = \int_0^{t_l} dt \int_0^{t} h_i\,(t)\,\mathrm{d}t, \tag{3}$$

where t_l are collocation points and $t_l = \frac{l-0.5}{2M}$.
H, P, and Q are matrices of order $2M \times 2M$. Taking $2M = 2$ and $2M = 4$, we have

$$H_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, (PH)_{2\times 2} = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}, P_{2\times 2} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$H_{4\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, (PH)_{4\times 4} = \frac{1}{8}\begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$P_{4\times 4} = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

Similarly,

$$(QH)_{2\times 2} = \frac{1}{32}\begin{bmatrix} 1 & 9 \\ 1 & 15 \end{bmatrix}, \quad Q_{2\times 2} = \frac{1}{32}\begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}, \text{ and}$$

$$(QH)_{4\times 4} = \frac{1}{128}\begin{bmatrix} 1 & 9 & 25 & 49 \\ 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 \\ 0 & 0 & 1 & 7 \end{bmatrix}, \quad (Q)_{4\times 4} = \frac{1}{128}\begin{bmatrix} 21 & -16 & -4 & -12 \\ 16 & -11 & -4 & -4 \\ 6 & -2 & -3 & 0 \\ 2 & -2 & 0 & -3 \end{bmatrix}.$$

Chen and Hsiao [2] have derived the following formula

$$P_{2M\times 2M} = \frac{1}{4M}\begin{pmatrix} 4M P_{M\times M} & -H_{M\times M} \\ H^{-1}_{M\times M} & O \end{pmatrix}.$$

Notation: We have used the symbols:

$$H_{2M}^{(0)} := H_{2M\times 2M},\, H_{2M}^{(1)} := (PH)_{2M\times 2M}\,,\, P_{2M}^{(1)} := H_{2M}^{(1)}\left(H_{2M}^{(0)}\right)^{-1}.$$

2.2 Method for First-Order Linear IVP

Consider the first-order linear ordinary differential equation

$$U^{'} = a\,(t)\,U + b\,(t)\,,\quad t \in [0, T]\,, U\,(0) = U_0. \tag{4}$$

Let us divide the interval $[0, T]$ into n equal subinterval such that $t_{i+1} - t_i = d_i$. Let introduce the local coordinate $\tau = \frac{t-t_i}{d_i}$ in the interval $\left[t_i, t_{i+1}\right]$. Define the collocation points in the interval [0, 1] by

$$\tau_j = \frac{\left(j - \frac{1}{2}\right)}{2M},\quad j = 1, 2, \ldots, 2M.$$

Now, define $u\,(\tau) = U\,(t)$ and the given IVP becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = d_i\,[a\,(\tau)\,u\,(\tau) + b\,(\tau)]\,,\quad u\,(0) = U_i. \tag{5}$$

Introducing the row vector of order $1 \times 2M$

$$\mathbf{u} = \left[\,u\,(\tau_1)\;u\,(\tau_2)\;\cdots\;u\,(\tau_{2M})\,\right],$$

the equation (2.5), can be written as

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau} = d_i\,[\mathbf{u}A\,(\tau) + B\,(\tau)] \tag{6}$$

where

$$A\,(\tau) = \begin{bmatrix} a\,(\tau_1 d_i + t_i) & 0 & 0 \cdots & 0 \\ 0 & a\,(\tau_2 d_i + t_i) & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \cdots & \vdots \\ 0 & 0 & 0 \cdots & a\,(\tau_{2M} d_i + t_i) \end{bmatrix},$$

and

$$B\,(\tau) = (b\,(\tau_1 d_i + t_i)\,, b\,(\tau_2 d_i + t_i)\,, \ldots, b\,(\tau_{2M} d_i + t_i))\,,$$

