Karlheinz Schüffler

Proportions and Their Music

What Fractions and Tone Sequences Have to Do with Each Other

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For my grandchildren
Anthea, Christian,
Helena and Hendrik

Foreword

finally he had recognized that not only in this modulatio (musica) the numbers rule, but they complete everything in the first place.

Augustine (de ordine XIV, From [5], p. 124)

There have always been numbers from which a mysterious magic emanated. Today they may be gigantic prime number monsters, which in their unimaginable size want to bring us closer to the unreachable edge of the comprehensible; in times of the awakening analysis, they were irrationalities and transcendences of famous numbers such as π and e as well as the kingdom of functions; in the many centuries of antiquity above all – chosen for higher things – they have been the counting numbers of our world of experience. Cultures and religions are connected with the numbers one, two and three; there is hardly a fairy tale without seven somehow playing its part, and the twelve apostles are certainly not the only example of the importance this number claims for itself. And even more specially the number six: it is the sum of its real divisors – and therefore perfect and the very first of its kind!

We can confidently assume that in former times there must have been a bickering among the numbers – certainly not without danger – about the undisputed sole significance. Reliable information about the outcome of this dispute is not known – but we have good reason to believe that some of them came to their senses and realized that together they were – eventually – capable of quite different things. The "1" had already recognized this in the beginning of its being: It connects the *rationes arithmetica* ($1 : n$) with the *rationes harmonia* ($n : 1$) since forever.

► So four numbers got together and decided to found the musica theoretica: 6 – 8 – 9 – 12.

These four numbers were the first to recognize they could still guard a totally different perfection as a treasure than if they had remained alone: The two inner ones could call themselves the arithmetic and harmonic mean of the two outer ones. And they discovered exciting magic: When they revealed their proportions to Queen One and they switched from the arithmetic to the harmonic world, the inner ones among them reversed their roles as mean values: Arithmetic became harmonic, and harmonic became arithmetic. But what they still sheltered as a higher secret was their belief that they truly lived in just relations among themselves. Since being able to predict the future, they were certain of a number-nuisance called the geometric mean, which claimed the place between the two inner ones. A brave man from Gerasa – Nicomachus was his name – told them that just as the outer ones were in inverse proportion to this number-nuisance, so were the two inner ones too.

Envious of all this perfect symmetry, the "10" decided to at least occasionally ask to be admitted to this alliance, which was indeed granted to it from time to time and then with reservations – even if not by the great Pythagoras. But instead, other greats realized where its contribution to musica theoretica could ultimately lead.

Thus the Harmonia perfecta maxima 6 : 8 : 9 : 12 has arisen, and with the participation of the 10 – which was given the title of a contra-harmonic mean – or mediety – and which also managed to smuggle its front geometric proportional into the family as a little contra-arithmetic sister – the Harmonia perfecta maxima founded what was later called diatonic music of perfect octaves, fifths and thirds.

In this book, we will undertake a tour on the connection between the two cultural sciences of mathematics and music, which since forever and throughout the ages has led to everything that has become the basis of the musical theory of tones, intervals, sounds and scales. This is the theory of the

- **proportions and their consonance**,

which, in its millennia-long alliance, has produced the whole edifice of concepts and their living anchors among themselves.

The book is intended for both mathematicians and musicians, but above all for **readers interested in cultural history**. We also hope that **schools** will want to make use of the **helpful impulses** that we have incorporated into the conception of the text.

- For "**mathematicians**", the book offers a new, comprehensive presentation of the ancient theory of proportions, its axiomatics and the laws arising from it. At the same time, a new interesting application potential arises from the interconnection with music-theoretical examples.
- For the "**musicians**", the book offers the technical background from whose laws the theory of scales, chords and, above all, terminology can be understood in a sustainable way; in doing so, we also give a great deal of space to the concerns of the organ and its timbres born out of proportions. In this way, we recommend to **students** in particular that they try to access the fundamental theoretical concepts of their subject also through an approach guided by mathematics.
- For those **interested in cultural history**, we would like to give an insight into the vitality of the relationship between these two sciences, which has been and still is the focus of heated debate from antiquity to the present day.
- In particular, we recommend reading this book to those **teachers** who, for example, would like to work with their students respectively their pupils on interdisciplinary – and

therefore particularly acceptable and memorable – areas of application within the framework of separate advanced courses in their subject areas.

From a **mathematical point of view**, experience has shown that it is an enormous help to see that behind a "grey theory of calculation" there is a practice that can be understood and described precisely through this – in our case, the multitude of sound constructs.

Conversely, from a **musical point of view**, it is not really wrong to add mathematical elements to the musical levels of the scale theories, which not only serve the understanding excellently, but also – together with others – give the subject of "music theory", apart from its artistic enjoyment, the attitude of a science with all the characteristics that go with it.

► In short, this book is devoted to the mathematics of the ancient theory of proportions; we develop the subject matter from the principles and concerns of the also ancient music theory. We describe this theory and we use a colloquial language – very often and whenever possible – to explain our maths. Beyond that we choose all our application references and examples exclusively from the field of music and its theory of diatonic and non-diatonic structures and constructions. The guiding principle of our ideas is the "Harmonia Perfecta Maxima".

Introduction

> The goddess Harmonia had been the daughter of Jupiter because of celestial harmony, and of Venus because of pleasure. . . .
>
> Remigius (From [5], p. 129).

