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Petr Hájek, Vicente Montesinos Santalucía,
Jon Vanderwerff and Václav Zizler

Biorthogonal Systems in Banach Spaces

Springer

Petr Hájek
Mathematical Institute of the
Czech Academy of Sciences
Žitná 25
Praha 1, 11567
The Czech Republic
hajek@math.cas.cz

Vicente Montesinos Santalucía
Departamento de Matemática Aplicada
E.T.S.I. Telecomunicación
Instituto de Matemática Pura y Aplicada
Universidad Politécnica de Valencia
C/Vera, s/n. 46022 - Valencia
Spain
vmontesinos@mat.upv.es

Jon Vanderwerff
Mathematics and Computer Science
Department
La Sierra University
4500 Riverwalk Parkway
Riverside, CA 92515
USA
jvanderw@lasierra.edu

Václav Zizler
Mathematical Institute of the
Czech Academy of Sciences
Žitná 25
Praha 1, 11567
The Czech Republic
zizler@math.cas.cz

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To Paola, Danuta, Judith, and Jarmila

Preface

The main theme of this book is the relation between the global structure of Banach spaces and the various types of generalized “coordinate systems”—or “bases”—they possess. This subject is not new; in fact, it has been investigated since the inception of the study of Banach spaces. The existence of a nice basis in a Banach space is very desirable. Bases are not only very useful in many analytic calculations and various constructions but can also be used to classify Banach spaces. The long-standing hope of having such a system in every Banach space was shattered first by Enflo’s construction of a separable Banach space without a Schauder basis and, more recently, by the work of Argyros, Gowers, Maurey, Schlumprecht, Tsirelson, and others that has produced hereditarily indecomposable Banach spaces and, in particular, Banach spaces containing no unconditional Schauder basic sequence. In light of these results, the classical rich structural theory of various special classes of Banach spaces, such as $\mathcal{L}_p$ spaces, separable $C(K)$ spaces, or Banach lattices, to name a few separable classes, as well as nonseparable weakly compactly generated or weakly countably determined (Vašák) spaces—and the coordinate systems they possess—increases in value, importance, and complexity. Of course, in order to obtain more general results, one has to weaken the analytic properties of the desired systems.

In this book, we systematically investigate the concepts of Markushevich bases, fundamental systems, total systems, and their variants. The material naturally splits into the case of separable Banach spaces, as is treated in the first two chapters, and the nonseparable case, which is covered in the remainder of the book.

Our starting point is that every separable Banach space has a fundamental total biorthogonal system. This was proved by Markushevich, and hence today such systems are called *Markushevich bases*. However, there are now several significantly stronger versions of this result. Indeed, using Dvoretzky’s theorem combined with orthogonal transformation techniques, Pełczyński and, independently, Plichko, obtained $(1+\varepsilon)$-bounded Markushevich bases in every separable Banach space. More recently, Terenzi has constructed several versions of strong Markushevich bases in all separable spaces that, in particular, allow one to recover every vector from its coordinates using permutations and blockings. These results, together with some background material, are treated in Chapter 1.

In Chapter 2, we present some classical as well as some recent results on the universality of spaces. This includes basic material on well-founded trees, applications of the Kunen-Martin theorem, theorems of Bourgain and Szlenk, and a thorough introduction to the geometric theory of the Szlenk index.

Chapter 3's material is preparatory in nature. In particular, it presents some results and techniques dealing with weak compactness, decompositions, and renormings that are useful in the nonseparable setting.

Chapter 4 focuses on the existence of total, fundamental, or, more generally, biorthogonal systems in Banach spaces. Among other things, we give Plichko's characterization of spaces admitting a fundamental biorthogonal system, the Godefroy-Talagrand results on representable spaces, a version under the clubsuit axiom ($\clubsuit$) of Kunen's example of a nonseparable $C(K)$ space without any uncountable biorthogonal system, and finally the recent result of Todorčević under Martin's Maximum (MM) axiom on the existence of a fundamental biorthogonal system for every Banach space of density ω_1. These latter results are typically obtained by using powerful infinite combinatorial methods—in the form of additional axioms in ZFC.

Many Banach spaces with nice structural and renormability properties can be classified according to the types of Markushevich bases they possess. Chapters 5 and 6 present, in detail, characterizations of several important classes of Banach spaces using this approach. This concerns spaces that are weakly compactly generated, weakly Lindelöf determined, weakly countably determined (Vašák), and Hilbert generated, as well as some others.

Chapter 7 deals with the class of spaces possessing long unconditional Schauder bases and their renormings. In particular, elements of the Pełczyński and Rosenthal structural theory of spaces containing $c_0(\Gamma)$, ℓ_∞, and operators that fix these spaces are discussed. The Pełczyński, Argyros, and Talagrand circle of results on the containment of $\ell_1(\Gamma)$ in dual spaces is also included.

The concluding chapter, Chapter 8, is devoted to some applications of biorthogonal and other weaker systems. Among other things, it presents some results on the existence of support sets, the theory of norm-attaining operators, and the Mazur intersection property.

It is our hope that the contents of this book reflect that nonseparable Banach space theory is a flourishing field. Indeed, this is a field that has recently attracted the attention of researchers not only in Banach space theory but also in many other areas, such as topology, set theory, logic, combinatorics, and, of course, analysis. This has influenced the choice of topics selected for this book. We tried to illustrate that the use of set-theoretical methods is, in some cases, unavoidable by showing that some important problems in the structural theory of nonseparable Banach spaces are undecidable in ZFC.

