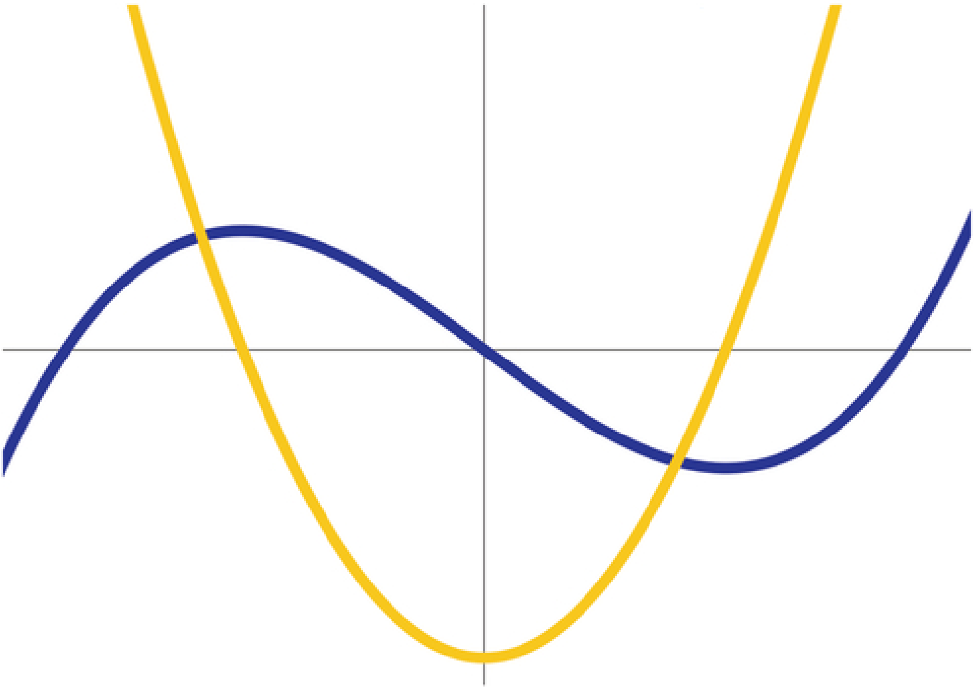
A SELF-TEACHING GUIDE

# Quick Calculus

3RD EDITION



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# Quick Calculus

## A Self-Teaching Guide

Third Edition

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# Preface

*Quick Calculus* is designed for you to learn the basic techniques of differential and integral calculus with a minimum of wasted effort, studying by yourself. It was created on a premise that is now widely accepted: in technical subjects such as calculus, students learn by *doing* rather than by *listening*. The book consists of a sequence of relatively short discussions, each followed by a problem whose solution is immediately available. One's path through the book is directed by the responses. The text is aimed at newcomers to calculus, but additional topics are discussed in the final chapter for those who wish to go further.

The initial audience for *Quick Calculus* was composed of students entering college who did not wish to postpone physics for a semester in order to take a prerequisite in calculus. In reality, the level of calculus needed to start out in physics is not high and could readily be mastered by self-study.

The readership for *Quick Calculus* has grown far beyond novice physics students, encompassing people at every stage of their career. The fundamental reason is that calculus is empowering, providing the language for every physical science and for engineering, as well as tools that are crucial for economics, the social sciences, medicine, genetics, and public health, to name a few. Anyone who learns the basics of calculus will think about how things change and influence each other with a new perspective. We hope that *Quick Calculus* will provide a firm launching point for helping the reader to achieve this perspective.

Daniel Kleppner
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*Cambridge, Massachusetts*

# Contents

# CHAPTER ONE
# Starting Out

In spite of its formidable name, calculus is not a particularly difficult subject. The fundamental concepts of calculus are straightforward. Your appreciation of their value will grow as you develop the skills to use them.

After working through *Quick Calculus* you will be able to handle many problems and be prepared to acquire more elaborate techniques if you need them. The important word here is *working,* though we hope that you find that the work is enjoyable.

*Quick Calculus* comprises four chapters that consist of sections and subsections. We refer to the subsections as *frames*. Each chapter concludes with a summary. Following these chapters there are two appendixes on supplementary material and a collection of review problems with solutions.

The frames are numbered sequentially throughout the text. Working *Quick Calculus* involves studying the frames and doing the problems. You can check your answers immediately: they will be located at the bottom of one of the following pages or, if the solutions are longer, in a separate frame. Also a summary of frame problems answers start on page 273.

Your path through *Quick Calculus* will depend on your answers. The reward for a correct answer is to go on to new material. If you have difficulty, the solution will usually be explained and you may be directed to another problem.

Go on to frame **1**.