Since early antiquity, the two sciences of mathematics and music have belonged to the circle of "arts" – namely to the "Septem artes liberales":

- **geometria, arithmetica, astronomia and musica**

were what the ancient worldview regarded as "the sciences and their culture". But musica was not – as is commonly assumed – the art of making music: rather, it consisted in describing the regularities of **tones and their relationships to one another** with the aid of the laws of arithmetic of whole numbers. A very precise distinction was made between

- **musica theoretica and musica practica**;

and the first was considered a mathematical science; its object of theory were the consonances. These, however, were again defined as "proportions of monochordal divisions".

► For this reason, it is not surprising that, when viewed in the light of day, we encounter many basic concepts of music theory in a seemingly mathematical guise: "semitones, whole tones, octaves, sevenths, fifths, thirds, seconds, circle of fifths" – these are some of the essentials of the structural instruments of scale theory. We also encounter the word "**harmonic**" – at first seemingly separately – in both disciplines; however, it is actually anchored there in – several – profound units of meaning.

Strictly speaking, then the unique connection between music and mathematics has its early historical anchorage in the interplay between the **theory of the "consonantiae" and the theory of the "proportiones".**

In fact, the historical "theory of music" – especially that of Greek antiquity – is an ongoing, continous discussion about consonances; it is just as finely detailed as it is, unfortunately, also inconsistent with many opposing terminologies and views. These consonances were – as mentioned – on the one hand and originally obtained as proportions of certain (permitted) ratios on stringed instruments, from which the Greek term *"chordōn symphōnia"* (interplay of musical instruments) originates. On the other hand, however, number games dominated ratios over the "musica practica".

This is mainly the "theory **of proportions**" as described by Aristotle (384–322 BC) and Euclid (330–275 BC). How closely this theory of proportions was connected with the science of music – i.e. the theory of consonantiae – can be seen from the fact that it originally started with Pythagoras as a purely musically motivated theory of intervals and scales; it then developed into a mathematical theory of number proportions and was finally developed in its most abstract form – in which even an abstraction from numbers to sets can be interpreted – essentially by Eudoxos (408–347 BC).

► It shows that the comparison of any "magnitudes" is based on a concept of proportions, which only knows the natural numbers and with them describes the commensurability of those magnitudes. Thereby, the core of the ancient "commensurability" consists in the assumption that two comparable things are always commensurable – this means: There is always one unit, so that both magnitudes are composed of this unit (as a whole number).

This assumption was – still at times of Pythagoras and long after – a law, and questioning this could end badly. For us it is certainly clear, that the demand for commensurability was equal to the sole existence of rational numbers – irrationalities were – so to say by decree – not existent. But the diagonal in the unit-square could have eliminated this error: Because there the length of the diagonal is $\sqrt{2}$ and this number cannot be written in the form *n*/*m*, the commensurability of $\sqrt{2}$ with 1 is not given. Characteristically, these numbers have always been called "not specifiable quantities".

The **science of music** thus corresponded to the science **of the proportions of tones –** the latter being quite free of a physical robe of existence. Thereby we can also use the analogy (equality) of

- "**Proportion of two tones**" and "**Interval**".

Thus, the Pythagorean (pure) fifth corresponds to the (ancient notated) proportion 2 : 3, the octave to the proportion 1 : 2 and the unison to the proportion 1 : 1. While Pythagoras created his music exclusively from these three building blocks by joining them together ("adjointing, iterating") and allowed nothing other than these three (prime) numbers 1, 2 and 3 and "derived numbers" from them, we encounter incredibly bizarre interval proportions in Greek tetrachordics with its doric, phrygian, (mixo-)lydian and many other scales. Thus, for example, the Phrygian scale of Archytas of Taranto (in the fourth century BC) knows the tiny enharmonion with the proportion 35 : 36 – i.e. of the rank of a very small quarter tone – as a "consonant" interval, apparently carrying the prime number 7 with it – one of many examples.

Both in the field of musical constructions and in the arithmetic of proportions itself, the "joining together" of proportions to form so-called

- **Proportion Chains**

plays an important role. This is how we derived – transferred to music – chords and scales. A particularly remarkable interweaving arises for these chains of proportions when, in addition, the classical mean values ("medieties") are brought into play: Already the kingdom of the universally known

- **Babylonian mean values (arithmetic – harmonic – geometric)**

offers all by itself a wealth of highly interesting symmetries between chains of proportions and their inversions, the "reciprocals", and this network of relationships can be described even more comprehensively if we take these and other – in antiquity considered to be a science on their own –

- **Medieties (mean values)**

into the consideration. To give a simple example:

► The arithmetically averaged proportion of the fifth 2 : 3 ≅ 20 : 30 yields the perfect **major** third 20 : 25 ≅ 4 : 5, the harmonically averaged proportion 20 : 30, on the other hand, yields the perfect **minor** third 20 : 24 ≅ 5 : 6. Conclusion: major and minor of the just diatonic are born; the proportion chain of the diatonic minor triad is the "reciprocal" of the proportion chain of the major triad.

In its theory of proportions, antiquity knew about ten mean values – "medieties" – one suspects that a music-theoretical game with intervals, scales and their chords opens up in its own world of inexhaustible relationships.

To the Chapter Overview

In **Chap. 1 (Proportions)** we begin with some reflections on the origins of the connections between mathematics and music – as far as it is of importance in the context of the theory of proportions**.** Then we begin this theory based on historical concepts. Here we develop the set of rules of ancient arithmetic laws from a minimal stock of given "plausible" basic rules, to which the role of "axioms" can be assigned and which can be identified in historical descriptions. The chapter concludes with a section on proportion fusion – that means the layering of two ore more intervals to its sum or its "Adjunction", respectively "Fusion" – and we describe the calculus of proportion equations and its algebra.