Given the breadth of this field and the diverse areas that impinge on the subject of this book, we have endeavored to compile a large portion of the relevant results into a streamlined exposition—often with the help of simplifications of the original proofs. In the process, we have presented a large variety of techniques that should provide the reader with a good foundation for future research. A substantial portion of the material is new to book form, and much of it has been developed in the last two decades. Several new results are included.

Unfortunately, for reasons of space and time, it has not been possible to include all relevant results in the area, and we apologize to all authors whose important results have been left out. Nevertheless, we believe that the present text, together with [ArTod05], [DGZ93a], [Fab97], [JoLi01h, Chap. 23 and 41], [MeNe92], [Negr84], and the introductory [Fa˜01], will help the reader to gain a clear picture of the current state of research in nonseparable Banach space theory.

We especially hope that this book will inspire some young mathematicians to choose Banach space theory as their field of interest, and we wish readers a pleasant time using this book.

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Prague, Valencia, and Riverside.
September, 2006

Petr Hájek
Vicente Montesinos
Jon Vanderwerff
Václav Zizler

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Standard Definitions, Notation, and Conventions

We will work with real Banach spaces in this book. We denote by $B_{(X,\|\cdot\|)}$ or simply B_X, the closed unit ball of a Banach space X under the norm $\|\cdot\|$; that is, $B_{(X,\|\cdot\|)} := \{x \in X;\ \|x\| \leq 1\}$. Similarly, the unit sphere $S_{(X,\|\cdot\|)}$, or simply S_X, is $\{x \in X;\ \|x\| = 1\}$. Unless stated otherwise, the topological dual space X^* (i.e., the space of all continuous linear functionals on X) is considered endowed with the canonical dual supremum norm; i.e., for $f \in X^*$, $\|f\| := \sup\{f(x);\ x \in B_{(X,\|\cdot\|)}\}$. We will use interchangeably $f(x)$, $\langle x, f\rangle$, or $\langle f, x\rangle$ for the action of an element $f \in X^*$ on an element $x \in X$. Whenever convenient, the space X will be assumed to be canonically a subspace of X^{**}. The w^*- (or weak*-) topology on X^* denotes the topology of pointwise convergence on the elements in X. The w-topology denotes the weak topology of X. We write c_0 and ℓ_p for $c_0(\mathbb{N})$ and $\ell_p(\mathbb{N})$, respectively, where $\mathbb{N}$ denotes the set of all positive integers. By a *copy* of a space we will usually mean an isomorphic copy of this space. By an *operator* we always mean a bounded linear operator. Unless stated otherwise, the word *subspace* will be understood to mean a closed linear subspace, and the fact that Y is a subspace of X will be denoted by $Y \hookrightarrow X$. The closure of a set A in a topological space (E, τ) is denoted by $\overline{A}^{\tau}$, or just by $\overline{A}$ if the topology τ is understood. Convergence in the topology τ will be denoted sometimes as $\xrightarrow{\tau}$. The *density* of a topological space T is the smallest cardinal $\aleph$ such that T has a dense subset of cardinality $\aleph$. The *span* of a subset A of a linear space X, denoted $\operatorname{span}(A)$, is the linear hull of A. The closed linear span of a set A in a topological vector space (E, τ) is denoted by $\overline{\operatorname{span}}^{\tau}(A)$ or, if the topology τ is understood, just by $\overline{\operatorname{span}}(A)$. For a completely regular topological space T, the space βT denotes the *Čech-Stone compactification* of the space T.

Cardinal numbers are usually denoted by $\aleph$, while ordinal numbers are denoted by α, β, etc. With the symbol $\aleph_0$ we denote the cardinal number of $\mathbb{N}$, and $\aleph_1$ is the first uncountable cardinal. Similarly, ω is the ordinal number of $\mathbb{N}$ under its natural order, and ω_1 is the first uncountable ordinal. The symbol c is used for the cardinal of the continuum.

In this book we use, unless stated otherwise, Zermelo-Fraenkel set theory with the axiom of choice (ZFC).

A *Schauder basis* is a sequence (x_n) in X such that every element $x \in X$ can be uniquely written as $x = \sum_{n=1}^{\infty} a_n x_n$ for some real numbers a_n, $n \in \mathbb{N}$. It then follows that the canonical projections $P_n(x) := \sum_{i=1}^{n} f_i(x) x_i$, where $f_i(x) = a_i$, $i = 1, 2 \ldots, n$, $n \in \mathbb{N}$, are uniformly bounded linear projections. If $\sum_{n=1}^{\infty} a_{\pi(n)} x_{\pi(n)}$ converges to x for every permutation π of $\mathbb{N}$, we speak of an *unconditional basis*. The basis is called *shrinking* if $\overline{\operatorname{span}}\{f_n; n \in \mathbb{N}\} = X^*$.

For information on Schauder bases, we refer the reader to[LiTz77], [Sing70b], [Sing81], [Woj91], [Fa˜01], and [JoLi01h]. In [JoLi01h, Chaps. 1, 7, 14, 41], several more general notions—the approximation property, finite-dimensional decompositions, etc.—are discussed.

1

Separable Banach Spaces

In this chapter, we introduce the basic definitions concerning biorthogonal systems in Banach spaces and discuss several results, mostly in the separable setting, related to this structure. When searching for a system of coordinates to represent any vector of a (separable) Banach space, a natural approach is to consider the concept of a Schauder basis. Unfortunately, not every separable Banach space has such a basis, as was proved by Enflo in [Enfl73]. However, all such spaces have a Markushevich basis (from now on called an M-basis), a result due to Markushevich himself that elaborates on the basic Gram-Schmidt orthogonal process. It will be proved in Chapter 5 that many nonseparable Banach spaces also possess M-bases, even with some extra features, allowing actual computations and opening a way to classification of Banach spaces.