## 1.1 A Few Preliminaries

**1**

Chapter 1 will review topics that are foundational for the discussions to come. These are:

- the definition of a mathematical function;
- graphs of functions;

*(continued)*

- the properties of the most widely used functions: linear and quadratic functions, trigonometric functions, exponentials, and logarithms.

A note about calculators: a few problems in *Quick Calculus* need a simple calculator. The calculator in a typical smartphone is more than adequate. If you do not happen to have access to a calculator, simply skip the problem: you can master the text without it.

Go on to frame **2**.

**2**

---

Here is what's ahead: this first chapter is a review, which will be useful later on; Chapter 2 is on differential calculus; and Chapter 3 introduces integral calculus. Chapter 4 presents some more advanced topics. At the end of each chapter there is a summary to help you review the material in that chapter. There are two appendixes—one gives proofs of a number of relations used in the book, and the other describes some supplementary topics. In addition, there is a list of extra problems with answers in the Review Problems on page 277, and a section of tables you may find useful.

First we review the definition of a function. If you are already familiar with this and with the idea of dependent and independent variables, skip to frame **14**. (In fact, in this chapter there is ample opportunity for skipping if you already know the material. On the other hand, some of the material may be new to you, and a little time spent on review can be a good thing.)

A word of caution about the next few frames. Because we start with some definitions, the first section must be somewhat more formal than most other parts of the book.

Go on to frame **3**.

## 1.2 Functions

**3**

---

The definition of a function makes use of the idea of a *set*. If you know what a set is, go to **4**. If not, read on.

A *set* is a collection of objects—not necessarily material objects—described in such a way that we have no doubt as to whether a particular object does or does not belong to it. A set may be described by listing its elements. Example: 23, 7, 5, 10 is a set of numbers. Another example: Reykjavik, Ottawa, and Rome is a set of capitals.

We can also describe a set by a rule, such as all the even positive integers (this set contains an infinite number of objects).

A particularly useful set is the set of all real numbers. This includes all numbers such as 5, −4, 0, ½, $\pi$, −3.482, $\sqrt{2}$. The set of real numbers does *not* include quantities involving the square root of negative numbers. Such quantities are called *complex numbers*; in this book we will be concerned only with real numbers.

The mathematical use of the word "set" is similar to the use of the same word in ordinary conversation, as "a set of chess pieces."

Go to **4**.

**4**

---

In the blank below, list the elements of the set that consists of all the odd integers between −10 and +10.

Elements: ____________________

Go to **5** for the correct answer.

**5**

---

Here are the elements of the set of all odd integers between −10 and +10:

$$-9, -7, -3, -5, -1,\ 1,\ 3,\ 5,\ 7,\ 9.$$

Go to **6**.

**6**

---

Now we are ready to talk about functions. Here is the definition.

A *function* is a rule that assigns to each element in a set $A$ one and only one element in a set $B$.

The rule can be specified by a mathematical formula such as $y = x^2$, or by tables of associated numbers, for instance, the temperature at each hour of the day. If $x$ is one of the elements in set $A$, then the element in set $B$ that the function $f$ associates with $x$ is denoted by the symbol $f(x)$. This symbol $f(x)$ is the value of $f$ evaluated at the element $x$. It is usually read as "$f$ of $x$."

The set $A$ is called the *domain* of the function. The set of all possible values of $f(x)$ as $x$ varies over the domain is called the *range* of the function. The range of $f$ need not be all of $B$.

*(continued)*

In general, $A$ and $B$ need not be restricted to sets of real numbers. However, as mentioned in frame **3**, in this book we will be concerned only with real numbers.

Go to **7**.

---

**7**

For example, for the function $f(x) = x^2$, with the domain being all real numbers, the range is ______________________________________.

Go to **8**.

---

**8**

The range is *all nonnegative real numbers*. For an explanation, go to **9**.

Otherwise, skip to **10**.

---

**9**

Recall that the product of two negative numbers is positive. Thus for any real value of $x$ positive or negative, $x^2$ is positive. When $x$ is 0, $x^2$ is also 0. Therefore, the range of $f(x) = x^2$ is all nonnegative numbers.

Go to **10**.

---

**10**

Our chief interest will be in rules for evaluating functions defined by formulas. If the domain is not specified, it will be understood that the domain is the set of all real numbers for which the formula produces a real number, and for which it makes sense. For instance,

(a) $f(x) = \sqrt{x}$ Range = ______________________.

(b) $f(x) = \dfrac{1}{x}$ Range = ______________________.

For the answers go to **11**.

---

**11**

The function $\sqrt{x}$ is real for $x$ nonnegative, so the answer to (a) is all nonnegative real numbers. The function $1/x$ is defined for all values of $x$ except zero, so the range in (b) is all real numbers except zero.

Go to **12**.