Chapter 2 (Proportion Chains) establishes all the constructive elements of dealing with proportions and the necessary rules and justifying theorems. Proportion chains are inherently compositions of several proportions into a more complex construction. We then develop, thanks to a structure-giving algebra and by means of the mathematical notions like

- Similarity
- Reciprocity
- Symmetry
- Composition - the ability to join

an ordering system for such chains of proportions. A musical counterpart of these constructions are intervals, scales and their chords. And it is precisely this ordering algebra of proportions that is also responsible for corresponding applications in these musical areas. The interplay of symmetry and reciprocity is particularly attractive; in the proportion chain theorem, this algebra is shown comprehensively and in as general a form as possible.

Chapter 3 (Medieties) is devoted to mean values. First, we present the three historically abstract and in the literature also readily available description possibilities for "mean value proportions", which we then combine by the "**Babylonian medietary trinity**"

- **geometric – arithmetic – harmonic**

with the theorems of **Nicomachus** and **Iamblichos**, which had acquired their perfect universal significance in the theory of music of that time with the description of the "**Harmonia perfecta maxima**" in the case of the Pythagorean octave canon, and we present the general version of these important principles of harmony and symmetry, which shows that all those formerly marvelled number games are general and natural.

The effort to obtain chord and scale structures directly from averages of existing proportions led to further – nowadays largely unknown – medieties, such as the "contra-harmonic mean" or the "contra-arithmetic mean" or to "higher front, middle or back proportionals" and many other deeper mysteries. At the heart of this mathematics are the **theorem on the symmetry of the third proportional** and the **theorem on the symmetry of the classical means**, which describes the chains of proportions – usually tripartite – formed from these means in terms of their symmetry and similarity properties and their relationships to each other, as well as their calculation formulas. This is followed by a section characterizing the **geometric mediety** as the "power center" of all mean values. We present the main symmetry mechanisms of this undoubtedly most important mediety in a goal-oriented **theorem on the harmonia perfecta maxima abstracta**. This is followed by **the theorem on the harmonia perfecta maxima diatonica** of pure diatonics, in which we describe and prove all symmetries of general five-step musical proportion chains.

In Chap. 4 (Proportion Sequences of Babylonian Medieties), we first discuss the basic possibilities of extending given proportions or chains to arithmetic, geometric or harmonic chains. This is now followed by a discussion of the **contra medieties sequences**, which can be defined as sequences of mean values to an exponential family of proportion parameters. The contra medieties show a deeper inner threefold structured symmetry, which is subordinated to the common (!) geometric mean, as long as one orders this infinite sequences of medieties accordingly.

▶ Here we have certainly entered uncharted territory, and the view of the musical interval ratios of all the infinitely many possible contra medieties described in some special literatures to biblical dimensions can certainly benefit from our tighter mathematical description.

The second focus consists of a presentation of the iteration procedure for sequences of **Babylonian medieties** – with the result that we obtain two-sided pairs of proportion chains with infinitely many members – with the remarkable observation, that their magnitudes all lie on one curve – the **hyperbola of Archytas –** and furthermore continue the symmetries of ordinary finite Babylonian proportion chains into their ramified infinities – a **Harmonia perfecta infinita** is achieved – both for the sequences of Babylonian medieties and for the sequences of contra medieties.

▶ This theorem on the "Harmonia perfecta infinita" embodies both an inspiring and unique combination of analysis and geometry on the one hand and ancient musicology and its modern theory on the other, and it represents – in some sense – the mathematical center of our text.

Finally, in the last **Chap. 5 (The Music of Proportions)**, we consider a variety of applications of the symmetries found around proportion chains and their Harmonia perfecta maxima in the musical empire of intervals, chords, and scales.

We start with the laws of string and tone – that indispensable instrument "**monochord**", which represents the connection of ***audible but not measurable with measurable but not audible proportions.***

The theory of proportion chains then accompanies us into the presentation of some musical tonal systems – such as the pythagorean, the diatonic and the so-called ekmelic systems.

▶ We have placed great emphasis on the development of the **ancient intervals** and their laws from the rules of calculation of the theory of proportions, and in particular we offer a systematic discovery, acquaintance with and practice in the use of all the significant whole, half and quarter tones as well as the commas of harmony from the idea of proportions and its laws.

The connection between music and proportions is also often seen in chordal music: The proportion chain 4 : 5 : 6 of the major triad yields in its reciprocal chain 10 : 12 : 15 – as already mentioned – the minor triad; that here also the arithmetic mean 5 (of the numbers 4 and 6) changes to the harmonic mean 12 (of the numbers 10 and 15), however, no longer surprises us after reading the central theorem about the symmetries of the **medieties** chains as well as Nicomachus' theorem.

An extraordinary role in the construction of all scales, from the ancient Greek and ecclesiastical modes to the temperaments of the Bach era and finally to the simplified major and minor of our time, is played by the **tetrachords** – those elements which characterize the fourth as a four-note scale composed of three steps. These structures can also be used to describe the architecture of **gregorian** and **ecclesiastical** tonal forms. The section **Modology** takes up this theme. A steep course accompanies the reader from the universal tonal system of antiquity, the "**systema teleion**" and its **octochords**, to the Greek and **ecclesiastical tonalities.**

And in the last section **(Proportions and the Organ)**, we explain the **arithmetic of the foot-number of stops ("Organ Stop Calculus")**, and this topic is typical of organ music and highly relevant to organists, including the phenomenon of acoustic "32-foot basses" and similar tricky physical-mathematical applications to acoustic proportions. As a result, it is more likely that we will gain a new understanding of the colorfulness of a given organ in its capacity as an orchestral instrument with its almost infinite sound possibilities.

