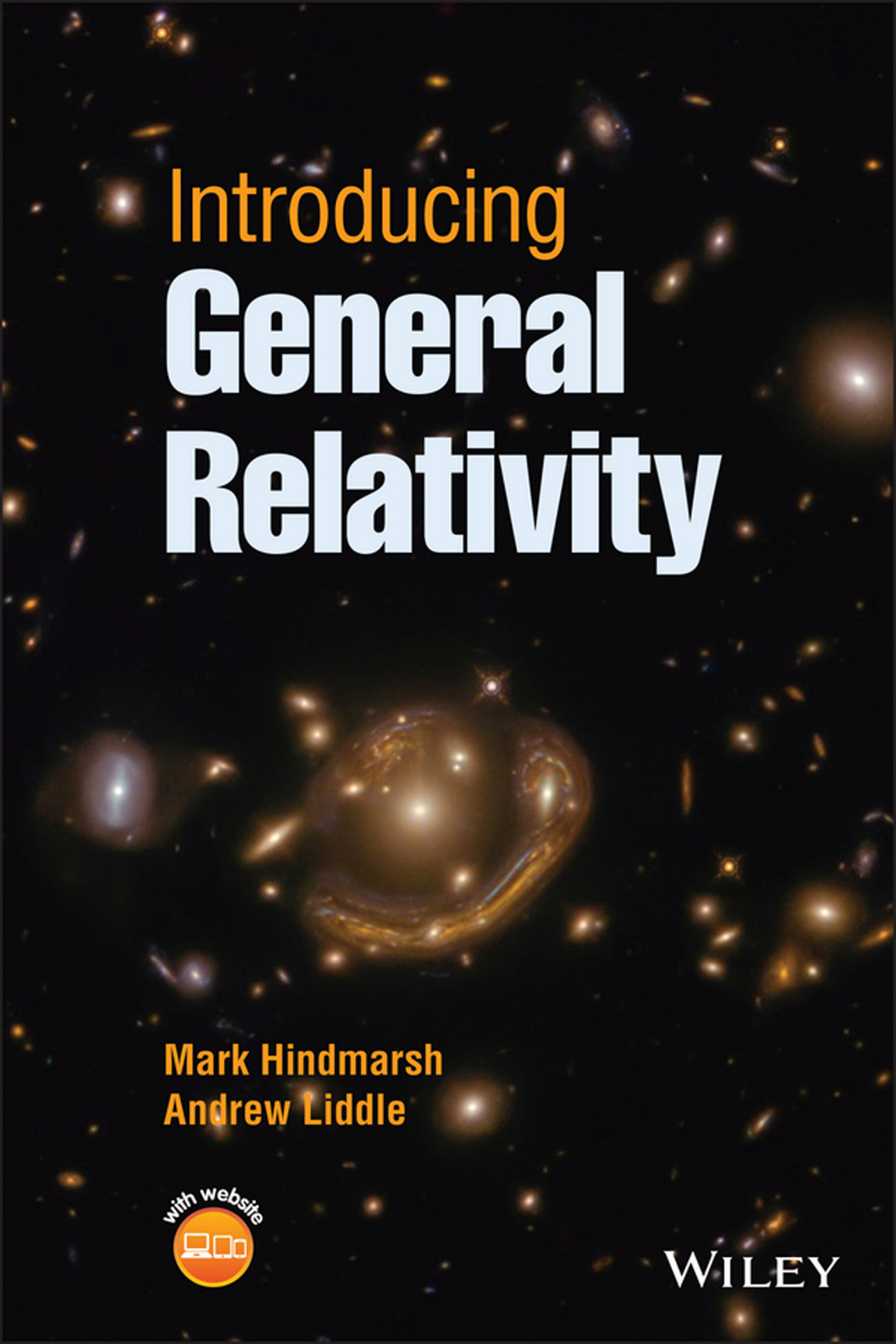


The background of the cover is a deep space image filled with numerous distant galaxies and stars. In the center, there is a prominent, bright, yellowish-white galaxy that appears to be gravitationally lensed, showing multiple distorted images of itself. The title text is overlaid on the upper half of the image.

Introducing **General Relativity**

**Mark Hindmarsh
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WILEY

Introducing General Relativity

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This edition first published 2022
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Library of Congress Cataloging-in-Publication Data applied for

ISBN: 9781118600719

Cover Design: Wiley

Cover Images: © ESA/Hubble & NASA, S. Jha;

Acknowledgment: L. Shatz

Set in 10/12pt NimbusRomNo9L by Straive, Chennai, India

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Preface

Find a physicist. Sit them down, treat them to a coffee, and ask for their opinion as to the most beautiful theory that the subject has to offer. The chances are that their answer will be General Relativity.