Following [2, 11], we take

$$\frac{\mathrm{d}\mathbf{u}}{d\tau} = cH_{2M}^{(0)}, \tag{7}$$

where $c = [c(1), c(2), \ldots c(2M)]$.
Integrating (2.7) we have

$$\mathbf{u} = cH_{2M}^{(1)} + U_i E, \tag{8}$$

where $E = [1, 1, 1, \ldots, 1]$ and $U_i = U(t_i)$.
Comparing (2.6) & (2.7) and putting the value of $\mathbf{u}$ from (2.8) to get c.
Now,

$$c = d_i U_i Y S^{-1} + d_i B A^{-1} \left(H_{2M}^{(0)}\right)^{-1} S^{-1},$$

where

$$S = \left(H_{2M}^{(0)} A^{-1} \left(H_{2M}^{(0)}\right)^{-1} - d_i P_{2M}^{(1)}\right),$$

and

$$Y = E \left(H_{2M}^{(0)}\right)^{-1}.$$

Taking all $\tau_j = 1$, the approximation is

$$U_{i+1} = c(1) + U_i.$$

2.3 Method for Second-Order Linear IVP

Let us consider the second-order linear differential equation

$$\frac{\mathrm{d}^2U}{\mathrm{d}t^2} = F\left(t, U, \frac{\mathrm{d}U}{\mathrm{d}t}\right), \quad t \in [0, T], U(t_0) = U_0, U^{'}(t_0) = V_0. \tag{9}$$

We follow [10] to find the solution of the above equation. Let $V = \frac{\mathrm{d}U}{\mathrm{d}t}$. Then the given differential equation becomes the first-order linear system as follows:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}^2U}{\mathrm{d}t^2} = F(t, U, V).$$

Here also, we divide the interval $[0, T]$ into n equal subinterval of length d_i. Let us consider the interval $\left[t_i, t_{i+1}\right]$ and define the collocation points τ_j as in previous section. In this interval, define $u(\tau) = U(t)$ and $v(\tau) = V(t)$, where τ is the local coordinate in $\left[t_i, t_{i+1}\right]$. Let $U(t_i) = U_i$ and $V(t_i) = V_i$ are known approximations. Now, the converted system of differential equations can be written as

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = d_i v \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = d_i F(t_i + \tau d_i, u, v).$$

Let us introduce the row vectors $\mathbf{u}$ and $\mathbf{v}$ as given below:

$$\mathbf{u} = \left[u(\tau_1)\ u(\tau_2)\ \cdots\ u(\tau_{2M}) \right] \text{and} \quad \mathbf{v} = \left[v(\tau_1)\ v(\tau_2)\ \cdots\ v(\tau_{2M}) \right].$$

Following Chen & Hsiao [2] and Lepik [11],

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau} = aH_{2M}^{(0)} \quad \text{and} \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau} = d_i\left(bH_{2M}^{(1)} + V_iE\right),$$

where a and b are row matrix of order $1 \times 2M$. The converted system of ODE becomes

$$aH_{2M}^{(0)} = d_i\left(bH_{2M}^{(1)} + V_iE\right), \tag{10}$$

and

$$bH_{2M}^{(0)} = d_iF\left(t_i + \tau d_i, aH_{2M}^{(1)} + U_iE, bH_{2M}^{(1)} + V_iE\right). \tag{11}$$

Solving Eqs. (10) and (11) we get a and hence b. Taking all $\tau_j = 1$, the approximations are

$$U_{i+1} = a(1) + U_i, \quad V_{i+1} = b(1) + V_i,$$

where $a(1)$ and $b(1)$ are first elements of a and b.

3 Frame Operational Matrix Method

The frame of linear cardinal B-spline is considered to construct the operational matrix. It is an operational matrix of integration.

3.1 A Short Literature Review

Recently, the frame operational matrix method has been used to solve the initial value problems [15]. This operational matrix is obtained from a frame of linear cardinal B-spline. Frames are considered as some kinds of alternatives to wavelets. They are useful when they have compact supports and are obtained from a refinable function.

Definition 2 (*Refinable Function*) [5] A function $\phi \in L^2(\mathbb{R})$ is called a refinable function if there exists scalars $p_k \in \mathbb{R}, \quad k \in Z$ such that

$$\phi(x) = \frac{1}{2}\sum_{k \in Z} p_k \phi(2x - k).$$

Definition 3 (*Multiresolution Analysis*) [5] Let $\phi \in L^2(\mathbb{R})$ is a refinable function and $V_j = closure\left\{\phi_{j,k} : k \in Z\right\}$. The collection of subspaces $\left\{V_j\right\}_{j\in Z}$ of $L^2(\mathbb{R})$ generates an multiresolution analysis (MRA) of $L^2(\mathbb{R})$ if they have the following properties:

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$
2. $span\left(\bigcup_{j\in Z} V_j\right) = L^2(\mathbb{R})$
3. $\bigcap_{j\in Z} V_j = \{0\}$
4. $V_{j+1} = V_j + W_j, \quad j \in Z$
5. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}, \quad j \in Z$

Definition 4 (*Tight Frame*) [5] A family $\Psi = \{\psi_1, \psi_2, \ldots, \psi_N\} \subset L^2(\mathbb{R})$ is called tight frame of $L^2(\mathbb{R})$ if it satisfies

$$\sum_{i=1}^{N} \sum_{j,k\in Z} \left|\left\langle f, \psi_{i;j,k}\right\rangle\right|^2 = \|f\|^2, \quad \text{all} \quad f \in L^2(\mathbb{R}),$$

where $\psi_{i;j,k} = 2^{j/2}\psi_i\left(2^j \cdot -k\right)$.

Definition 5 (*Linear Cardinal B-spline*) [5] Let us define the linear cardinal B-spline by

$$\phi(x) = \begin{cases} x, x \in [0,1] \\ 2-x, x \in [1,2] \end{cases}.$$

It is refinable with $p_0 = \frac{1}{2}$, $p_1 = 1$, $p_2 = \frac{1}{2}$ and $p_k = 0$ for $k \neq 0, 1, 2$.

Definition 6 (*MRA Tight(wavelet)Frame*) [5] A family $\Psi = \{\psi_1, \psi_2, \ldots, \psi_N\} \subset L^2(\mathbb{R})$ is called an MRA tight(wavelet) frame if it is a tight frame and is associated with a refinable function that generates an MRA and $\Psi \subset V_1$.

We now consider linear cardinal B-spline ϕ to construct an operational matrix method. Define

$$\begin{aligned} &\psi_0(x) = \phi(2x), \\ &\psi_1(x) = \frac{1}{\sqrt{2}}\left(\psi_0(2x) - \psi_0(2x-1)\right), \\ \text{and} \quad &\psi_2(x) = \frac{1}{2}\left(\psi_0(2x) - 2\psi_0\left(2x - \frac{1}{2}\right) + \psi_0(2x-1)\right). \end{aligned}$$

Note that ψ_0 generates an MRA. Moreover, $\Psi \subset V_1$ and is a tight frame [5] and also minimum energy(tight)frame [5]. All functions ψ_0, ψ_1 and ψ_2 have support in [0, 1].

3.2 Frame Operational Matrices

The collection $\Delta = \{\psi_0, \psi_{l,j,k} : l = 1, 2 \text{ and } j, k \in Z\}$, where $\psi_{l,j,k} = 2^{j/2}\psi_l(2^j x - k)$ forms a minimum energy (tight)frame for $L^2(\mathbb{R})$ [5]. The parameter $j \geq 0$ in Δ is called the resolution level. Let J denotes the maximal resolution. Let $M = 1 + 2(1 + 2 + \cdots + 2^J)$. Suppose the grid points are $t_i = (i-1)/M,\ i = 1, 2, \ldots, M + 1$ and the collocation points are

$$\tau_n = \frac{t_n + t_{n+1}}{2},\ n = 1, 2, \ldots, M.$$

Following Sahu and Jena [15], we have the frame matrix and the frame operational matrix as given below:

For fixed J, the frame matrix F_0 is a matrix of order $M \times M$, defined by

$$F_0 := \begin{pmatrix} \Psi_0 \\ \Psi_{1,0} \\ \Psi_{1,1} \\ \vdots \\ \Psi_{1,J} \\ \Psi_{2,0} \\ \Psi_{2,1} \\ \vdots \\ \Psi_{2,J} \end{pmatrix}$$

where $\Psi_0 = (\psi_0(\tau_1), \psi_0(\tau_2), \ldots, \psi_0(\tau_M))$ and for $l = 1, 2$ and $j = 0, 1, \ldots, J$,

$$\Psi_{l,j} = \begin{pmatrix} \psi_{l,j,0}(\tau_1) & \cdots & \psi_{l,j,0}(\tau_M) \\ \vdots & & \vdots \\ \psi_{l,j,2^j-1}(\tau_1) & \cdots & \psi_{l,j,2^j-1}(\tau_M) \end{pmatrix}.$$