We begin this chapter by introducing in Section 1.1, those special properties of biorthogonal systems that will be used later. Every n-dimensional normed space has a biorthogonal system $\{x_j; x_j^*\}_{j=1}^n$ that, in some sense, has an optimal location: every vector x_j and every functional x_j have norm 1. This system in called an Auerbach basis. The infinite-dimensional counterpart of the former result is an open problem. In Section 1.2, a preliminary approach to this question is presented. Section 1.3 deals with the basic Markushevich construction, a building block for subsequent developments. The control on the size of vectors and functionals in a biorthogonal system leads to the concept of a bounded M-basis. The question of its existence in every separable Banach space was open for some time and solved in the positive by Ovsepian and Pełczyński [OvPe75], a result later sharpened by Pełczyński and, independently, by Plichko. This, together with some hints on a possibly negative solution of the Auerbach system problem in separable Banach spaces, is presented in Section 1.4. Strong M-bases, a natural concept in view of the kind of convergence of the partial sums of developments generated by Schauder bases and Fourier series, are defined, and their existence in every separable Banach space is discussed in Section 1.5, where Terenzi's results, in a utility-grade version, are presented. We follow also Terenzi's approach in extending bounded M-bases from subspaces to bounded M-bases in overspaces in Section

1.6, preceded by the general extension theorem of Gurarii and Kadets and followed by V.D. Milman's result on extensions in directions of quasicomplements. The chapter ends with a discussion, which will be enlarged in Chapter 8, on ω-independent families.

1.1 Basics

This section introduces some basic definitions that will be used throughout this book. Although this chapter deals primarily with separable Banach spaces, general (possibly uncountable) biorthogonal systems are presented in order to cover the nonseparable case as well. It is shown that the minimality of a family of vectors is equivalent to the biorthogonal behavior. We also collect some simple facts about decompositions of a space with a biorthogonal system. Fundamentality, totality and the shrinking or norming character of a biorthogonal system are also defined.

Definition 1.1. *Let* X *be a Banach space. Let* Γ *be a nonempty set. A family* $\{(x_\gamma, x_\gamma^*)\}_{\gamma\in\Gamma}$ *of pairs in* $X \times X^*$ *is called a* biorthogonal system *in* $X \times X^*$ *if* $\langle x_\alpha, x_\beta^*\rangle = \delta_{\alpha,\beta}$*, where* $\delta_{\alpha,\beta}$ *is the Kronecker* δ*, for all* $\alpha, \beta \in \Gamma$*. For simplicity, a biorthogonal system in* $X \times X^*$ *will be denoted by* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$*. A family* $\{x_\gamma\}_{\gamma\in\Gamma} \subset X$ *is called a* minimal system *if there exists a family* $\{x_\gamma^*\}_{\gamma\in\Gamma} \subset X^*$ *such that* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *is a biorthogonal system.*

In the case where $\Gamma = \mathbb{N}$, we shall use the notation $\{x_n; x_n^*\}_{n=1}^\infty$ (resp. $\{x_n\}_{n=1}^\infty$) instead of $\{x_n; x_n^*\}_{n\in\mathbb{N}}$ (resp. $\{x_n\}_{n\in\mathbb{N}}$). By the Hahn-Banach theorem, a family $\{x_\gamma\}_{\gamma\in\Gamma} \subset X$ is a minimal system if and only if, for every $\gamma \in \Gamma$, $x_\gamma \notin \overline{\operatorname{span}}\{x_\alpha; \alpha \in \Gamma, \alpha \neq \gamma\}$.

The following simple facts will be used frequently throughout this book.

Fact 1.2. *If* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *is a biorthogonal system in* $X \times X^*$*, if* $x \in X$ *and* $\gamma \in \Gamma$ *are such that* $\langle x, x_\gamma^*\rangle \neq 0$*, and if for some finite set* $A \subset \Gamma$

$$\left\| \sum_{\alpha\in A} \langle x, x_\alpha^*\rangle x_\alpha - x \right\| < \frac{|\langle x, x_\gamma^*\rangle|}{\|x_\gamma^*\|},$$

then $\gamma \in A$.

Proof. If $\gamma \notin A$, then

$$|\langle x, x_\gamma^*\rangle| = \left| \left\langle \left(\sum_{\alpha\in A} \langle x, x_\alpha^*\rangle x_\alpha - x \right), x_\gamma^* \right\rangle \right| \leq \|x_\gamma^*\| \cdot \left\| \sum_{\alpha\in A} \langle x, x_\alpha^*\rangle x_\alpha - x \right\|. \quad \square$$

Fact 1.3. *Let* X *be a Banach space. Let* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *be a biorthogonal system in* $X \times X^*$*. Let* A *be a nonempty finite subset of* Γ *and* B *its complement. Then, denoting by* $\oplus$ *the topological direct sum:*

(i) $X = \operatorname{span}\{x_a; a \in A\} \oplus \{x_a^*; a \in A\}_\perp$.
(ii) $\{x_b^*; b \in B\}_\perp = \operatorname{span}\{x_a; a \in A\} \oplus \{x_\gamma^*; \gamma \in \Gamma\}_\perp$.

Proof. (i) Given $x \in X$, we have $x - \sum_{a\in A}\langle x, x_a^*\rangle x_a \in \{x_a^*; a \in A\}_\perp$, so $X = \operatorname{span}\{x_a; a \in A\} + \{x_a^*; a \in A\}_\perp$. Obviously $\operatorname{span}\{x_a; a \in A\} \cap \{x_a^*; a \in A\}_\perp = \{0\}$, so the sum above is a topological direct sum.