---

**12**

When a function is defined by a formula such as $f(x) = ax^3 + b$, $x$ is called the *independent variable* and $f(x)$ is called the *dependent variable*. One advantage of this notation is that the value of the dependent variable, say for $x = 3$, can be indicated by $f(3)$.

Often, a single letter is used to represent the dependent variable, as in

$$y = f(x).$$

Here $x$ is the independent variable, and $y$ is the dependent variable.

Go to **13**.

**13**

In mathematics the symbol $x$ frequently represents an independent variable, $f$ often represents the function, and $y = f(x)$ usually denotes the dependent variable. However, any other symbols may be used for the function, the independent variable, and the dependent variable. For example, we might have $z = H(r)$, which is read as "$z$ equals $H$ of $r$." Here $r$ is the independent variable, $z$ is the dependent variable, and $H$ is the function.

Now that we know what a function means, let's describe a function with a graph instead of a formula.

Go to **14**.

## 1.3 Graphs

**14**

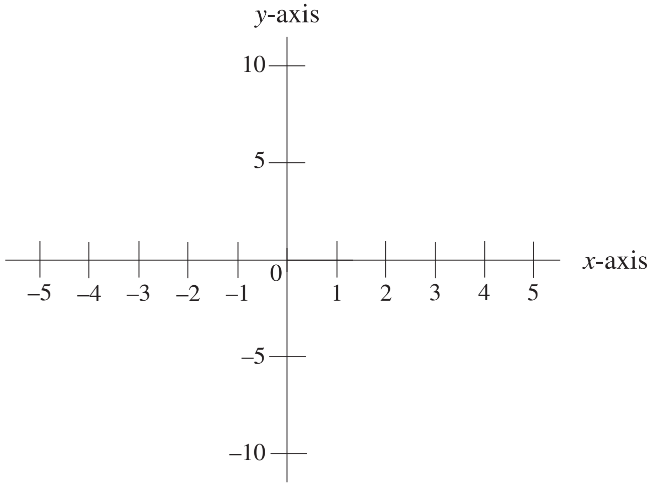
If you know how to plot graphs of functions, skip to frame **19**.

Otherwise, go to **15**.

**15**

We start by constructing coordinate axes. In the most common cases we construct a pair of mutually perpendicular intersecting lines, one horizontal, the other vertical. The horizontal line is often called the *x-axis* and the vertical line the *y-axis*. The point of intersection is the origin, and the axes together are called the *coordinate axes*.

*(continued)*

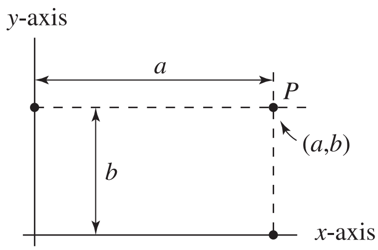



Next we select a convenient unit of length and, starting from the origin, mark off a number scale on the $x$-axis, positive to the right and negative to the left. In the same way, we mark off a scale along the $y$-axis with positive numbers going upward and negative downward. The scale of the $y$-axis does not need to be the same as that for the $x$-axis (as in the drawing). In fact, $y$ and $x$ can have different units, such as distance and time.

Go to **16**.

**16**

We can represent one specific pair of values associated by the function in the following way: let $a$ represent some particular value for the independent variable $x$, and let $b$ indicate the corresponding value of $y = f(x)$. Thus, $b = f(a)$.




We now draw a line parallel to the $y$-axis at distance $a$ from the $y$-axis and another line parallel to the $x$-axis at distance $b$ from that axis. The point $P$ at which these two lines intersect is designated by the pair of values $(a, b)$ for $x$ and $y$, respectively.

The number $a$ is called the $x$-coordinate of $P$, and the number $b$ is called the $y$-coordinate of $P$. (Sometimes the x-coordinate is called the *abscissa*, and the $y$-coordinate is called the *ordinate*.) In the designation of a typical point by the notation $(a, b)$ we will always designate the $x$-coordinate first and the $y$-coordinate second.

As a review of this terminology, encircle the correct answers below. For the point (5, −3):

$x$-coordinate: [−5 | − 3 | 3 | 5]

$y$-coordinate: [−5 | − 3 | 3 | 5]

Go to **17**.

**17**

---

The most direct way to plot the graph of a function $y = f(x)$ is to make a table of reasonably spaced values of $x$ and of the corresponding values of $y = f(x)$. Then each pair of values $(x, y)$ can be represented by a point as in the previous frame. A graph of the function is obtained by connecting the points with a smooth curve. Of course, the points on the curve may be only approximate. If we want an accurate plot, we just have to be very careful and use many points. (On the other hand, crude plots are pretty good for many purposes.)