Let us define α-th order integrations ψ_0^α and $\psi_{l;j,k}^\alpha$, $\alpha \geq 1$ by [11]

$$\psi_0^\alpha(x) = \int_0^x \int_0^x \cdots \int_0^x \psi_0(t)\,\mathrm{d}t^\alpha$$
$$= \frac{1}{(\alpha-1)!} \int_0^x (x-t)^{\alpha-1}\,\psi_0(t)\,\mathrm{d}t,$$
$$\psi_{l;j,k}^\alpha(x) = \int_0^x \int_0^x \cdots \int_0^x \psi_{l;j,k}(t)\,\mathrm{d}t^\alpha$$
$$= \frac{1}{(\alpha-1)!} \int_0^x (x-t)^{\alpha-1}\,\psi_{l;j,k}(t)\,\mathrm{d}t.$$

The higher-order frame matrices $F_\alpha, \alpha \geq 1$ are matrices of order $M \times M$, defined by

$$F_\alpha := \begin{pmatrix} \Psi_0^\alpha \\ \Psi_{1,0}^\alpha \\ \Psi_{1,1}^\alpha \\ \vdots \\ \Psi_{1,J}^\alpha \\ \Psi_{2,0}^\alpha \\ \Psi_{2,1}^\alpha \\ \vdots \\ \Psi_{2,J}^\alpha \end{pmatrix}$$

where $\Psi_0^\alpha = \left(\psi_0^\alpha(\tau_1), \psi_0^\alpha(\tau_2), \ldots, \psi_0^\alpha(\tau_M)\right)$ and for $l = 1, 2$ and $j = 0, 1, \ldots, J$,

$$\Psi_{l,j}^\alpha = \begin{pmatrix} \psi_{l;j,0}^\alpha(\tau_1) & \ldots & \psi_{l;j,0}^\alpha(\tau_M) \\ \vdots & & \vdots \\ \psi_{l;j,2^{j-1}}^\alpha(\tau_1) & \cdots & \psi_{l;j,2^{j-1}}^\alpha(\tau_M) \end{pmatrix}.$$

The α-th-order frame operational matrix P_α is now defined by

$$P_\alpha = F_\alpha F_0^{-1}.$$

In particular, frame matrices F_0, F_1 and frame operational matrix P_1 are given below.

Frame Matrix for $J = 1$ $(M = 7)$

$$F_0 = \begin{pmatrix} \frac{2}{7} & \frac{6}{7} & \frac{10}{7} & 2 & \frac{10}{7} & \frac{6}{7} & \frac{2}{7} \\ -\frac{398}{985} & -\frac{1194}{985} & -\frac{796}{985} & 0 & \frac{796}{985} & \frac{1194}{985} & \frac{398}{598} \\ -\frac{796}{985} & -\frac{398}{985} & \frac{1194}{985} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1194}{985} & \frac{398}{985} & \frac{796}{985} \\ -\frac{2}{7} & -\frac{6}{7} & \frac{2}{7} & 2 & \frac{2}{7} & -\frac{6}{7} & -\frac{2}{7} \\ -\frac{4}{7} & \frac{8}{7} & -\frac{4}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{4}{7} & \frac{8}{7} & -\frac{4}{7} \end{pmatrix}.$$

$$F_1 = \begin{pmatrix} \frac{1}{98} & \frac{9}{98} & \frac{25}{98} & \frac{1}{2} & \frac{73}{98} & \frac{89}{98} & \frac{97}{98} \\ -\frac{226}{15661} & -\frac{253}{1948} & -\frac{447}{1511} & -\frac{1189}{3363} & -\frac{447}{1511} & -\frac{253}{1948} & -\frac{226}{15661} \\ -\frac{253}{4383} & -\frac{371}{1094} & -\frac{393}{1757} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{393}{1757} & -\frac{371}{1094} & -\frac{253}{4383} \\ -\frac{1}{98} & -\frac{9}{98} & -\frac{8}{49} & 0 & \frac{8}{49} & \frac{9}{98} & \frac{1}{98} \\ -\frac{2}{49} & -\frac{11}{98} & \frac{15}{98} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{15}{98} & \frac{11}{98} & \frac{2}{49} \end{pmatrix}.$$