Einstein's General Theory of Relativity, to give it its full and proper name, is over a hundred years old. Yet it offers a defiantly modern viewpoint, defining principles of how the Universe ought to work and establishing a mathematical framework upon them. It gives a radical, even shocking, reconception of a fundamental force, gravity. And it has maintained an exquisite agreement with observational data, making predictions of such subtlety that one of its key implications, the existence of gravitational waves, took over a hundred years to be directly verified.

This book is based on a lecture course at the University of Sussex, given by each of us at various times. The course is taken by final-year physics undergraduates, and has no special prerequisites, so we have attempted to limit the coverage of topics and to be as explicit as possible. For many undergraduates, a lecture course on General Relativity is the pinnacle of their theoretical education, and their main exposure to the modern methodology of physics as based on principles and symmetries.

Our aim is to make that experience as enjoyable as possible, while accepting that the pleasure comes not just from the astonishing physical implications, such as black holes, singularities, and gravitational waves, but from the elegance of the underlying mathematical structure. By the standards of textbooks on the topic, we have sought to create something that is genuinely introductory, yet which provides the mathematical tools to see the theory work in a quantitative way. We hope that something of the elegance of the theory emerges from the technical difficulty, along with an understanding of the physical predictions that have been so beautifully confirmed by decades of experiment and observation. Enjoy the challenge!

Mark Hindmarsh and Andrew Liddle
Helsinki and Lisbon, October 2021

Some Fundamental Constants and Astronomical Values

Newton's gravitational constant	G	$6.672 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Speed of light	c	$2.998 \times 10^8 \text{ m s}^{-1}$
Reduced Planck constant	$\hbar = h/2\pi$	$1.055 \times 10^{-34} \text{ m}^2 \text{ kg s}^{-1}$
Boltzmann constant	k_{B}	$1.381 \times 10^{-23} \text{ J K}^{-1}$
Solar mass	$M_{\odot}$	$1 M_{\odot} = 1.989 \times 10^{30} \text{ kg}$
Solar radius	$R_{\odot}$	$1 R_{\odot} = 6.957 \times 10^8 \text{ m}$
Parsec	pc	$1 \text{ pc} = 3.086 \times 10^{16} \text{ m}$
Electron volt	eV	$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$

Commonly-Used Symbols

t, x, y, z	coordinates	<i>defined on page</i> 4
c	speed of light	5
v	velocity (usually the relative velocity between two frames)	6
γ	Lorentz boost factor, $\gamma = 1/\sqrt{1-v^2/c^2}$	7
s or Δs	space–time distance (or interval) between two events	7, 19
u	velocity (of an object measured in a frame)	11
x^μ	coordinates (as a 4-vector)	18
Λ^μ_ν	Lorentz transformation	18
$\eta_{\mu\nu}$	Minkowski metric (for Special Relativity)	19
$g_{\mu\nu}$	metric of space–time	20, 50
δ^μ_ρ	Kronecker delta	21
τ	proper time	24
u^μ	4-velocity	24
p^μ	4-momentum	25, 83
a^μ	4-acceleration (this notation is also used for a generic 4-vector)	26
∂_μ	4-derivative	27
$\square$	D’Alembertian	27
G	Newton’s gravitational constant	37
M	mass (usually of a gravitating body)	37
m	mass (usually of a test body near a larger mass M)	37
Φ	gravitational potential	39
ds	infinitesimal interval in space–time	50
$D a^\mu$	covariant differential	60
$\Gamma^\nu_{\mu\sigma}$	connection/Christoffel symbols	60
,	partial derivative	63
;	covariant derivative	63
$R^\mu_{\nu\alpha\beta}$	Riemann tensor	69
$R_{\mu\nu}$	Ricci tensor	72
R	Ricci scalar	72
ρ	density	80
n	(particle) number density	81
j^μ	(particle) flux 4-vector	82
$T^{\mu\nu}$	energy–momentum tensor	83
p	pressure	85
$G_{\mu\nu}$	Einstein tensor	92
Λ	cosmological constant	95
r_S	Schwarzschild radius	104
t^μ	tangent vector (to a trajectory)	110
$a(t)$	scale factor of the Universe	158
$h_{\mu\nu}$	small perturbation to the metric $g_{\mu\nu}$	209