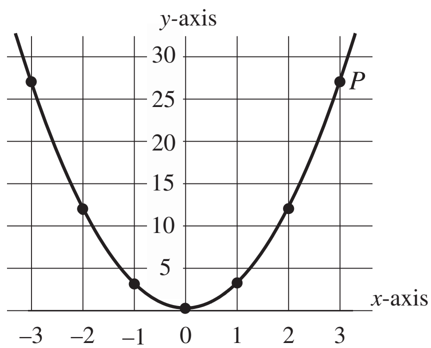
Go to **18**.

**18**

---

As an example, here is a plot of the function $y = 3x^2$. A table of values of $x$ and $y$ is shown, and these points are indicated on the graph.

| $x$ | $y$ |
|---|---|
| –3 | 27 |
| –2 | 12 |
| –1 | 3 |
| 0 | 0 |
| 1 | 3 |
| 2 | 12 |
| 3 | 27 |




To test yourself, encircle the pair of coordinates that corresponds to the point $P$ indicated in the figure.

[(3, 27) | (27, 3) | none of these]

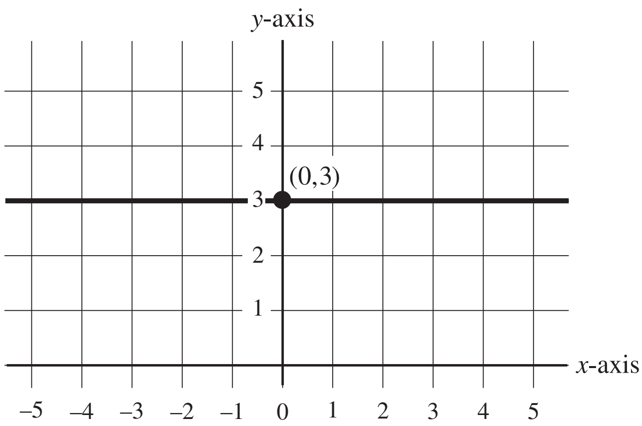
If incorrect, study frame **16** once again and then go to **19**. If correct,

Go to **19**.

**19**

---

Here is a rather special function. It is called a *constant function* and assigns a single fixed number $c$ to every value of the independent variable, $x$. Hence, $f(x) = c$.




This is a peculiar function because the value of the dependent variable is the same for all values of the independent variable. Nevertheless, the relation $f(x) = c$ assigns exactly one value of $f(x)$ to each value of $x$ as required in the definition of a function. All the values of $f(x)$ happen to be the same.

Try to convince yourself that the graph of the constant function $y = f(x) = 3$ is a straight line parallel to the $x$-axis passing through the point (0,3) as shown in the figure.

Go to **20**.

**20**

---

Another special function is the *absolute value function*. The absolute value of $x$ is indicated by the symbol $|x|$. The absolute value of a number $x$ determines the size, or magnitude, of the number without regard to its sign. For example,

$$|-3| = |3| = 3$$

---

Answers: Frame 16: 5, −3
Frame 18: (3, 27)

Now we will define $|x|$ in a general way. But first we need to recall the inequality symbols:

$a > b$ means $a$ is greater than $b$.

$a \geq b$ means $a$ is greater than or equal to $b$.

$a < b$ means $a$ is less than $b$.

$a \leq b$ means $a$ is less than or equal to $b$.

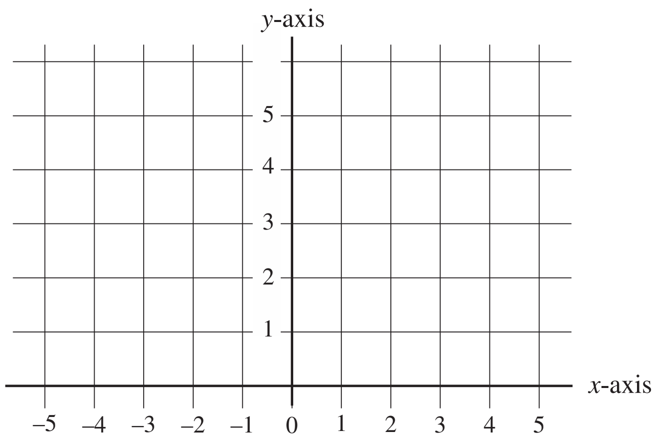
With this notation we can define the absolute value function, $|x|$, by the following two rules:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Go to **21**.

**21**

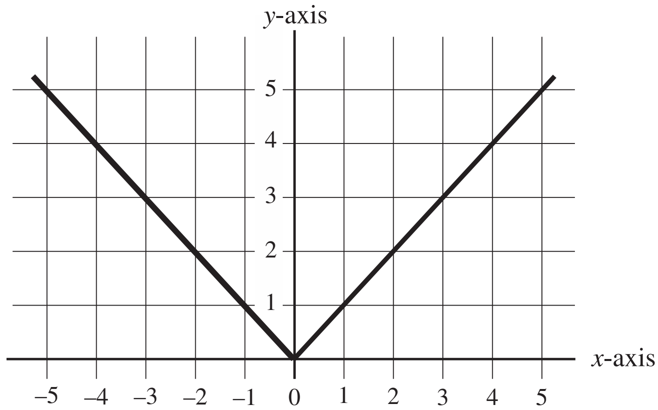
A good way to show the behavior of a function is to plot its graph. Therefore, as an exercise, plot a graph of the function $y = |x|$ in the figure below.