(ii) Given $x \in \{x_b^*; b \in B\}_\perp$, it is obvious that $x - \sum_{a\in A}\langle x, x_a^*\rangle x_a \in \{x_\gamma^*; \gamma \in \Gamma\}_\perp$. Then

$$\{x_b^*; b \in B\}_\perp \subset \operatorname{span}\{x_a; a \in A\} + \{x_\gamma^*; \gamma \in \Gamma\}_\perp \, (\subset \{x_b^*; b \in B\}_\perp)\,.$$

It follows that $\{x_b^*; b \in B\}_\perp = \operatorname{span}\{x_a; a \in A\} + \{x_\gamma^*; \gamma \in \Gamma\}_\perp$; moreover, $\operatorname{span}\{x_a; a \in A\} \cap \{x_\gamma^*; \gamma \in \Gamma\}_\perp = \{0\}$, so the sum above is a topological direct sum. □

Definition 1.4. *A family $\{x_\gamma\}_{\gamma\in\Gamma}$ of vectors in the Banach space X is called* fundamental *if* $\overline{\operatorname{span}}\{x_\gamma\}_{\gamma\in\Gamma} = X$.

In the case of a fundamental minimal system $\{x_\gamma\}_{\gamma\in\Gamma}$ in X, there exists a unique system $\{x_\gamma^*\}_{\gamma\in\Gamma}$ (and it is called its system of *functional coefficients*) in X^* such that $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ is a biorthogonal system. The corresponding biorthogonal system $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ is also called *fundamental.* Whenever convenient, we will use the abbreviated notation $\{x_\gamma\}_{\gamma\in\Gamma}$ in the case of a fundamental biorthogonal system $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$.

Fact 1.5. *Let X be a Banach space. Let $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ be a biorthogonal system in $X \times X^*$. Then the following are equivalent:*

(i) $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *is fundamental.*
(ii) *For some nonempty finite set $A \subset \Gamma$ and its complement B, we have* $\operatorname{span}\{x_a^*; a \in A\} = \{x_b; b \in B\}^\perp$.
(iii) *For some nonempty finite set $A \subset \Gamma$ and its complement B, we have* $X = \operatorname{span}\{x_a; a \in A\} \oplus \overline{\operatorname{span}}\{x_b; b \in B\}$.

Moreover, if (i) *holds, then* (ii) *and* (iii) *hold for every nonempty finite set $A \subset \Gamma$.*

Proof. (i)⇒(ii) Take any nonempty finite set $A \subset \Gamma$. From (i) in Fact 1.3, it follows that $X^* = \{x_a; a \in A\}^\perp \oplus \operatorname{span}\{x_a^*; a \in A\}$. From the fundamentality of the system, $\{x_a; a \in A\}^\perp \cap \{x_b; b \in B\}^\perp = \{0\}$. Moreover, $\operatorname{span}\{x_a^*; a \in A\} \subset \{x_b; b \in B\}^\perp$, so we get (ii).

(ii)⇒(iii) From (ii) we get $\{x_a^*; a \in A\}_\perp = \overline{\operatorname{span}}\{x_b; b \in B\}$. Now use (i) in Fact 1.3.

(iii)⇒(i) is trivial.

The last statement is a consequence of the two following observations: (a) (ii) for some nonempty finite set A implies (iii) for the same A, and (b) (i)⇒(ii) for every nonempty finite set A. □

Definition 1.6. *A biorthogonal system* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *in* $X \times X^*$ *is called* total *if* $\overline{\operatorname{span}}^{w^*}\{x_\gamma^*; \gamma \in \Gamma\} = X^*$.

Definition 1.7. *A fundamental and total biorthogonal system* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *in* $X \times X^*$ *is called a* Markushevich basis *for* X*, henceforth called an* M-basis *in this book. Whenever convenient, we will use the abbreviated notation* $\{x_\gamma\}_{\gamma\in\Gamma}$ *for an M-basis in* X*.*

Fact 1.8. *Let* X *be a Banach space and let* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *be a fundamental biorthogonal system in* $X \times X^*$*. Let* $(B_i)_{i\in I}$ *be a family of subsets of* Γ*.*

(i) *If* $\bigcap_{i\in I} B_i = \emptyset$ *and* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *is an M-basis in* $X\times X^*$*, then* $\bigcap_{i\in I} \overline{\operatorname{span}}\{x_\gamma; \gamma \in B_i\} = \{0\}$*.*
(ii) *If* $A_i := \Gamma \setminus B_i$ *is finite for every* $i \in I$ *and* $\bigcap_{i\in I} \overline{\operatorname{span}}\{x_\gamma; \gamma \in B_i\} = \{0\}$*, then* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *is an M-basis in* $X \times X^*$*.*

Proof. (i) Let $x \in \bigcap_{i\in I} \overline{\operatorname{span}}\{x_\gamma; \gamma \in B_i\}$. Fix $\gamma_0 \in \Gamma$. There exists $i \in I$ such that $\gamma_0 \notin B_i$. As $x \in \overline{\operatorname{span}}\{x_\gamma; \gamma \in B_i\}$, we get $\langle x, x_{\gamma_0}^*\rangle = 0$. This happens for every $\gamma_0 \in \Gamma$, so $x = 0$ since $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ is an M-basis.