About the Companion Website

This book is accompanied by a companion website:
www.wiley.com/go/hindmarsh/introducingGR

The website includes: Instructor's Manual

Chapter 1

Introducing General Relativity

It is now more than a hundred years since Albert Einstein presented the final form of the General Theory of Relativity to the Prussian Academy of Sciences, in November 1915. Since then, it has migrated from an extraordinary achievement at the frontiers of physics, reputedly understood by only a very few, to a standard advanced undergraduate course. General Relativity (as it is usually called, commonly shortened to simply GR) is essential for the understanding of the Universe as a whole, wherever gravity is strong, and also whenever precise time measurements are made. The Global Positioning System (GPS), now built in to billions of devices around the world, would not work without the General Relativistic prediction that clocks run more slowly on Earth than in the satellites defining the GPS reference frame.

Part of the fascination of General Relativity lies in the personality of Einstein, and the way he is often presented as a lone genius working for years in isolation, finally to reemerge with the fully formed and beautiful theory we know today. In reality, he was in constant communication with other scientists, and others were working towards a relativistic theory of gravitation. The first such theory was actually written down by Gunnar Nordström in 1913, who attempted a direct relativistic generalisation of the Newtonian gravitational potential. Einstein was the first to understand that the appropriate dynamical quantity is the space–time metric itself, but the geometric aspect of General Relativity was probably first appreciated by the mathematicians Marcel Grossmann and David Hilbert. Einstein worked with his friend Grossmann, and had crucial correspondence with Hilbert before coming up with the final and correct formulation. Einstein himself took several wrong turnings during the years between 1907 and 1915 when he was working most intensively on the theory. The lone genius is a myth, but it is fair to consider that General Relativity is Einstein’s own, and crowning, achievement.

The technical complexity of General Relativity comes from several sources.

The fundamental objects of relativity are tensors, because the relativity principles (both special and general) are statements about the properties of physical laws under transformations between coordinates. Thus any General Relativity course must start with tensor calculus. General Relativity is a geometrical theory, treating space–time as a manifold, describing its dynamics in terms of geometrical quantities. Thus in approaching General Relativity the basic geometrical concepts developed by Bernhard Riemann and others from the mid 1800s — of connection, geodesic, parallel transport, and curvature — must be introduced. Tensor calculus and Riemannian geometry are not part of the standard mathematical equipment of a physics undergraduate. This was also true in Einstein’s undergraduate career, although there were courses on offer. It has been speculated that had he gone to an advanced geometry course, he would have later saved himself several years’ work.

A final difficulty is the one of translating the mathematical concepts into physical observables. General Relativity rethinks the fundamentals of space and time, which take part in physical processes rather than being a framework on which things happen. So deciding what is observable, rather than simply an artefact of a particular choice of coordinates, is difficult. Indeed, Einstein changed his mind a couple of times as to whether gravitational waves were real or not, and it took about fifty years for a unanimous view to emerge.

Gravitational waves, an early prediction of General Relativity, are amongst the hottest topics in physics following their direct detection by the LIGO/Virgo collaboration, announced in 2016. Their real significance is not so much as a triumphant vindication of Einstein’s theory; there was no serious doubt that gravitational waves existed following the careful measurements of the orbital decay of a binary pulsar system discovered by Russell Hulse and Joseph Taylor in the 1970s. Rather, the detection signals the beginning of a new branch of astronomy, which has the prospect of detecting violent astronomical events right back to the very earliest stages of the Big Bang. New detectors, similar to LIGO and Virgo, are being built in Japan and India, and a space-based gravitational wave detector called LISA is planned for the early 2030s. General Relativity will continue to be at the forefront of scientific research in the 21st century, as it was throughout the 20th.

Chapter 2

A Special Relativity Reminder

The Special Theory unites space and time · shorter lengths and longer times · seeing it with diagrams

Before launching into our account of General Relativity, we give a brief reminder of the main characteristics of its predecessor theory, the Special Theory of Relativity. This was introduced by Einstein in 1905, and is usually referred to by the shorthand Special Relativity. These theories have a rather different status to traditional physics topics, such as electromagnetism or atomic physics, which seek to understand phenomena of a particular type or within a certain domain. Instead, the relativity theories set down principles which apply to *all* physical laws and restrict the ways in which they can be put together. Whether those principles are actually true is something that needs to be tested against experiment and observation, but the assumption that they do hold has far-reaching implications for how physical laws can be constructed. In particular, the role of symmetries of Nature is highlighted, which is a defining feature of how modern physics is constructed; as such the relativity theories often give students the first glimpse of how contemporary theoretical physics is done.