To check your answer, go to **22**.

**22**

The graph for | $x$ | is




This can be seen by preparing a table of $x$ and $y$ values as follows:

| $x$ | $y = \mid x \mid$ |
|---|---|
| −4 | +4 |
| −2 | +2 |
| 0 | 0 |
| +2 | +2 |
| +4 | +4 |

These points may be plotted as in frames **16** and **18** and the lines drawn with the results in the above figure.

The graph and $x$, $y$ coordinates described here are known as a *Cartesian coordinate system*. There are other coordinate systems better suited to other geometries, such as cylindrical or spherical coordinate systems, but Cartesian coordinates are the best known.

With this introduction on functions and graphs, we are now going to familiarize ourselves with some important elementary mathematical functions.

These functions are the linear, quadratic, trigonometric, exponential, and logarithmic functions.

Go to **23**.

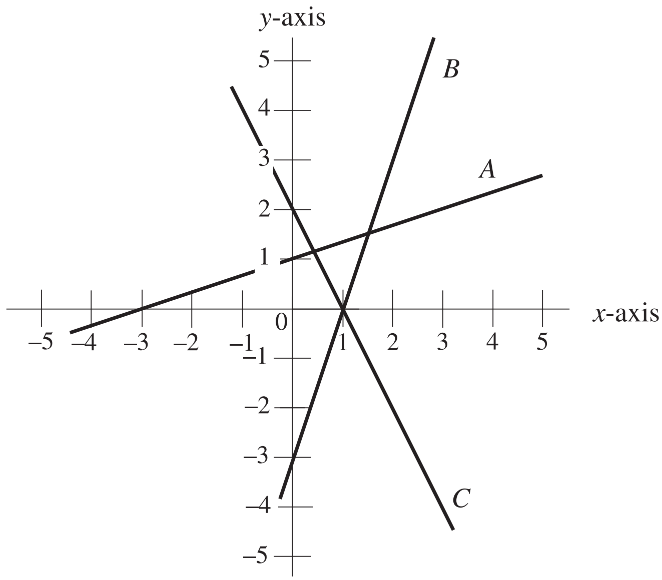
# 1.4 Linear and Quadratic Functions

**23**

A function defined by an equation in the form $y = mx + b$, where $m$ and $b$ are constants, is called a *linear function* because its graph is a straight line. This is a simple and useful function, and you need to become familiar with it.

Here is an example: Encircle the letter that identifies the graph (as labeled in the figure) of

$$y = 3x - 3. \qquad [A \mid B \mid C]$$




If you missed this or if you do not feel entirely sure of the answer, go to **24**.

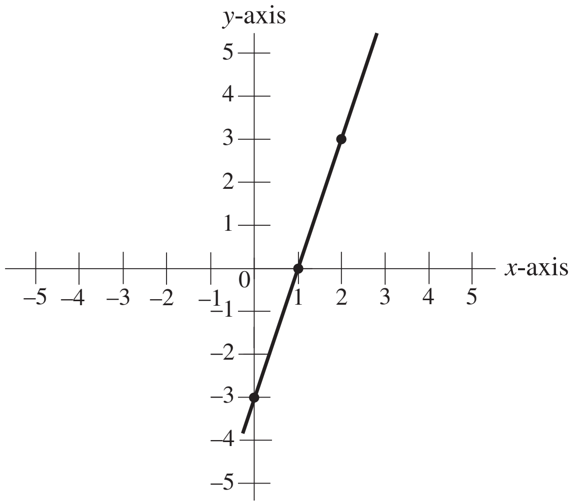
Otherwise, go to **25**.

**24**

---

The table below gives a few values of $x$ and $y$ for the function $y = 3x - 3$.

| $x$ | $y$ |
|---|---|
| –2 | –9 |
| –1 | –6 |
| 0 | –3 |
| 1 | 0 |
| 2 | 3 |




A few of these points are shown on the graph, and a straight line has been drawn through them. This is line $B$ of the figure in frame **23**.

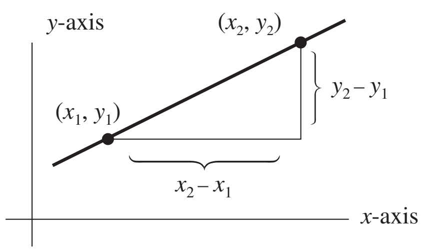
Go to **25**.