► This arithmetic of the organ stops is also of an original nature and defined by the concept of proportions of intervals and combinations of sounds.

This special topic is rounded off by some examples from the exciting real world of the **dispositions** of this instrument.

In an **appendix**, there is also, among other things, a reference table of almost all the ancient intervals of the diatonic with their proportions, their frequency measures and their cent measures. We have also added those four functions whose graphic progressions offer both the mean value relationships and the symmetry laws of the Harmonia perfecta to the eye as memorable elements of the theory.

Finally Our exposition will make use of quite common forms of mathematical reading, which, apart from a way of writing and designating that has become customary, also consists in defining terms – especially the decisive ones – (reasonably) sharply wherever possible, and in presenting all facts as well as their internal connections, discoveries and conclusions in a familiar orderly system of definitions, propositions (most of which are called "**theorem**"), their proofs as well as helpful remarks and **musical examples**. In fact, we have deliberately avoided other applications of the theory of proportions, especially geometrically motivated ones, except in a single case. Thus, the reader will find suitable examples from the world of music theory not only in the last large Chap. 5, but also accompanying the entire reading.

► **Reading Suggestions**

We would also like to point out that this book tries to meet the different interests as well as knowledge and inclinations of its different readership: We can very well imagine that especially the fifth chapter is able to establish the entry as well as the usual start in the first chapter – and whoever then subsequently wants to get to the bottom of mathematical things will certainly soon be on the desired course of success by turning back the pages. Certainly, our diverse music-theoretical examples, which accompany the complete text, may also contribute to this.

For Use: Proportion Convention *a*:*b* or *b*:*a*?

What is an octave, what is a fifth?

Wikipedia says about the former one: "**Octave** (rarely *octav,* from **Latin** octava 'the eighth') is the term used in music for the **interval** between two notes whose **frequencies** act like 2 : 1."

In fact, this "ratio" has become the predominantly common answer to the question posed at the beginning – and in the case of the fifth, the indication of the ratio also follows 3 : 2. What is missing from the above definition, however, is an indication of which arrangement this ratio refers to. If we have two tones a and b that form an octave, which frequency ratio is then meant by 2 : 1 – b to a or a to b (b/a or a/b)?

Furthermore, contrary to the Wikipedia definition above, a musical interval $[a, b]$ is not the "space between two tones", but the "ordered tone pair (a, b)" itself. Here a is a starting tone or reference tone and the second tone is the target tone.

Question How should we interpret a statement that an interval has the "ratio" $x : y$ – for example 1 : 2 or 2 : 1?

Anticipating later explanations, we find the following concept in the idea of proportions and its symbolic notation: If two quantities (magnitudes – for example, tones) a and b are given, then the proportion equation

$$a : b \cong 1 : 2$$

(read "a relates to b like (the number) 1 to (the number) 2") means, that the formative characteristic (pitch, frequency, length, content, weight. . .) of the magnitude b is twice as large as that of the magnitude a. And the equivalent fraction-arithmetically interpreted equation "$a/b = 1/2$" also has exactly the solution "$b = 2a$".

The question concerning the indication of proportions is thus the question of how tones are "measured". For this there are – in principle – the two forms:

A. **The pitch proportion**

Here we connect (but basically unconsciously) the frequencies of the tones (that is their fundamental physical vibrations). If we write the ratio of the vibrations of two ordered tones (a, b) as a proportion, it follows that the musical interval $[a, b]$ has the proportion

$$a : b \cong (\text{frequency of the tone } a) : (\text{frequency of the tone } b)$$

and we speak of the pitch proportion. Here we interpret the sign ≅ as "corresponding to" – which means that the corresponding magnitudes are equal in the case of numbers – modulo a suitable multiplication with a common factor – in short, that the fraction (a/b) of the left side is equal to the fraction (frequency of the tone a /frequency of the tone b) of the right side.

→*The octave has the pitch proportion* 1 : 2.

B. **The monochord proportion (string/pipe length proportion)**

At all times when one knew nothing about vibrations and frequencies, tone relationships were usually measured by the length ratio of the vibrating monochord strings: If we have a tensioned string with the (fundamental) tone *a* and if we shorten this string by half, for example, the newer tone *b* of each of these halves will sound "an octave higher". The ratio of the lengths of the tone-producing strings L_a (fundamental) and L_b (octave tone) is therefore 2 : 1.

→*The octave has the monochord proportion* 2 : 1.

We have decided to use the pitch proportion **(A)** throughout this text. The proportions (2 : 3), (3 : 4), (4 : 5) therefore stand for perfect fifths, fourths, major thirds and so on. Thus, if we say $a : b \cong 1 : 2$, we have described an (upward) octave.

There are several reasons for doing this, for one thing: By noting the proportion $a : b$ for a pair of notes, we think more musically than geometrically: our mind's eye is (always) accompanied by a keyboard that quite spontaneously and without being asked transforms the proportion into two notes and which we also imagine audibly in an inner world.