Both the theories focus on how physical phenomena are viewed in different coordinate systems, with the underlying principle that the outcome of physical processes should not depend on the choice of coordinates that we use to describe them. Special Relativity restricts us to so-called **inertial frames**, where the term **frame** means a set of coordinates to be used for describing physical laws. As we will see, this restricts us to coordinate transformations which are linear in the coordinates, corresponding to coordinate systems moving relative to one another with constant velocity, and/or rotated with respect to one another. This turns out to be a suitable framework for considering all known physical laws *except for those corresponding to gravity*.

Einstein's remarkable insight, leading to the General Theory of Relativity, was

that allowing arbitrary non-linear coordinate transformations would allow gravity to be incorporated. Indeed, if we want to allow non-linear transformations, we *have* to include gravity. Understanding the motivations for, and implications of, this extraordinary statement is the purpose of this book. But for now, we place the focus on Special Relativity, emphasising those features that will later generalise.

2.1 The need for Special Relativity

In Newtonian dynamics, the equations are invariant under the Galilean transformation which takes us from one set of coordinates (t, x, y, z) to another (t', x', y', z') according to the rule

$$t' = t; \quad x' = x - vt; \quad y' = y; \quad z' = z, \quad (2.1)$$

where v is the relative speed between the two coordinate systems, which have been aligned so that the velocity is entirely along the x direction. [NB primes are not derivatives!] Each coordinate frame is idealised as extending throughout space and time, providing the scaffolding that lets us locate physical processes in space and time. We introduce an **event** as something which happens at a specific location in space and at a specific time, such as the collision of two particles.

Typically any observer will want to choose a coordinate system to describe events, and will be located somewhere within the coordinate system. Commonly, though not always, observers will decide to choose coordinate frames that move along with them as a natural way to describe the phenomena as they see them, and so it can be useful to sometimes think of a coordinate system as being associated to a particular observer who carries the coordinate system along with them. For instance, we might consider two different observers moving at a constant velocity with respect to one another, and ask how they would describe the same physical process from their differing points of view.

When we refer to **invariance** of a physical quantity, we mean that a physical quantity expressed in the new coordinates is identical to the same quantity expressed in the old ones. That means that observers in relative motion agree on its value.

In particular, acceleration is invariant in Newtonian dynamics; it depends on second time derivatives of the coordinates of, for example, a moving particle, and the second time derivatives of x and of x' are equal. An everyday example is that an object dropped in a train moving at constant velocity appears, to an observer in the carriage, to follow exactly the same trajectory as it would were the train stationary.

The Galilean transformation is characterised by a single universal time coordinate that all observers agree upon. Combining relative velocities in each of the

coordinate directions means that generally $x' \neq x$, $y' \neq y$, and $z' \neq z$, but t' always remains equal to t . The idea of a universal time sits in good agreement with our everyday experience. However, our own direct perceptions of physical laws probe only a very restrictive set of circumstances. For example, we are unaware of quantum mechanics in our day-to-day life, because quantum laws such as Heisenberg's Uncertainty Principle are significant only on scales far smaller than we can personally witness. Hence, we cannot immediately conclude that invariance under the Galilean transformation should apply to all physical laws.

Indeed, it was already known in Einstein's time that Maxwell's equations, describing electromagnetic phenomena including the propagation of light waves, are not consistent with Galilean invariance. For example, they state that the speed of light is independent of the motion of a source, whereas the Galilean transformation would predict that light would emerge more rapidly from a torch if its holder were running towards you. In a famous thought experiment (i.e. an experiment carried out only in the mind, not in the laboratory), Einstein tried to envisage what would happen if one tried to catch up with a light wave by matching its velocity, knowing that Maxwell's equations would not permit a stationary wave.

One possible resolution of this would be if there were special frame of reference in which Maxwell's equations were valid, a frame that came to be known as the aether. However, since the Earth revolves around the Sun, it cannot always be stationary with respect to this aether. In the late 1880s, Albert Michelson and Edward Morley sought to detect the motion of the Earth relative to this aether, using an interferometer experiment. It should have had the sensitivity to easily see the effect, given the known properties of the Earth's orbit, yet no signal was found, putting the existence of the aether in doubt.