**25**

---

Here is the graph of a typical linear function. Let us take any two different points on the line, $(x_2, y_2)$ and $(x_1, y_1)$. We define the *slope* of the line in the following way:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

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Answer: Frame 23: $B$

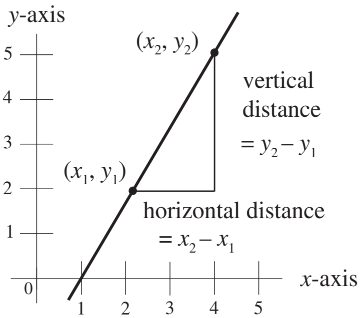
The idea of slope will be important in our later work, so let's spend a little time learning more about it.

Go to **26**.

**26**

---

If the $x$ and $y$ scales are the same, as in the figure, then the slope is the ratio of the vertical distance $y_2 - y_1$ to the horizontal distance $x_2 - x_1$ as we go from the point $(x_1, y_1)$ on the line to $(x_2, y_2)$. If the line is vertical, the slope is infinite (or, more strictly, undefined). Test for yourself that the slope is the same for any pair of two separate points on the line.




Go to **27**.

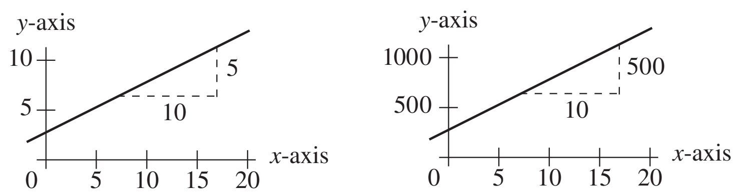
**27**

---

If the vertical and horizontal scales are not the same, the slope is still defined by

$$\text{slope} = \frac{\text{vertical distance}}{\text{horizontal distance}},$$

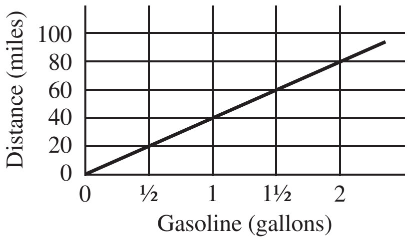
but now the distance is measured using the appropriate scale. For instance, the two figures below may look similar, but the slopes are quite different. In the first figure the $x$ and $y$ scales are identical, and the slope is ½. In the second figure the $y$ scale has been changed by a factor of 100, and the slope is 50.




*(continued)*

Because the slope is the ratio of two lengths, the slope is a pure number if the lengths are pure numbers. However, if the variables have different dimensions, the slope will also have a dimension.

Below is a plot of the distance traveled by a car vs. the amount of gasoline consumed.




Here the slope has the units of miles per gallon (mpg). What is the slope of the line shown?

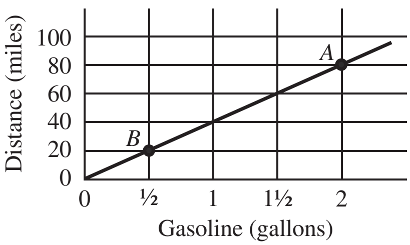
Slope = [20 | 40 | 60 | 80] mpg

If right, go to **29**.
Otherwise, go to **28**.

**28**

---

To evaluate the slope, let us find the coordinates of any two points on the line.




For instance, *A* has the coordinates (2 gallons, 80 miles) and *B* has the coordinates (½ gallon, 20 miles). Therefore, the slope is

$$\frac{(80 - 20)\text{ miles}}{(2 - 1/2)\text{ gallon}} = 40\,\frac{\text{miles}}{\text{gallon}} = 40\text{ mpg}.$$

We would have obtained the same value for the slope no matter which two points we used, because two points determine a straight line.

Go to **29**.

---

**29**

---

If the line is described by an equation of the form $y = mx + b$, then the slope is given by

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Substituting in the above expression for $y$, we have

$$\text{slope} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$

What is the slope of $y = 7x - 5$?

[5/7 | 7/5 | −5 | −7 | 5 | 7]

If right, go to **31**.
Otherwise, go to **30**.

**30**

---

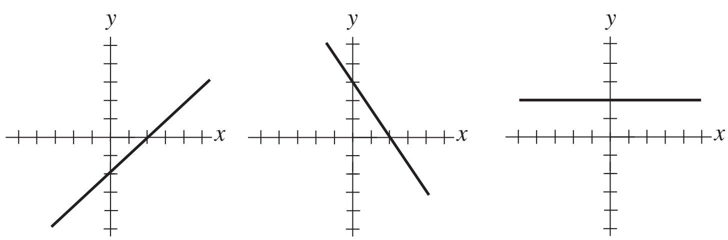
The equation $y = 7x - 5$ can be written in the form $y = mx + b$ if $m = 7$ and $b = -5$. Because slope $= m$, the line given has a slope of 7.