(ii) Let x be a nonzero element of X. There exists $i \in I$ such that $x \notin \overline{\operatorname{span}}\{x_\gamma; \gamma \in B_i\}$. From (iii) in Fact 1.5, we can write $x = y + z$, where $0 \neq y \in \operatorname{span}\{x_\gamma; \gamma \in A_i\}$ and $z \in \overline{\operatorname{span}}\{x_\gamma; \gamma \in B_i\}$. We can find $\gamma \in A_i$ such that $\langle y, x_\gamma^*\rangle (= \langle x, x_\gamma^*\rangle) \neq 0$. Thus $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ is an M-basis. □

Definition 1.9. *A subset* N *of the dual* X^* *of a Banach space* $(X, \|\cdot\|)$ *is called* λ-norming*, for some* $0 < \lambda \leq 1$*, if* $|||\cdot|||$ *defined on* X *by* $|||x||| := \sup\{\langle x, x^*\rangle; x^* \in N \cap B_{(X^*,\|\cdot\|)}\}$*,* $x \in X$*, is a norm satisfying* $\lambda\|x\| \leq |||x||| \; (\leq \|x\|)$*. If* N *is* λ*-norming for some* $0 < \lambda \leq 1$*,* N *is just called* norming.

Definition 1.10. *Let* X *be a Banach space. A biorthogonal system* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *in* $X \times X^*$ *is called* λ-norming *(for some* $0 < \lambda \leq 1$*) if* $\overline{\operatorname{span}}^{\|\cdot\|}\{x_\gamma^*\}_{\gamma\in\Gamma}$ *is a* λ*-norming subspace. If* $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *is* λ*-norming for some* $0 < \lambda \leq 1$*, the system is just called* norming.

Remark 1.11. 1. *Every infinite-dimensional separable Banach space contains a fundamental system that is not an M-basis.* Indeed, let $\{y_i\}_{i=1}^\infty$ be a linearly independent system in X such that $\overline{\operatorname{span}}\{y_i\}_{i=1}^\infty = X$. Pick $x \in X \setminus \operatorname{span}\{y_i\}_{i=1}^\infty$. Using the Hahn-Banach theorem, we find $g_i \in X^*$ such that $g_i(x) = 0$, $g_i(y_i) = 1$, and $g_i(y_j) = 0$ for $j = 1, \dots, i-1$. As in Lemma 1.21, we find a biorthogonal system $\{x_i; f_i\}_{i=1}^\infty$ such that $\operatorname{span}\{x_i\}_{i=1}^\infty = \operatorname{span}\{y_i\}_{i=1}^\infty$ and $\operatorname{span}\{f_i\}_{i=1}^\infty = \operatorname{span}\{g_i\}_{i=1}^\infty$. Then $\overline{\operatorname{span}}\{x_i\}_{i=1}^\infty = X$ and $\{f_i\}_{i=1}^\infty$ does not separate points of the space X.

2. *Every separable Banach space* X *has a 1-norming M-basis* (see Theorem 1.22). The problem of the existence of a norming M-basis in every *weakly compactly generated* Banach space (WCG) (i.e., a space with a weakly compact and linearly dense subset), is still open. Partial negative results are given in [Gode95], [Vald94] and [VWZ94] (see Theorems 5.21, 5.22, and 5.23).

3. *A norming subspace of X^* is w^*-dense in X^*. However, not every w^*-dense subspace of X^* is norming.* In fact, if X is a separable Banach space such that X^{**}/X is infinite-dimensional, there exists a subspace $N \subset X^*$ that is w^*-dense and not norming (see, e.g., [Fa˜01, Exer. 3.40]). From this and from Theorem 1.22, it follows that every separable space contains an M-basis that is not norming.

Definition 1.12. *Let X be a Banach space. A biorthogonal system $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ in $X \times X^*$ is called* shrinking *whenever $X^* = \overline{\operatorname{span}}^{\|\cdot\|}\{x_\gamma^*; \gamma \in \Gamma\}$.*

Definition 1.13. *Let X be a Banach space. A biorthogonal system $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ in $X \times X^*$ is called* boundedly complete *if given a bounded sequence (y_n) in $\overline{\operatorname{span}}\{x_\gamma; \gamma \in \Gamma\}$ such that $a_\gamma := \lim_n \langle y_n, x_\gamma^*\rangle$ exists for all $\gamma \in \Gamma$, then there exists an element $y \in \overline{\operatorname{span}}\{x_\gamma; \gamma \in \Gamma\}$ such that $\langle y, x_\gamma^*\rangle = a_\gamma$ for all $\gamma \in \Gamma$.*

Remark 1.14. It is shown in [PeSz65] that there exists a separable Banach space X with a normalized unconditional basis $\{e_n\}_{n\in\mathbb{N}}$ such that $\{e_n; n \in \mathbb{N}\} \cup \{0\}$ is weakly compact and the basis is not shrinking.

1.2 Auerbach Bases

It is well known that a finite-dimensional Banach space X has a Hamel basis. It can be taken as normalized (i.e., all their vectors have norm 1). However, without additional effort, there is no control over the size of the functional coefficients. The following theorem shows that it is possible to choose the Hamel basis such that both their vectors and the functionals have norm 1. The construction has a clear geometric meaning: a parallelepiped of maximal volume is inscribed in the closed unit ball of X. The endeavor to reproduce this behavior in infinite-dimensional Banach spaces has been a recurrent theme in Banach space theory, and this theme will be analyzed later in this book. In this direction, this section also presents a basic construction due to Krein, Krasnosel'skiĭ, and Milman—which is in a sense opposite to Mazur's approach for building a basic sequence in every Banach space—and gives Day's procedure for producing infinite Auerbach systems in every Banach space. However, the goal of having an Auerbach basis in every separable space remains unrealized.