From the viewpoint of wanting a unified view of physical laws, it makes little sense that different types of physical laws should respect different invariance properties. After all, electromagnetic phenomena lead to dynamical motions. This incompatibility posed a stark problem for physics.

Einstein's 1905 paper resolved this seeming paradox decisively in favour of electromagnetism. Based on his thought experiments, he demanded that physical laws satisfied two postulates:

- 1. The laws of physics are the same in all inertial frames.**
- 2. The speed of light, denoted c , is the same in all inertial frames and independent of the motion of the source.**

As remarked above, inertial frames are those which move with a constant velocity with respect to one another. The requirement that the laws of physics be the same in each is inherited from the Galilean transformation, which also requires it. Another way of expressing this first postulate is to say that there is no possible experiment an observer can carry out to measure their absolute velocity.

But the second postulate then requires that the coordinate transformation between frames must mix space and time, as we are about to see. It is inconsistent with the notion of a universal time coordinate, and requires that invariance under the Galilean transformation be abandoned. If Nature's laws are to be invariant under coordinate transformations, the invariance must be of another type.

2.2 The Lorentz transformation

Hendrik Lorentz, in 1904, had already discovered a transformation that left Maxwell's equations invariant, and it now bears his name. We will derive it under the assumption that the transformation is linear, like a Galilean transformation, and reduces to a Galilean transformation in the limit of relative velocities much less than that of light.

Consider a frame, which we call S' , moving relative to the original frame S with velocity v along the x -axis, so that we can assume $y' = y$ and $z' = z$.¹ This is shown in Figure 2.1. Linearity lets us write

$$t' = At + Bx; \quad x' = Ct + Dx, \quad (2.2)$$

where A , B , C , and D are constants. Now, the origin of S' is moving relative to S at velocity v , so $x' = 0$ corresponds to $x = vt$, implying $C = -vD$. So the second of the above equations becomes

$$x' = D(x - vt). \quad (2.3)$$

By symmetry, the same equation must hold for transforming back from S' to S , exchanging $v \rightarrow -v$, so

$$x = D(x' + vt'). \quad (2.4)$$

Now, we use the assumption of a constant speed of light. At the instant when the two coordinate systems agree, send out a pulse of light along the x -axis. Then $x = ct$ in the original frame, and $x' = ct'$ in the new one. Substitute this in to equation (2.3) to get

$$x' = D(1 - v/c)x, \quad (2.5)$$

¹We will refer to such a transformation as a **Lorentz boost**. The general linear transformation satisfying Einstein's postulates, referred to as a Lorentz transformation, also permits rotations of the spatial coordinate frame.

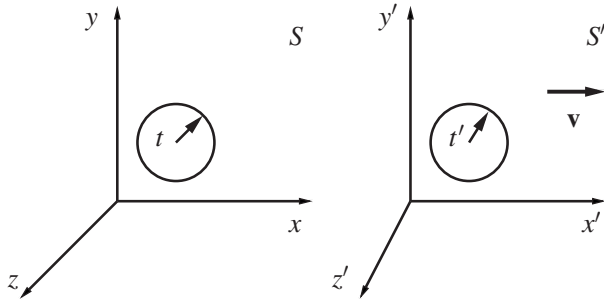


Figure 2.1 A frame S' moving relative to another frame S , with velocity $\mathbf{v}$ along the x -axis. Demanding that the transformation relating the coordinates (t', x', y', z') to (t, x, y, z) is linear and preserves the speed of light c , uniquely fixes it to be the Lorentz boost, equation (2.7).

and into equation (2.4) to get

$$x = D(1 + v/c)x'. \quad (2.6)$$

Consistency of these two requires $D^2 = 1/(1 - v^2/c^2)$. This rules out the Galilean transformation, which would have required D to be equal to one.

We write this as $x' = \gamma(x - vt)$, where the famous γ -factor is $\gamma = 1/\sqrt{1 - v^2/c^2}$. This measures the strength of relativistic effects. Again by symmetry, $x = \gamma(x' + vt')$. Combining these gives $vt' = \gamma^{-1}x - x' = \gamma^{-1}x - \gamma(x - vt)$ which we can rearrange to get $t' = \gamma(t - vx/c^2)$. We can finally write the Lorentz boost (in the x -direction) as

$$\begin{aligned} ct' &= \gamma \left(ct - \frac{v}{c}x \right); \\ x' &= \gamma(x - vt); \\ y' &= y; \quad z' = z. \end{aligned} \quad (2.7)$$

Notice the strong symmetry if we think of ct as a coordinate. The Lorentz boost is the unique linear transformation relating time and space coordinates satisfying the postulates of Special Relativity. It also reduces to the Galilean transformation in the limit $v \ll c$.