Go to **31**.

**31**

---

The slope of a line can be positive (greater than 0), negative (less than 0), or 0. An example of each is shown graphically below.




Positive slope
Figure 1

Negative slope
Figure 2

Zero slope
Figure 3

Note how a line with positive slope rises in going from left to right, a line with negative slope falls in going from left to right, and a line of zero slope is horizontal. (It was pointed out in frame **26** that the slope of a vertical line is not defined.)

*(continued)*

Indicate whether the slope of the graph of each of the following equations is positive, negative, or zero by encircling your choice.

| | Equation | Slope |
|---|---|---|
| 1. | $y = 2x - 5$ | { + \| − \| 0 } |
| 2. | $y = -3x$ | { + \| − \| 0 } |
| 3. | $p = q - 2$ | { + \| − \| 0 } |
| 4. | $y = 4$ | { + \| − \| 0 } |

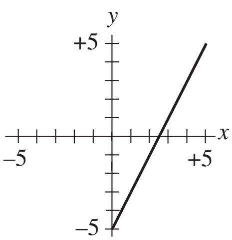
The answer is in the next frame.

**32**

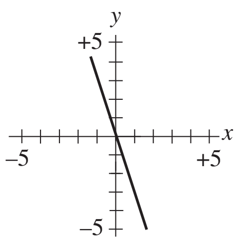
Here are the answers to the questions in frame **31**.

In frame **29** we saw that for a linear equation in the form $y = mx + b$ the slope is $m$.

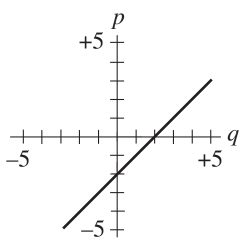
1. $y = 2x - 5$. Here $m = 2$ and the slope is 2. Clearly this is a positive number. See Figure 1 below.
2. $y = -3x$. Here $m = -3$. The slope is $-3$, which is negative. See Figure 2 below.
3. $p = q - 2$. In this equation the variables are $p$ and $q$, rather than $y$ and $x$. Written in the form $p = mq + b$, it is evident that $m = 1$, which is positive. See Figure 3 below.
4. $y = 4$. This is an example of a constant function. Here $m = 0$, $b = 4$, and the slope is 0. See Figure 4 below.



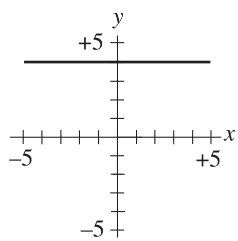
Positive slope
$y = 2x - 5$
Figure 1



Negative slope
$y = -3x$
Figure 2



Positive slope
$p = q - 2$
Figure 3



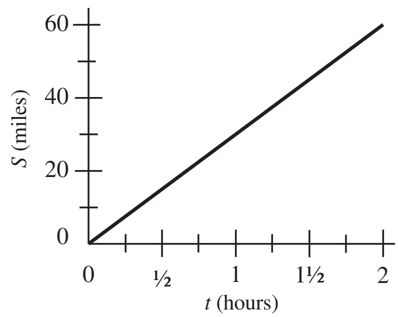
Zero slope
$y = 4$
Figure 4

Go to **33**.

Answers: Frame 27: 40 mpg
Frame 29: 7

**33**

Here is a linear equation in which the slope has a familiar meaning. The graph below shows the position $S$ of a car on a straight road at different times. The position $S = 0$ means the car is at the starting point.




Try to guess the correct word to fill in the blank below:
The slope of the line has the same value as the car's ____________________.

Go to **34**.

**34**

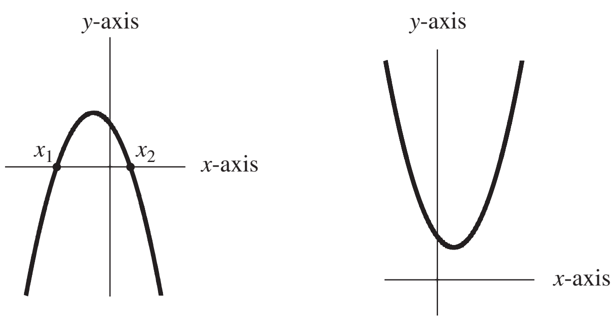
The slope of the line has the same value as the car's *speed* (or, for this one-dimensional motion *velocity*).

The slope is the ratio of the distance traveled to the time required. But, by definition, the speed is also the distance traveled divided by the time. Thus the value of the slope of the line is equal to the speed.

Go to **35**.

**35**

Now let's look at another type of equation. An equation in the form $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants ($a \neq 0$), is called a *quadratic function* and its graph is called a *parabola*. Two typical parabolas are shown in the figure.




Go to **36**.