Definition 1.15. *Let X be a Banach space. A biorthogonal system $\{e_\gamma; e_\gamma^*\}_{\gamma\in\Gamma}$ in $X \times X^*$ such that $\|e_\gamma\| = \|e_\gamma^*\| = 1$ for all $\gamma \in \Gamma$ is called an* Auerbach system *in X. If it is, moreover, an M-basis, it is called an* Auerbach basis *for X.*

Theorem 1.16 (Auerbach). *Every finite-dimensional Banach space contains an Auerbach basis.*

Proof. Let $\{x_i\}_{i=1}^n$ be an algebraic basis of X. Given a finite sequence $(u_i)_{i=1}^n$ of vectors in X, let $\det(u_1, u_2, \ldots, u_n)$ be the determinant of the matrix whose

j-th column consists of the coordinates of u_j in the basis $\{x_i\}_{i=1}^n$. The function $|\det|$ is continuous on the compact set $B_X \times \ldots \times B_X$, so it attains its supremum at some $(e_1, e_2, \ldots, e_n) \in B_X \times \ldots \times B_X$.

Determinants are multilinear mappings of their columns. Then $\|e_i\| = 1$, $i = 1, 2, \ldots, n$. Moreover, the vectors $\{u_i\}_{i=1}^n$ are linearly independent if and only if $\det(u_1, u_2, \ldots, u_n) \neq 0$. Then the vectors $\{e_i\}_{i=1}^n$ are linearly independent.

For $i = 1, 2, \ldots, n$, let us define $e_i^* \in X^*$ by

$$\langle x, e_i^* \rangle := \frac{\det(e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n)}{\det(e_1, \ldots, e_n)} \quad \text{for all } x \in X.$$

Then $e_i^* \in B_{X^*}$ and $\langle e_k, e_i^* \rangle = \delta_{k,i}$ for all $1 \leq k, i \leq n$, so $\{e_i; e_i^*\}_{i=1}^n$ is an Auerbach basis. □

Corollary 1.17. *Let X be a Banach space, and let Y be an n-dimensional subspace. Then there exists a linear projection P from X onto Y such that $\|P\| \leq n$.*

Proof. Let $\{e_i; e_i^*\}_{i=1}^n$ be an Auerbach basis for Y. Extend e_i^* to an element of S_{X^*} (still denoted by e_i^*) for all i and define

$$P(x) := \sum_{i=1}^n \langle x, e_i^* \rangle e_i \text{ for all } x \in X.$$

Then $P : X \to Y$ is a linear projection and $\|P(x)\| \leq n\|x\|$, $\forall x \in X$. □

Theorem 1.16 means in particular that in two-dimensional spaces, there always exist *monotone* Schauder bases (i.e., bases where the canonical projections have norm 1). This is no longer true for three-dimensional spaces; Bohnenblust proved in [Bohn41] that *there is a three-dimensional Banach space that does not admit any monotone Schauder basis.* It is still unknown if every infinite-dimensional Banach space contains an infinite-dimensional Banach space with a monotone Schauder basis.

Definition 1.18. *Let X be a Banach space. We will say that $x \in X$ is* orthogonal *to $y \in X$ (and write $x \perp y$) if $\|y\| \leq \|y + \lambda x\|$ for all $\lambda \in \mathbb{R}$. If Y is a subspace of X and $x \perp y$ for all $y \in Y$, we will say that x* is orthogonal to *Y and write $x \perp Y$. Analogously, we will say that Y* is orthogonal to *x, and write $Y \perp x$, if $y \perp x$ for all $y \in Y$.*

It is obvious that $Y \perp x$ if and only if $\operatorname{dist}(x, Y) = \|x\|$.

The following basic result will be applied several times in this book.

Lemma 1.19 (M.G. Krein, M.A. Krasnosel'skiĭ, and D.P. Milman [KKM48]; see, e.g., [Sing70a] p. 269**).** *Let E be a normed space and G_1 and G_2 two subspaces such that $\dim G_1 < \infty$ and $\dim G_1 < \dim G_2$. Then there exists $y \in S_{G_2}$ such that $G_1 \perp y$.*

Proof. Without loss of generality, we may and do assume that $\dim G_1 = n$, $\dim G_2 = n+1$, and $E = \operatorname{span}\{G_1 \cup G_2\}$. Suppose first that the norm of E is strictly convex ($\equiv$ rotund). Then π_{G_1}, the metric projection from E onto G_1 that associates to $x \in E$ elements in G_1 of best approximation, is well defined, single-valued, and continuous; moreover, $\pi_{G_1}(-x) = -\pi_{G_1}(x)$ for every $x \in E$. Assume that the result is false. Then $\pi_{G_1}(g_2) \neq 0$ for every $g_2 \in G_2 \setminus \{0\}$. The mapping $\psi : S_{G_2} \to S_{G_1}$ given by $\psi(g_2) := \pi_{G_1}(g_2)/\|\pi_{G_1}(g_2)\|$ is continuous and $\psi(-g_2) = -\psi(g_2)$ for every $g_2 \in S_{G_2}$. Choose a basis $\{y_{k,i}\}_{i=1}^{n+k-1}$ in G_k and define a homeomorphism $\varphi_k : S_{G_k} \to \Sigma_k$, where Σ_k denotes the unit sphere in ℓ_2^{n+k-1}, by $\varphi(\sum \alpha_i y_{k,i}) := (\alpha_i/(\sum \alpha_i^2)^{1/2})_{i=1}^{n+k-1}$, $k = 1, 2$. Then, $\chi := \Sigma_2 \to \Sigma_1$ given by $\chi := \varphi_1 \circ \psi \circ \varphi_2^{-1}$ is a continuous mapping such that $\chi(-z) = -\chi(z)$ for every $z \in \Sigma_2$, which is impossible by the Borsuk Antipodal Theorem. To deal with the general case, define a norm $\|\cdot\|_2$ in E by $\|x\|_2 = \|\sum_{i=1}^m \alpha_i z_i\|_2 := \left(\sum_{i=1}^m \alpha_i^2\right)^{1/2}$, where $\{z_i\}_{i=1}^m$ is a basis of E. Then, for every $\delta > 0$, $\|\cdot\|_\delta := \|\cdot\| + \delta\|\cdot\|_2$ defines a strictly convex equivalent norm on E and $\|\cdot\| \leq \|\cdot\|_\delta \leq (1+\delta\gamma)\|\cdot\|$, where $\gamma := \max\{\|x\|_2; x \in E, \|x\| = 1\}$. Let y_n be the solution to the problem for the $\|\cdot\|_{1/n}$ norm, $n \in \mathbb{N}$. If y is the limit of a convergent subsequence of (y_n), the vector $y/\|y\|$ solves the problem. □