Operational Matrix for $J = 1$ ($M = 7$)

$$P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1121}{2378} & \frac{66}{19601} & \frac{65}{152} & -\frac{1}{4} & \frac{2}{105} & -\frac{44}{105} \\ -\frac{580}{3361} & 0 & -\frac{17}{420} & \frac{17}{420} & -\frac{165}{39202} & -\frac{33}{19601} & -\frac{33}{19601} \\ -\frac{336}{3713} & \frac{43}{336} & -\frac{13}{420} & 0 & \frac{336}{3713} & -\frac{83}{2293} & 0 \\ -\frac{336}{3713} & -\frac{43}{336} & 0 & \frac{13}{420} & \frac{336}{3713} & 0 & -\frac{83}{2293} \\ 0 & \frac{529}{4834} & -\frac{130}{2413} & -\frac{130}{2413} & 0 & \frac{1}{60} & -\frac{1}{60} \\ 0 & 0 & \frac{195}{2032} & 0 & 0 & -\frac{9}{140} & 0 \\ 0 & 0 & 0 & \frac{195}{2032} & 0 & 0 & \frac{9}{40} \end{pmatrix}.$$

3.3 Method for First-Order Linear IVP

This method is very much similar to the HWOM method. Let us consider the ODE

$$U^{'}(t) = A(t)U(t) + B(t), \quad U(t_0) = U_0, \quad t \in [t_0, T]. \tag{12}$$

We divide the whole interval of discretization into n equal segments with $h_i = t_{i+1} - t_i, \quad i = 0, 1, \ldots, n-1$. Let us consider the interval $\left[t_i, t_{i+1}\right]$. Assume that U_i is a known approximation to $U(t_i)$. The local coordinate $\tau = \frac{t-t_i}{h_i}$ to the interval $[t_i, t_{i+1}]$. This is now belongs to [0, 1]. The given Eq. (12) becomes a new IVP with local coordinate,

$$\dot{u}(\tau) := \frac{\mathrm{d}u}{\mathrm{d}\tau} = h_i(a(\tau)u(\tau) + b(\tau)), \quad u(0) = U_i. \tag{13}$$

Introduce

$$\mathbf{u} = [u(\tau_1), u(\tau_2), \ldots, u(\tau_M)],$$

From (13), we get

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau} = h_i\,[\mathbf{u}A + \mathbf{b}], \tag{14}$$

where

$$\mathbf{A} = \begin{pmatrix} a(t_1^*) & 0 & \cdots & 0 \\ 0 & a(t_2^*) & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & a(t_M^*) \end{pmatrix},$$

$t_j^* = t_i + h_i\tau_j$
and

$$\mathbf{b} = \left[b(t_1^*), b(t_2^*), \ldots, b(t_M^*)\right].$$

Following Sahu and Jena [15], and simplifying,

$$\mathbf{c} = h_i\left(dY + \mathbf{b}\mathbf{A}^{-1}F_0^{-1}\right)(F_0\mathbf{A}^{-1}F_0^{-1} - h_iP_1)^{-1}, \tag{15}$$

where $Y = EF_0^{-1}$ and $P_1 = F_1F_0^{-1}$.

The approximation is $u(1) = c(1) + u(0) = c(1) + U_i$, where $c(1)$ is the first entry of $\mathbf{c}$.

3.4 Method for Second Order Linear IVP

Let us consider the differential equation

$$U'' + p^* U' + q^* U = f(t), \quad U(t_0) = U_0, \quad U'(t_0) = V_0, \tag{16}$$

where p^*, q^*, and $f(t)$ are function of t. The equation (16) is reduced to a system of first-order ODE by taking

$$\frac{\mathrm{d}U}{\mathrm{d}t} = V, \qquad \frac{\mathrm{d}V}{\mathrm{d}t} = -p^* V - q^* U + f(t). \tag{17}$$

Let us consider the interval $\left[t_i, t_{i+1}\right]$. Assume that the known U_i and V_i are approximation to $U\ (t_i)$ and $V\ (t_i)$=$U^{'}\ (t_i)$, respectively. The local coordinate in the interval $\left[t_i, t_{i+1}\right]$ is $\tau = (t - t_i)/h_i$, where $h_i = t_{i+1} - t_i$. In terms of this local coordinate, we have $u(\tau) = U(t), \quad v(\tau) = V(t)$. Introduce

$$\mathbf{u} = \left[u(\tau_1), u(\tau_2), \ldots, u(\tau_M)\right],$$
$$\mathbf{v} = \left[v(\tau_1), v(\tau_2), \ldots, v(\tau_M)\right].$$