A second thought supports this choice in a quite different way: Harmonics – especially that of antiquity – is the science of tonal relations, which result from the proportions to the natural numbers. Thus we read, for example, in [Hans Kayser (10), p. 48 ff:]

> If I set the oscillation number of the fundamental equal to 1, then we look to see which tones come out for the oscillation numbers 2, 3, 4, 5, 6… these are then the octave above the fundamental then the fifth above this octave (2 : 3), then the double octave above the fundamental (1 : 4), then the pure (perfect) major third above the double octave (4 : 5) and then above this another minor third (5 : 6)…

From the so-called overtone spectrum of a tone, the proportions line up in the form $(1 : n)$ to the fundamental, and consequently the proportion representations of the intervals arise, which do not describe the relationship of the final tone to the reference tone, but follow the written direction of the proportion.

A third idea: harmony is concerned with far more than simple one-step proportions, but above all with multi-step chains. Thus the simple numerical proportion chain 4 : 5 : 6 stands for the "major chord of just diatonicism", or more precisely: three tones *a*, *b* and *c* form a major triad if the similarity equation

$$a : b : c \cong 4 : 5 : 6$$

is true. And just as you read from left to right, you get the proportions 4 : 5 for *a* : *b, 5 : 6 for b : c* and 4 : 6 for *a* : *c*, that is, *b* has 5/4 times the frequency of *a* and c has 3/2 times it.

► Conclusion: By reading longer chains, the eye sees the sequence of intervals simultaneously in the sequence of proportions – without having to reverse the roles of end and start point.

Throughout the book, therefore, we use the (pitch) proportion measure 1 : 2 for the octave, and all intervals described by numerical proportions are to be interpreted accordingly.

The "frequency measure" *b*/*a* of a musical interval, on the other hand, remains unaffected: it is defined precisely by specifying the factor by which the frequency of a given (fundamental) tone *a* must be changed in order to produce the tone *b* – this factor is obviously *b*/*a*.

► Thus, the frequency measure of an interval $[a, b]$ focuses on the target note (*b*), whereas the proportion measure (A) focuses on the "ratio of the initial note to the final note".

From a superficial point of view, then, the frequency measure is apparently directly connected to the monochord proportion: Fifth 3 : 2 ↔ Frequency Measure 3/2.

Incidentally, both forms (A) and (B) are connected via the musical mechanism of action "up – down": The intervals *a* : *b* to (A) and *a* : *b* to (B) have the same frequency measure – but run in opposite directions. Thus, for example, the interval 2 : 3 to (A) is a perfect upward fifth, while the interval 2 : 3 to (B) describes a perfect downward fifth.

To illustrate all this in a final example, let us consider the chain of proportions – called the "**Senarius**", which was very famous in antiquity,

$$1 : 2 : 3 : 4 : 5 : 6.$$

We see that all number elements of their proportions come from the prime numbers (1), 2, 3 and 5, from which all intervals called "**emmelic**" arose. Intervals whose proportions also require other prime numbers besides 2, 3 and 5 as factors are, by the way, called "**ekmelic**" intervals. If we write notes over them, the notes played in just tuning at a given start c_0 (the "tonic") result $c_0 - c_1 - g_1 - c_2 - e_2 - g_2$, and we read off the step intervals from left to right (from note to next note) **and** concordantly to the chain of proportions:

Octave (1 : 2) – Fifth (2 : 3) – Fourth (3 : 4) – major Third (4 : 5) – minor Third (5 : 6),

which corresponds to the score

We hope that this reading will serve to speed up the acquisition, because it is just perfectly aligned with the concept of proportions and their chains.

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I thank my readers if they would like to accompany me on this path – lined by the science of mathematics – into the ancient world of music theory.

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Düsseldorf and Willich, Germany — Karlheinz Schüffler

Contents

1 Proportions

> Harmony consists of tones and intervals, and indeed the tone is the one and the same thing, the intervals are the otherness and the difference of the tones, and by mixing this, song and melody result. . . . *Aristoxenes (From [5], p. 79)*

There is no doubt about it: hardly any other term like that of "proportion" accompanies the musical theory of intervals – thus also of tonal intervals – with all its meanings. Thus we read everywhere that

> the interval of the pure fifth is in the ratio 2 : 3,
> the octave is defined by the condition 1 : 2

and so on. Almost everything connected with scales, their characteristic features, and distinctions, is ubiquitously permeated by a language which – in wanting to give precise descriptions – makes use of the terminology of "proportions" all around. In this chapter we present a foundation of arithmetic with proportions. We begin with a classification of various mathematical-musical concepts, oriented on some ancient ideas. Then we develop the **notion of proportions** from these fundaments and introduce an **axiomatic of the laws of arithmetic** – called the "theory of proportions" respectively the "calculus with proportions" – which is centered mainly in Theorems 1.1 and 1.2. In doing so, we anchor this theory of proportions on basic rules, which leads to the catalogue of ancient arithmetic rules. Finally, we also adress the multiplication of proportions, viewing it as a process of merging (**fusing**) two (or more) proportions - which musically corresponds to the layering of intervals to form a new interval.

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1.1 Arithmetica and Harmonia: The Genesis

Unfortunately, of Greek music – the oldest source of occidental culture – only theoretical material (writings) has survived – quite in contrast to architecture, sculpture, painting, coinage, gemology, and poetry. If there were only a single audio example – how much more revealing would such a stroke of luck be about the way music was understood. Certainly: our knowledge of ancient instruments allows us to guess at much of this and to recognize it as certain – but how it really felt in the ancient “musica practica” still gives us plenty of room for imaginative ideas and research. And a mere theoretical description of musical processes might be of very limited use to us. And yet: the more we know about the metrical data of the “musica theoretica”, the more we can – also thanks to modern technology – listen into the sound structures of the “musica practica”.