If we define the distance of a point in space and time from the origin as

$$s^2 = -c^2t^2 + x^2 + y^2 + z^2, \quad (2.8)$$

this quantity, which is known as the space–time **interval**, is invariant under the

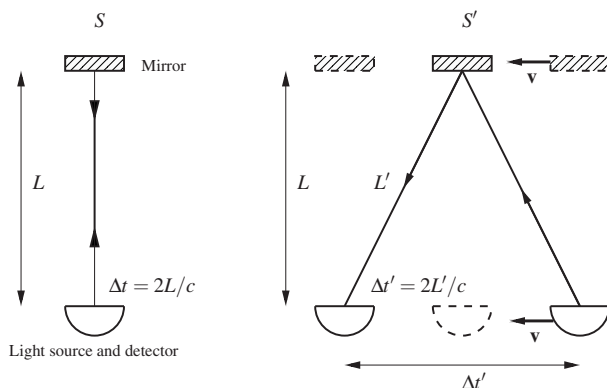


Figure 2.2 Light pulses are sent out from the origin in the y direction in frame S , bouncing off a mirror at a distance L , and returning to the origin at time $t = 2L/c$ later. As soon as they are received another pulse is sent. In frame S' , the path is longer, and the arrival times back at the detector correspondingly later. If we think of the arrival times of the pulses as the ticking of a clock, the conclusion is that the moving clock runs slower, by a factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ (see equation (2.9)).

Lorentz boost (see Problem 2.1). Hence, all inertial observers agree on the interval between two events, even though they will not generally agree on how much of that interval is in the space direction and how much in the time direction. Invariant quantities such as this will play an important role throughout our book. Indeed, a more modern and fundamental point of view would be to start with the principle that the interval is to be invariant, and to show that the most general linear transformation consistent with that property is the Lorentz transformation.

2.3 Time dilation

A consequence of the Lorentz transformation is that moving clocks run more slowly. Consider a train of pulses of light sent out from the origin of a frame S , reflected from a mirror a distance L away, and returning a time Δt later. As soon as a pulse returns another is sent.

Consider now how this looks in frame S' , moving parallel to the mirror with speed v (see Figure 2.2). In the latter frame, the path taken by the light is longer, and due to the constancy of the speed of light in all frames, the time taken to return $\Delta t'$ is longer.

In Frame S , the pulse return time is $\Delta t = 2L/c$. In Frame S' , it is $\Delta t' = 2L'/c$,

where $L'^2 = L^2 + (v\Delta t'/2)^2$. Combining these gives

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}. \quad (2.9)$$

Consider the train of light pulses as the ticking of a clock in the S frame. As $\Delta t' > \Delta t$, the S' observer (who sees a moving clock) notices that its ticks are slower. The situation is symmetric, so whenever we look at a moving clock, we will see it running slower than our own.

A classic example of time dilation comes from muons created in the upper atmosphere through impact of highly energetic cosmic rays. Muons are unstable particles whose lifetime, as measured in the laboratory, is only 2.2×10^{-6} seconds, which suggests that they could travel at most $ct \simeq 660$ m before decaying. But in practice, muons created high in the atmosphere are detected at the Earth's surface. The reason is that they are moving close to the speed of light and hence, from our perspective, their evolution is heavily time-dilated with the apparent lifetime much greater than the rest-frame lifetime.

2.4 Lorentz–Fitzgerald contraction

A complementary phenomenon to time dilation is that moving objects appear shorter, known as the Lorentz–Fitzgerald contraction. To derive it, take a rod that is stationary in frame S with ends at $x = 0$ and $x = L$, and choose a moving frame S' such that the spatial origins coincide at $t = t' = 0$. In this frame, the rod appears to move with speed v in the $-x'$ direction. What is the length of the rod in the moving frame?

To answer the question, we need to be a bit more precise about how we determine lengths. We can define the length of the object to be the difference in the x coordinate between the ends, when measured at the same time. In the S frame, where the problem was set up, this is trivially L , at any time t .