We have the following relation from (17)

$$\dot{\mathbf{u}} = h_i \mathbf{v}, \tag{18}$$
$$\dot{\mathbf{v}} = -\mathbf{vp} - \mathbf{uq} + \Phi, \tag{19}$$

where

$$\mathbf{p} = h_i \begin{bmatrix} p^*(t_1^*) & 0 & 0 \cdots & 0 \\ 0 & p^*(t_2^*) & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \ \vdots & \vdots \\ 0 & 0 & 0 \cdots & p^*(t_M^*) \end{bmatrix},$$

$$\mathbf{q} = h_i \begin{bmatrix} q^*(t_1^*) & 0 & 0 \cdots & 0 \\ 0 & q^*(t_2^*) & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \ \vdots & \vdots \\ 0 & 0 & 0 \cdots & q^*(t_M^*) \end{bmatrix},$$

and

$$\Phi = h_i \left[\, f(t_1^*)\ f(t_2^*)\ f(t_3^*) \cdots f(t_M^*) \right].$$

Here $t_j^* = \tau_j h_i + t_i, \quad j = 1, 2, \ldots, M$. Following Chen [2], Lepik [11], and simplifying as [15] we have

$$a = h_i b P_1 + h_i V_i Y \tag{20}$$

and

$$b = -V_i E\mathbf{p}F_0^{-1}S^{-1} - h_i V_i Y F_1\mathbf{q}F_0^{-1}S^{-1} - U_i E\mathbf{q}F_0^{-1}S^{-1} + \Phi F_0^{-1}S^{-1}, \quad (21)$$

where $S = I + F_1\mathbf{p}F_0^{-1} + h_i P_1 F_1\mathbf{q}F_0^{-1}$. Here $Y = EF_0^{-1}$ and $P_1 = F_1F_0^{-1}$. The approximation becomes

$$U_{i+1} = a\,(1) + U_i$$
$$V_{i+1} = b\,(1) + V_i.$$

4 Legendre's Wavelet Operational Matrix Method

The "Legendre wavelets" $\psi_{n,m}(t) = \psi(k, n, m.t)$ is defined as follows: [9, 12, 14]

Definition 7 Legendre Polynomial

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} P_m\left(2^k t - 2n + 1\right), & t \in [\xi_1, \xi_2] \\ 0 & \text{otherwise} \end{cases},$$

where $\xi_1 = \frac{2n-2}{2^k}$, $\xi_2 = \frac{2n}{2^k}$, $m = 0, 1, \ldots, M-1, n = 1, 2, \ldots, 2^{k-1}$, and $P_m(t)$ are Legendre polynomial of degree m. In particular, $P_0(t) = 1$ and $P_1(t) = t$.

Any function $f(t)$ can be represented in Legendre wavelet series in [0, 1) by Razzaghi and Yousefi [14]

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m}\psi_{n,m}(t),$$

where $c_{n,m} = \langle f(t), \psi_{n,m}(t)\rangle$, in which $\langle\cdots\rangle$ is the inner product.

4.1 Function Approximation

Consider the Legendre wavelet as [9, 14]

$$\Psi = \left(\psi_{1,0} \cdots \psi_{2^{k-1},0}\ \psi_{1,1} \cdots \psi_{2^{k-1},1} \cdots \psi_{1,M-1} \cdots \psi_{2^{k-1},M-1}\right)^T$$

Let f be an arbitrary function in $L^2[0, 1]$ then there exist unique coefficients $c_{n,m}$ such that

$$f(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = C^T\Psi(t),$$

where $c_{n,m} = \int_0^1 f(t)\,\psi_{n,m}(t)\,dt$, and

$$C = \left(c_{1,0} \cdots c_{2^{k-1},0}\; c_{1,1} \cdots c_{2^{k-1},1} \cdots\; c_{1,M-1} \cdots c_{2^{k-1},M-1}\right)^T,$$

Collocation points are given by [9]

$$t_i = \frac{2i-1}{2^k M}, \quad i = 1, 2, \ldots, 2^{k-1}M$$

Following [9, 14] we get the operational matrices as below:

$$\int_0^t \Psi(t)\,\mathrm{d}t = P\Psi(t),$$

where P is the operational matrix of order $2^{k-1}M \times 2^{k-1}M$.