A chronologically ordered list of important older written literature of music-theoretical information contains many well-known names – like for example

- Aristotle (fourth century BC): 19th chapter of the “Problemata” and 5th chapter of the 8th book of the “Republica”
- Aristoxenes (pupil of Aristotle, end of fourth century BC): three books: Elements of Music
- Euclides (third century BC): Introduction to music as well as the division of the string
- Philodemus (first century BC): On music
- Plutarch (first century AD): Writing about music
- Aristides Quintilianus (second century AD): Work on music (in three volumes)
- Claudius Ptolemaeus (second century AD): Harmonics (three books)
- Julius Pollux (second century AD): Instrumentology
- Theo of Smyrna (second century AD): Mathematical description of music
- Nicomachus of Gerasa (60–120 AD): two books on music theory
- Anitius Manlius Severinus Boethius (sixth century AD): five books on music
- Michael Constantin Psellus (eleventh century AD): The Quadrivium
- Manuel Bryennius (fourteenth century AD): Harmonics

On the other hand, our current conception of music – namely as an art that reveals itself directly to acoustic aesthetics – was by no means valid in those times: music-making and the theory of music must be regarded as almost entirely separate areas. While making music on the instruments of the time, such as flutes, lyres and alike, was regarded as a low-level activity and “musicians” therefore tended to earn their bread at the lower end of a social order, music theory was part of the **“quadrivium**” – that supreme human science consisting of the four “mathematical” disciplines

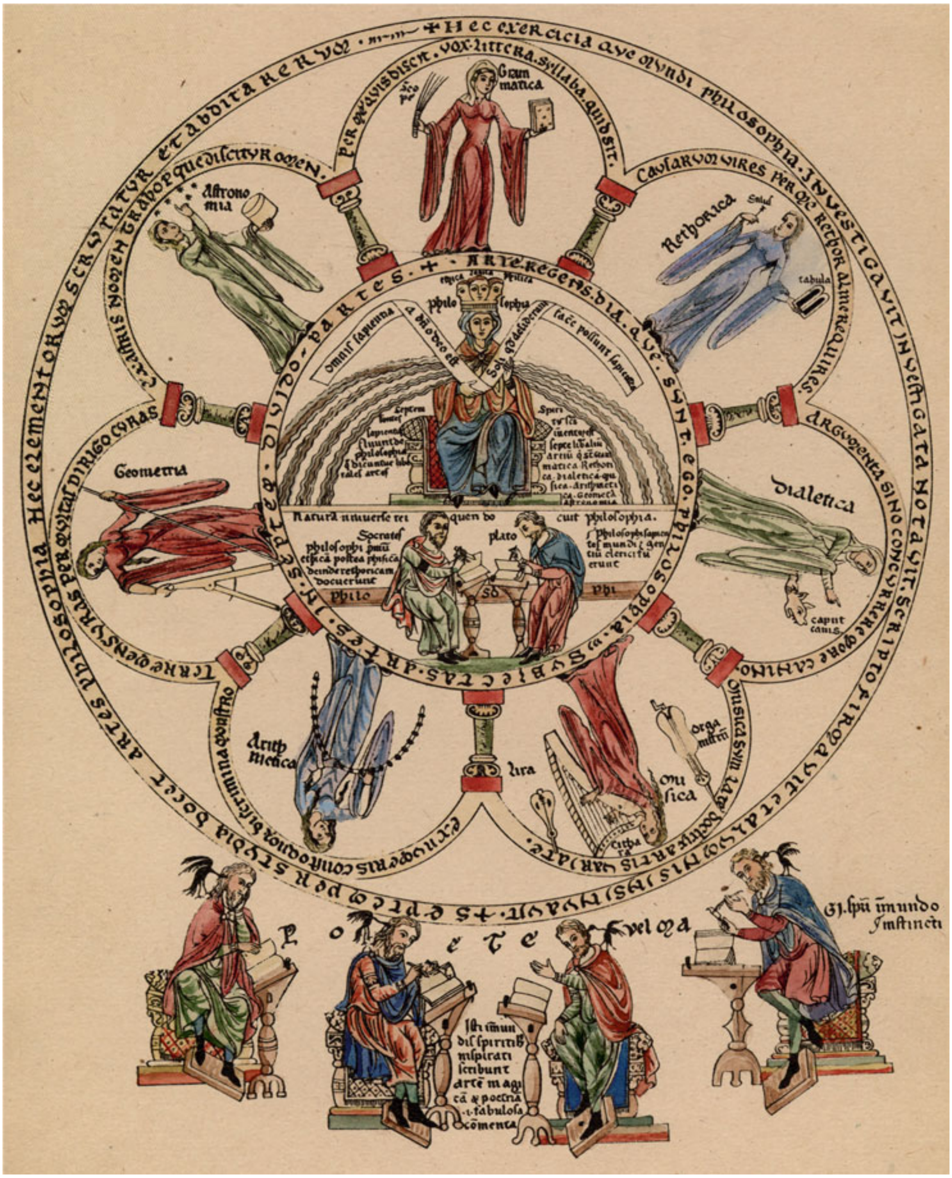

Fig. 1.1 The Septem Artes Liberales. (From the "Hortus deliciarum" of Herrad of Landsberg, twelfth century) (© akg-images/picture alliance)

• **Arithmetica – Geometria – Astronomia – Musica.**

These four liberal arts ranked even higher than those of the **"trivium,"** which consisted of the three philosophical arts