Theorem 1.20 (Day [Day62]). *Let X be an infinite-dimensional Banach space and let (c_n) be a sequence of positive numbers. Then there exists a countable infinite Auerbach system $\{b_n; b_n^*\}_{n=1}^\infty$ in $X \times X^*$ such that $\{b_n\}_{n=1}^\infty$ is a Schauder basic sequence and $\|P_n\| \leq 1 + c_n$, where $P_n(x) := \sum_{k=1}^n \langle x, b_k^*\rangle b_k$, for $x \in \overline{\operatorname{span}}\{b_n; n \in \mathbb{N}\}$ are the canonical projections associated to the basic sequence.*

Proof. As in Mazur's classical construction of a basic sequence (see, e.g., [Fa˜01, Thm. 6.14]), choose $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ such that $\prod_{k=n}^\infty (1+\varepsilon_k) < (1+c_n)$ for all $n \in \mathbb{N}$. The construction will be done inductively. Let us start by choosing any $b_1 \in S_X$ and some $b_1^* \in S_{X^*}$ such that $\langle b_1, b_1^*\rangle = 1$. Assume that, for some $n \in \mathbb{N}$, elements $b_i \in S_X$ and $b_i^* \in S_{X^*}$, $i = 1, 2, \ldots, n$, have already been defined such that $\langle b_i, b_j^*\rangle = \delta_{i,j}$, $i, j = 1, 2, \ldots, n$, and $\{b_i\}_{i=1}^n$ is a basic sequence in X with $\|Q_n\| \leq 1 + \varepsilon_n$, where $Q_n : \operatorname{span}\{b_i\}_{i=1}^n \to \operatorname{span}\{b_i\}_{i=1}^{n-1}$ denotes the canonical projection. Pick a finite $\varepsilon_{n+1}/2$-net $\{x_1, \ldots, x_k\}$ for $S_{\operatorname{span}\{b_i\}_{i=1}^n}$ and select $x_1^*, \ldots, x_k^*$ in S_{X^*} such that $\langle x_i, x_i^*\rangle = 1$, $i = 1, 2, \ldots, k$. Now apply Lemma 1.19 to $G_1 := \operatorname{span}\{b_i\}_{i=1}^n$ and $G_2 := \left(\bigcap_{i=1}^k \operatorname{Ker} x_i^*\right) \cap \left(\bigcap_{i=1}^n \operatorname{Ker} b_i^*\right)$. We can then find an element $b_{n+1} \in S_{G_2}$ such that $G_1 \perp b_{n+1}$. The element $b_{n+1}^* \in (\operatorname{span}(G_1, b_{n+1}))^*$ given by $\langle b_{n+1}, b_{n+1}^*\rangle = 1$ and $b_{n+1}^* \restriction G_1 \equiv 0$ has norm 1. Extend it to an element in S_{X^*} still denoted by b_{n+1}^*. At the same time, given $x \in S_{\operatorname{span}\{b_i\}_{i=1}^n}$, there exists $i \in \{1, 2, \ldots, k\}$ such that $\|x - x_i\| \leq \varepsilon_{n+1}/2$. Then

$$\begin{aligned}\|x + \lambda b_{n+1}\| &\geq \|x_i + \lambda b_{n+1}\| - \|x - x_i\| \\ &\geq \langle x_i + \lambda b_{n+1}, x_i^*\rangle - \|x - x_i\| \geq 1 - \frac{\varepsilon_{n+1}}{2} \geq \frac{1}{1+\varepsilon_{n+1}}.\end{aligned}$$

This completes the inductive step. It is clear now that $\{b_n; b_n^*\}_{n=1}^\infty$ is an Auerbach system and at the same time a Schauder basic sequence. Moreover, $\|P_n\| \leq \prod_{k=n}^\infty (1+\varepsilon_k) < 1 + c_n$ for all $n \in \mathbb{N}$. □

For a remark on Theorem 1.20, see Exercise 1.4. The existence of an Auerbach basis for every separable Banach space is an open problem.

1.3 Existence of M-bases in Separable Spaces

We present here the classical Markushevich construction of a (countable) M-basis in every separable Banach space. A careful choice of the proof's ingredients yields norming (and if the dual space is separable, shrinking) M-bases.