In general, the operational matrix P is given by Razzaghi and Yousefi [14]

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ 0 & 0 & L & \cdots & F \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & L \end{bmatrix},$$

where F and L are square matrix of order M as follows:

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$F = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-5}} & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}.$$

In particular, the matrices Ψ and P for $M = 3$ and $k = 2$ are given in the following:

$$\Psi_{6\times 6} = \begin{pmatrix} \sqrt{2} & \frac{-2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} & 0 & 0 & 0 \\ \sqrt{2} & 0 & \frac{\sqrt{10}}{2} & 0 & 0 & 0 \\ \sqrt{2} & \frac{2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \frac{-2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & \frac{\sqrt{10}}{2} \\ 0 & 0 & 0 & \sqrt{2} & \frac{2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} \end{pmatrix}$$

$$P_{6\times 6} = \frac{1}{4}\begin{pmatrix} 1 & \frac{\sqrt{2}}{\sqrt{6}} & 0 & 2 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\sqrt{2}}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 \end{pmatrix}$$

5 Results and Discussions

In this section, we consider some numerical examples to compare the solutions obtain from different operational matrix methods described above.

Example 1 Consider the singular initial value problem [9]

$$U^{''} + \frac{4}{t}U^{'} + \left(\frac{2}{t^2} + t\right)U = 20t + t^4, \quad U(0) = U^{'}(0) = 0.$$

The exact solution of the given IVP is $U(t) = t^3$. Numerical comparisons of solutions obtain from LWOM, HWOM, and FOM are presented in Table 1. Comparison of the solution from HWOM, FOM, and exact solution presented graphically in Fig. 1.

Example 2 Consider the singular initial value problem [9]

$$U^{''} + \frac{1}{t}U^{'} = \left(\frac{8}{8-t^2}\right)^2, \quad U(0) = 0, U^{'}(0) = 0.$$

The exact solution of the given IVP is $U(t) = 2\log\left(\frac{7}{8-t^2}\right)$. Numerical comparison of solution from LWOM, HWOM, and FOM is presented in the Table 2. Comparison of the solution from HWOM, FOM, and exact solution is presented graphically in Fig. 2.

Table 1 L_2-norm Comparison of solution of Example 1

t	LWOM [9]	HWOM	FOM	Exact
0.1	0.001000	0.0010	0.0010	0.001000
0.2	0.008003	0.0079	0.0080	0.008000
0.3	0.027008	0.0269	0.0269	0.027000
0.4	0.064045	0.0639	0.0639	0.064000
0.5	0.125131	0.1249	0.1249	0.125000
0.6	0.216534	0.2158	0.2158	0.216000
0.7	0.345017	0.3428	0.3428	0.343000
0.8	0.513002	0.5118	0.5118	0.512000
0.9	0.730000	0.7288	0.7288	0.729000

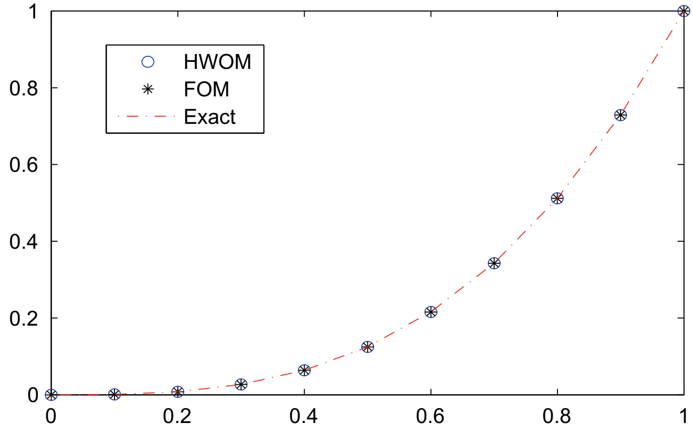

Fig. 1 Comparison of solution Example 1

Table 2 L_2-norm comparison of solution of Example 2

t	LWOM [9]	HWOM	FOM	Exact
0.1	−0.26456123	−0.2646	−0.2646	−0.26456122
0.2	−0.25703772	−0.2570	−0.2571	−0.25703770
0.3	−0.24443526	−0.2444	−0.2446	−0.24443526
0.4	−0.22665738	−0.2267	−0.2268	−0.22665737
0.5	−0.20356540	−0.2036	−0.2037	−0.20356538
0.6	−0.17497491	−0.1750	−0.1751	−0.17497490