- **Dialectica – Grammatica – Rhetorica.**

Together they formed the **"Septem Artes liberales",** as can be seen in Fig. 1.1 – the seven liberal arts, which until the beginning of the modern age – together with theology and medicine – defined what was called "science" and what was consequently the teaching at all historical universities. In the image of the four mathematical arts – i.e. the quadrivium – an inner structure emerges, which presented geometry with arithmetic and astronomy with music as closely connected:

- **Geometria ⇄ Arithmetica and Astronomia ⇄ Musica.**

It is precisely from the close relationship of these pairs of siblings that the richness of the common conceptual worlds is ultimately drawn. Indeed, in ancient writings we encounter a wealth of thought in which homage is paid to the significance of these connections with such unwavering devotion that we can (or could) gain a new approach to "scientific" thinking in ancient times. The following quotation from **Aristides Quintilianus** may serve as a sample, and we recall very briefly the Harmonia perfecta maxima already mentioned in the preface with its "sacred musical numbers" 6 – 8 – 9 – 12, to which the following refers:

> *It is said of the musical numbers that they are all perfect numbers and sacred. This is especially true of the ration of the whole tone (the great second)* 8 : 9. *This numerical ratio expresses the harmony of the universe; for the planets are seven; the Zodiac adds an eighth sphere; thus the outstanding cosmic significance of the number nine is shown. If, however,* 8 : 9 *were substituted for* 16 : 18*, the number 17 would result in a very meaningful division of the whole tone into two semitones. But because the number 17 is the arithmetical middle between 16 and 18, so it indicates the original reason of the connection of the moon with the earth (etc.)...* (From [6], p. 16).

The motor of this network of relations consisted in the fact that almost all forms of determining, describing and justifying made use of both language and the elements of the "theory of proportions", that form of mathematics of the time which pervaded both geometry and arithmetic in equal measure. If we think, for example, of the well-known **"ray theorems" of geometry**, we have a fairly good idea of how **thinking in "ratios"** came to be a formative form of the sciences of the Quadrivium. This thinking in ratios had its own laws together with their rules based on them – but a reliable and universally valid basis could not – how could it – really develop.

In the following sections of this first chapter, we will therefore introduce the calculus of the theory of proportions in more detail.

What Is the Theory of Proportions About?

In the theory of proportions "magnitudes" are compared and "set in relation to each other". Here two things are worth noting:

First: As already mentioned in the introduction, the notion of "commensurability" plays a role that should not be underestimated; it states that for two magnitudes a and b – if they are at all in proportion to each other (or: if they form a *ratio* at all) – one can say:

$$a \text{ relates to } b \text{ as (the natural number) } n \text{ relates to (the natural number) } m.$$

If we write $a : b \cong n : m$, then the sign $\cong$ already expresses, that the area of the common number-calculation is not necessarily present here.

Secondly, that among the objects to be compared – i.e. those magnitudes which one brings into proportion with one another – all kinds of abstract things such as surfaces, bodies, sounds, angles etc. – and of course also numbers – are found, so that a symbol "$a : b$" would not necessarily be interpreted as a "fraction" obeying the simplest laws of calculation. In fact, it can be observed that very subtle descriptions of what

$$a : b \cong n : m$$

should be, in retrospect also lead to the surprising consequence that the commensurability requirement with its consequent rationality of all number relations can be replaced by a more universal – giving space to irrational numbers – level of argumentation: The limits of commensurability are thus exceeded after all – perhaps unconsciously – at least in one or the other case.

The theory of proportions now led directly to the **theory of medieties** and their abundance of "mean values" **(medieties),** which in turn determined the path of the "arithmetica" and the "harmonia" – which we will discuss later – towards the "musica". At first, these were the three Babylonian mean values, as they were presumably already known to the Pythagoreans; at least in late antiquity, Archytas ($\approx 430 - 350$ BC) still lists the medieties that were also called **"Babylonian"**

- arithmetic mean,
- geometric mean,
- harmonic mean.

In antiquity, these were also known as **"musical medieties"**; we read, for example, in **Nicomachus of Gerasa**:

> *...that the knowledge of the theory of the different kinds of medieties (first the arithmetical, second the geometrical, and third the inverse, which is also called harmonic) is exceedingly necessary for natural history and the theory of music, for the contemplation of the spheres (astronomy) as well as the laws of geometrical measurement in the planes (i.e. for all physics and branches of the quadrivium), but most of all for the understanding of the reading of the ancient scriptures...* (From [6], p. 118 et seq. 118 ff.)

To these classical medieties, which are also encountered in other ancient cultures such as the Egyptian and Persian, are added other medieties: first of all, these are

- the contra-harmonic mean and
- the contra-arithmetic mean,

whose role in pure diatonicism is quite decisive. Finally, some other mean values can be added; these are partly those that arise from iterated arithmetic and harmonic averages – i.e. proportions. In antiquity there were (broadly speaking) ten medieties.

In **music theory**, however, it is especially and almost exclusively these five aforementioned medieties that define the diatonic tonal space – with the geometric medieties only appearing in the background, but being in charge there – the later Sect. 3.5 gives ample justification.

The Theory of Music (Musica Theoretica)

The Theory of Music (“Musica Theoretica”) now grew entirely out of the theory of proportions, namely by establishing music as an ordered cosmos of the basic building blocks **“intervals”**, whereby all this was expressed in the language of the theory of proportions.