However, in the S' frame, the calculation is not so trivial, as we need to establish the positions of the ends of the rod at the same value of t' . It is convenient to take this time to be $t' = 0$, as we set up the coordinate systems so that their origins coincide at this time. The Lorentz transformation (2.7) tells us that measurements taken at time $t' = 0$ must happen at times $t = xv/c^2$ in the frame S . Hence, the measurement the position of the end of the rod ($x = L$) at $t' = 0$ in the frame S' occurs at time $t = Lv/c^2$ in frame S (see Figure 2.3).

The Lorentz transformation (2.7) of the coordinates corresponding to the mea-

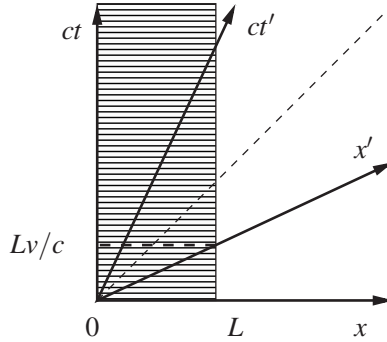


Figure 2.3 A rod of length L is stationary in a frame with coordinates (ct, x) , with its left-hand end at the origin. A frame is moving at speed v in the $+x$ direction, with coordinates (ct', x') coinciding with the original frame at $t' = 0$ and $x' = 0$. A measurement of the position of the end of the rod at time $t' = 0$ happens at time $t = Lv/c^2$ in the original frame. Knowing that the end of the rod is at $x = L$, the position of the end of the rod in the moving frame at time $t' = 0$, and hence its length L' , can be calculated from the Lorentz transform (2.7). The conclusion is that $L' = L/\gamma$, i.e. a moving rod is shorter.

surement $(ct, x) = (Lv/c, L)$ tells us that it happened at S' coordinate

$$x' = \gamma L \left(1 - \frac{v^2}{c^2} \right). \quad (2.10)$$

Hence, the length of the rod in the moving frame is

$$L' = L \sqrt{1 - v^2/c^2}. \quad (2.11)$$

The rod is therefore shorter, when measured in the moving S' frame.

Returning to the example of the atmospheric muons of the previous section, in the muon rest-frame the lifetime to decay is indeed 2.2×10^{-6} seconds. Nevertheless, muons can still reach the ground from the upper atmosphere, which from their point of view is because the atmosphere is approaching them at relativistic speeds and is Lorentz–Fitzgerald contracted. Ultimately, the physical outcome, muons reaching the ground and being detected there, has to be the same from either our point of view or the muons. Hence, we see that time dilation and Lorentz–Fitzgerald contraction are really two different sides of the same coin; it couldn't make sense to have one without the other.

2.5 Addition of velocities

We now ask how velocities change between moving frames. Consider an object moving with velocity $\mathbf{u}$ along the x axis in frame S . What is the velocity $\mathbf{u}'$ measured in frame S' , which is moving with velocity $\mathbf{v}$ along the x axis?

In Newtonian dynamics, where the Galilean transformation applies, velocities simply add. But we can already expect that this will not hold in Special Relativity, because of the postulate that the speed of light is constant in all frames regardless of their relative velocity.

In the S' frame, the component of velocity along the x -axis is given by

$$u' = \frac{\Delta x'}{\Delta t'}, \quad (2.12)$$

where the distance $\Delta x'$ is travelled in a time $\Delta t'$. Using the Lorentz boost given by equation (2.7), we can write (the γ factors immediately cancel out)

$$u' = \frac{\Delta x - v\Delta t}{\Delta t - (v/c^2)\Delta x} = \frac{\Delta x/\Delta t - v}{1 - (v/c^2)\Delta x/\Delta t}. \quad (2.13)$$

Since the x component of velocity measured in frame S is $u \equiv \Delta x/\Delta t$, this becomes

$$u' = \frac{u - v}{1 - (v/c)(u/c)}. \quad (2.14)$$

Going between frames in the opposite direction simply swaps v for $-v$ in the Lorentz transformation, so we can also write the equivalent equation

$$u = \frac{u' + v}{1 + (v/c)(u'/c)}. \quad (2.15)$$

In the non-relativistic limit $v \ll c$ and $u \ll c$, we recover the Galilean law $u' \simeq u - v$ that applies in our everyday experience. Taking $u = c$, we find that $u' = c$ regardless of the value of v , confirming successful implementation of the frame-independence of the speed of light. With a little more work, we can show that as long as both v and u are less than c , then so is u' . There is no way to generate a speed greater than that of light from speeds which are less, so the speed of light forms a barrier that cannot be crossed.