**36**

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*Roots of an Equation:*

The values of $x$ for which $f(x) = 0$ are called the *roots* of the equation. The values at $y = 0$, shown by $x_1$ and $x_2$ in the figure on the left in frame **35**, correspond to values of $x$ which satisfy $ax^2 + bx + c = 0$ and are thus the roots of the equation. Not all quadratic equations have real roots. For example, the curve on the right represents an equation with no real value of $x$ when $y = 0$.

Although you will not need to find the roots of any quadratic equation later in this book, you may want to know the formula anyway. If you would like to see a discussion of this, go to frame **37**.

Otherwise, skip to frame **39**.

**37**

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The equation $ax^2 + bx + c = 0$ has two roots. These are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The subscripts 1 and 2 serve merely to identify the roots. The two roots can be summarized by a single equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We will not prove these results, though they can be checked by substituting the values for $x$ in the original equation.

Here is a practice problem on finding roots: Which answer correctly gives the roots of $3x - 2x^2 = 1$?

(a) $\frac{1}{4}\left(3 + \sqrt{17}\right); \frac{1}{4}\left(3 - \sqrt{17}\right)$

(b) $-1; -\frac{1}{2}$

(c) $\frac{1}{4}; -\frac{1}{4}$

(d) $1; \frac{1}{2}$

---

Answers: Frame 31: +, −, +, 0

Encircle the letter of the correct answer.

$$[a \mid b \mid c \mid d]$$

If you got the right answer, go to **39**.
The answer is in the following frame.

**38**

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Here is the solution to the problem in frame **37**.
The equation $3x - 2x^2 = 1$ can be written in the standard form

$$2x^2 - 3x + 1 = 0.$$

Here $a = 2$, $b = -3$, $c = 1$.

$$x = \frac{1}{2a}\left[-b \pm \sqrt{b^2 - 4ac}\right] = \frac{1}{4}\left[-(-3) \pm \sqrt{(-3)^2 - (4)(2)(1)}\right]$$

$$= \frac{1}{4}(3 \pm 1).$$

$$x_1 = \frac{1}{4}(3 + 1) = 1.$$

$$x_2 = \frac{1}{4}(3 - 1) = \frac{1}{2}.$$

Go to **39**.

**39**

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This ends our brief discussion of linear and quadratic functions. Perhaps you would like some more practice on these topics before continuing. If so, try working Review Problems 1–5 on page 277. At the end of this chapter there is a concise summary of the material we have had so far, which you may find useful.

Whenever you are ready, go to **40**.

## 1.5 Angles and Their Measurements

**40**

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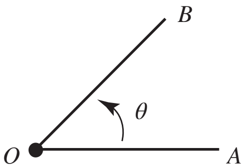
*Elementary features of rotations and angles:*

If you are already familiar with rotations, angle, and degrees and radians, you can jump to frame **50**.

*(continued)*




The concept of the angle is the bedrock of trigonometry. Although the general idea of an angle is probably familiar, it is important to agree on the conventions and units for describing angles. For two straight-line segments *OA* and *OB* that intersect at a point *O*, the angle between them is a measure of how far the line segment *OA* must be rotated about the point *O* to coincide with the line segment *OB*.

If the two segments initially coincide, for instance, half a revolution of either segment will leave them pointing in opposite directions and a full revolution will bring them back to their original positions.

The Greek letter $\theta$ (theta) symbolizes the *rotation angle*. The sense of rotation is shown by the curved arrow. We will follow the convention that if the segment *OA* is rotated in the counterclockwise direction to coincide with the segment *OB*, then the rotation is positive. Conversely, a rotation in the clockwise direction is negative. The direction can be indicated by a small arrowhead on the curve between the two segments. If the sense of rotation is unimportant, the arrowhead is usually omitted.

*Measuring the size of rotations:*

There are two conventions used for measuring the size of rotation. The first divides one revolution into 360 parts called *degrees* (not to be confused with degrees of temperature). The number 360 has the attraction of being large—providing good angular resolution—and possessing many divisors. The symbol for a degree of rotation is °; hence, a quarter turn is 90°. The degree can be subdivided into 60 *minutes* (60′), and the minute subdivided into 60 *seconds* (60″). Until recent years degrees, minutes, and seconds (*DMS coordinates*) were commonly used in map-making and navigation. With advent of GPS and laser metrology, the convention for subdividing the degree has been changed: instead of minutes and seconds, the convention is to express the fraction of a degree by a decimal number, typically with four digits.

*Location on Earth: latitude and longitude:*

Two numbers are needed to describe positions on a surface. Because Earth is spherical, Cartesian coordinates are not useful for specifying a position. Instead, a system based on coordinates known as *latitude* and *longitude* is employed. One of the coordinates is based on the idea of a *meridian*. This is an imaginary half circle on Earth that connects Earth's

Answers: Frame 37: d