> **Important**
>
> A musical interval – as we would call it today – is an **ordered** pair of two tones *(a, b),* and musica theoretica describes the interval as a proportion of these two tones (which we interpret, with our present knowledge, as the ratio of the vibrational frequencies (of a versus b)).
>
> It is not the individual tone – its absolute “pitch”, colour, frequency (“oscillation mass”) and so on – that is decisive, but the ratio of those magnitudes of input (front member a) to output (back member b) of the interval in question.

Finally, intervals are joined together to form tetrachords and scales together with their chords: this is how the edifice of musica theoretica is built. If, on the other hand, we add proportions to one another, we create chains of proportions – the bridge from the tonal scales and their chords to the numerical relationships is built.

In these construction mechanisms from interval to scale, as in the theory of consonance, the calculus of medieties plays an outstandingly important role: in the most elementary context, it mediates the change from interval to scale

- **Major** and **minor** in the ordinary musical sense,
- **authentic** and **plagal** in the Gregorian context

as a number game, which is accompanied by astonishing symmetries. In the arithmetic of two intervals placed next to each other, it is above all the Babylonian mean values with their contra-partners which accomplished the construction of the Greek tetrachordic and the later diatonic.

All these medieties are defined as chains of proportions, and the musical interplay of their inner proportions with one another is most impressively discernible in the **harmonia perfecta maxima –** that "most perfect harmony":

► Since the Harmonia perfecta maxima draws its almost mystical power from the inner symmetry of arithmetic ("authentic") and harmonic ("plagal") chains of mean-value proportions, which in turn possess their symmetries primarily in the fact that they are reciprocal to one another and that the geometric medieties represents a centre of symmetry to all other medieties – irrespective of whether that mean value exists as a rational number or not.

We will not only encounter this Harmonia perfecta maxima very frequently in our text – rather, we have developed the description of ancient music theory, if it forms a scientific twin with the theory of proportions, from the idea of a **Harmonia perfecta universale.**

After this brief overview, we will now describe what can (but does not need to) be considered the basis of music-mathematical elements in the **linguistic, philosophical** and **arithmetical** sense. Here we can – understandably – only consider a very small part of the historical considerations – and even this only in the style of an overview. Interested readers will find a huge space in the vast literatures and will also encounter the most diverse views – may they appear scientific, speculative or even bizarre, strange in nature.

The Genesis: Arithmetica and Harmonia

The entire richness of the ancient intervals as well as the theory of mean values is revealed when the model of two reciprocal proportion sequences "Arithmetica" and "Harmonia" is taken as the basis for all considerations. This model is presented primarily in [6, 7], and the extremely demanding reading of these investigations, which date from the nineteenth century, develops over many hundred pages an unprecedented as well as admirable overall presentation of musical proportions. Accordingly, it is emphasized that the two sequences of proportions

$$1:1,\ 1:2,\ 1:3,\ \ldots \text{etc.} - \text{ short } 1:n\ (\text{Arithmetica})$$

$$1:1,\ 2:1,\ 3:1,\ \ldots \text{etc.} - \text{ short } n:1\ (\text{Harmonia}),$$

which, in our present way of writing and looking at things,we rather associate with the two sequences

$$(n)_{n\in\mathbb{N}} = 1,\ 2,\ 3,\ 4,\ \ldots (\text{Arithmetica})$$

$$\left(\frac{1}{n}\right)_{n\in\mathbb{N}} = 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots \text{(Harmonia)}.$$

They form an (better: the) essential component of the ancient number-mystical harmonic symbolism and, as a consequence, are also to be counted among the anchors of the Aristotelian philosophy of that time.

Note These two forms of proportion are also a consequence of our way of seeing and notating we explained at the beginning of the book in the section "For Use":

A magnitude *(b),* which enters into the proportion $a : b \cong 1 : n$ with the magnitude $a \cong 1$, is identifiable with the natural number "n"; likewise, it follows from the ratio $a : b \cong n : 1 \cong 1 : \frac{1}{n}$, that now the magnitude b can be regarded as a stem fraction "$\frac{1}{n}$".

And indeed the consequence

$$(n)_{n\in\mathbb{N}} = 1,\ 2,\ 3,\ 4,\ \ldots$$

is the prototype of **all arithmetic sequences,** and that of their reciprocal values, the "aliquot fraction sequence"

$$\left(\frac{1}{n}\right)_{n\in\mathbb{N}} = 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,$$

has been called **"harmonic sequence"** since time immemorial. Certainly, the relationship with music can be considered as a source par excellence for this naming.

From the point of view of arithmetic – as well as from the point of view of the theory of proportions – these two sequences of proportions are both inverse and reciprocal to one another.

"Inverse": In musical terms, inverted proportions can be interpreted as opposing intervals: For example, if the proportion 1 : 2 means the interval of an upward octave to a given tonic, its inverse 2 : 1 is precisely the downward octave from that chosen tonic.

"Reciprocal" means that the step sequence of multi-membered proportion chains reverses its order when one changes from "arithmetica to harmonia" and vice versa – we describe this process in detail in Chap. 2.

The model of the two sequences Arithmetica and Harmonia, sketched in Fig. 1.2, thus generates the musica theoretica and is called the **harmonic-arithmetic proportion**

$0 = (\infty : 1) \ldots (n : 1) \ldots (3 : 1)\ (2 : 1)$ ↘ $(1 : 1)$ ↗ $(1 : 2)\ (1 : 3) \ldots (1 : n) \ldots (1 : \infty) = \infty$

HARMONIA — ARITHMETICA

Fig. 1.2 Proportion model of Nicomachus