Some further useful properties of the Special Relativistic velocity composition law are explored in Problem 2.4.

2.6 Simultaneity, colocality, and causality

We begin with a few definitions. We recall that an event refers to something which takes place at a specific location in space at a particular time. The set of all possible events is the **space–time**. Events are said to be **simultaneous** if they happen at the same time, and to be **colocal** if they happen at the same location. Note that different observers do not agree on which events are simultaneous or colocal. **Coincident** events happen at the same time and place, and everyone agrees on that.

Consider two events. Unless they are simultaneous, there is always a Galilean transformation which makes them happen at the same location, while in contrast there is no such transformation that can make two events appear simultaneous unless they are so in all frames. In Special Relativity, the situation is more complicated.

Consider the following question. Suppose an observer S sees event A, at location x_A , happening before event B at x_B (i.e. $t_B - t_A > 0$). We can assume both x_A and x_B are on the x -axis with $x_B > x_A$. What condition needs to be satisfied so that *all* observers see event A happening first?

We use the Lorentz transformation

$$ct'_B - ct'_A = \gamma \left[ct_B - ct_A - \frac{v}{c}(x_B - x_A) \right]. \quad (2.16)$$

This is positive provided $ct_B - ct_A > (v/c)(x_B - x_A)$. So, if $(x_B - x_A)$ is not too large, the order of events is observer-independent,

In the new frame, the distance between the events is

$$x'_B - x'_A = \gamma[x_B - x_A - v(t_B - t_A)]. \quad (2.17)$$

We see that if event A is able to cause event B, then we can always find a frame in which $x'_B - x'_A = 0$, that is, there is a frame in which the events are colocal.

Conversely, if $ct_B - ct_A < (v/c)(x_B - x_A)$, we can always find a frame in which $ct'_B - ct'_A = 0$. In this frame, the events are simultaneous.

A crucial notion is **causality**. Event A can only influence event B if it can send a signal, which cannot travel faster than light. Event A can therefore cause event B only if

$$\frac{x_B - x_A}{ct_B - ct_A} \leq 1. \quad (2.18)$$

This is exactly the condition as to whether there is a well-defined order, as no observer can have $v > c$. Relativity therefore respects causality; if it is possible for event A to physically influence event B, then all observers will agree that event

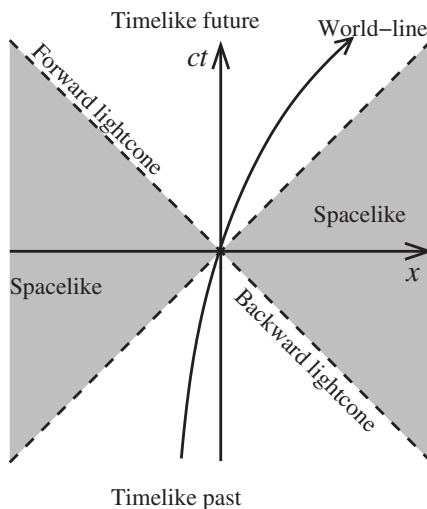


Figure 2.4 Diagram of Minkowski space-time, showing how the light cone divides it into regions separated from the origin by timelike and spacelike (shaded) intervals. Also shown is the world-line of a particle which is steadily accelerating in the $+x$ direction.

A happened first. The general statement is that event A can influence event B if the space-time interval satisfies $s^2 < 0$. Note that s can be imaginary.

2.7 Space-time diagrams

We recall that an event occurs at a given point in space-time, and an observer specifies it by the values of four coordinates. There is a useful graphical way of representing events and the causal relationship between them, called a space-time diagram, illustrated by Figure 2.4 for a simplified situation with one space coordinate. The convention is to scale the axes so that light travels at a 45° angle.

The collection of all possible paths taken by light rays emanating from and arriving at a space-time point $\mathcal{O}$ is known as the **light cone**. It consists of all points satisfying $s = 0$, where s is the interval defined above. Note that each point in space-time has its own light cone.

The light cone divides space-time into three regions, **timelike future**, **timelike past**, and **spacelike**. Only events in the timelike past of $\mathcal{O}$ can influence it, and it can only affect points in its timelike future. Space-time points on the light cone have **lightlike** intervals.

Timelike events are separated by an interval $s^2 < 0$, spacelike ones have $s^2 > 0$,