Lemma 1.21 (Markushevich [Mark43]). *Let X be an infinite-dimensional Banach space. If $\{z_n\}_{n=1}^\infty \subset X$ and $\{z_n^*\}_{n=1}^\infty \subset X^*$ are such that* $\operatorname{span}\{z_n\}_{n=1}^\infty$ *and* $\operatorname{span}\{z_n^*\}_{n=1}^\infty$ *are both infinite-dimensional and*

$$\text{(M1)} \qquad \{z_n\}_{n=1}^\infty \textit{separates points of } \operatorname{span}\{z_n^*\}_{n=1}^\infty,$$

$$\text{(M2)} \qquad \{z_n^*\}_{n=1}^\infty \textit{ separates points of } \operatorname{span}\{z_n\}_{n=1}^\infty,$$

then there exists a biorthogonal system $\{x_n; x_n^*\}_{n=1}^\infty$ in $X \times X^*$ such that

$$\operatorname{span}\{x_n\}_{n=1}^\infty = \operatorname{span}\{z_n\}_{n=1}^\infty \textit{ and } \operatorname{span}\{x_n^*\}_{n=1}^\infty = \operatorname{span}\{z_n^*\}_{n=1}^\infty.$$

Proof. Define $x_1 = z_{n_1}$, where n_1 is the first index n such that $z_n \neq 0$. By (M2) there exists $z_{m_1}^*$ such that $\langle x_1, z_{m_1}^* \rangle \neq 0$. Put

$$x_1^* := \frac{z_{m_1}^*}{\langle x_1, z_{m_1}^* \rangle}.$$

Then $\langle x_1, x_1^* \rangle = 1$. Find the first index m (call it m_2) such that $z_m^* \notin \operatorname{span}\{x_1^*\}$. Let $x_2^* := z_{m_2}^* - \langle x_1, z_{m_2}^* \rangle x_1^*$. Then $\langle x_1, x_2^* \rangle = 0$. Obviously $x_2^* \neq 0$; hence, by (M1), we can find z_{n_2} such that $\langle z_{n_2}, x_2^* \rangle \neq 0$. Put

$$x_2 := \frac{z_{n_2} - \langle z_{n_2}, x_1^* \rangle x_1}{\langle z_{n_2}, x_2^* \rangle}.$$

Then $\langle x_2, x_1^* \rangle = 0$ and $\langle x_2, x_2^* \rangle = 1$. Find the first index n (call it n_3) such that $z_n \notin \operatorname{span}\{x_1, x_2\}$, and put $x_3 := z_{n_3} - \langle z_{n_3}, x_1^* \rangle x_1 - \langle z_{n_3}, x_2^* \rangle x_2$. Then $\langle x_3, x_1^* \rangle = 0$, $\langle x_3, x_2^* \rangle = 0$, and $x_3 \neq 0$. Using (M2), find $z_{m_3}^*$ such that $\langle x_3, z_{m_3}^* \rangle \neq 0$. Continue in this way to get $\{x_n; x_n^*\}_{n=1}^\infty$ with the required properties. □

Theorem 1.22 (Markushevich [Mark43]). *Every separable Banach space has an M-basis (which can be taken to be 1-norming). If, moreover, X has a separable dual, the M-basis can be taken to be shrinking. Every Banach space X such that (X^*, w^*) is separable has a total biorthogonal system.*

Proof. For the first assertion, take two sets $\{z_n\}_{n=1}^\infty$ and $\{z_n^*\}_{n=1}^\infty$, the first dense in $(X, \|\cdot\|)$ and the second dense in (B_{X^*}, w^*) (a metrizable compact space), and apply Lemma 1.21. Observe that in this case it is always possible to choose a 1-norming system $\{z_n^*\}_{n=1}^\infty$ in X^*, and this gives the assertion in parentheses. If X^* is separable, just take $\{z_n^*\}_{n=1}^\infty$ to be $\|\cdot\|$-dense in X^*. For the last statement, take $\{z_n^*\}_{n=1}^\infty$ dense in (X^*, w^*) and put $Y := \overline{\operatorname{span}}^{\|\cdot\|}\{z_n^*\}_{n=1}^\infty$. Let $\{d_n^*\}_{n=1}^\infty$ be a dense set in $(B_Y, \|\cdot\|)$. For each $n \in \mathbb{N}$, select a countable set $M_n \subset B_X$ such that $\|d_n^*\| = \sup\{|\langle x, d_n^*\rangle|; x \in M_n\}$. Let $Z := \overline{\operatorname{span}} \bigcup_{n=1}^\infty M_n$ and choose a dense set $\{z_n\}_{n=1}^\infty$ in Z. Then $\{z_n\}_{n=1}^\infty$ separates points of $\operatorname{span}\{z_n^*\}_{n=1}^\infty$and $\{z_n^*\}_{n=1}^\infty$ separates points of $\operatorname{span}\{z_n\}_{n=1}^\infty$. Now apply Lemma 1.21. □

1.4 Bounded Minimal Systems

The procedure for constructing M-bases, described in the previous section, does not produce automatically bounded M-bases (i.e., bases $\{x_n; x_n^*\}_{n=1}^\infty$ where $\sup\{\|x_n\|.\|x_n^*\|; n \in \mathbb{N}\} < \infty$). The existence of such an M-basis in every separable Banach space was an open problem for many years and was solved in the positive by Ovsepian and Pełczyński. It was later adjusted by Pełczyński and independently Plichko, to produce an "almost" Auerbach (even norming) M-basis in every separable Banach space. Prior to presenting this result, we follow the lead of Davis and Johnson to produce a special Auerbach system in every separable Banach space; together with ideas of Singer, this process gives "almost" Auerbach fundamental (resp. total) systems in every separable Banach space. In order to illustrate the difficulties in obtaining a true Auerbach system in every separable Banach space (answering in the negative a question of Singer [Sing81, Problem 8.2.b]), we also present a result due to Plichko that says that no such system exists in certain separable $C(K)$ spaces if we request that the space generated by the functional coefficients contain the Dirac deltas.

Definition 1.23. *A biorthogonal system* $\{x_n; x_n^*\}_{n=1}^\infty$ *for a separable Banach space* X *is called* λ-bounded *for some* $\lambda \geq 1$ *if* $\sup\{\|x_n\|.\|x_n^*\|; n \in \mathbb{N}\} \leq \lambda$. *The biorthogonal system is called* bounded *if it is* λ*-bounded for some* $\lambda \geq 1$.

Remark 1.24. Clearly, by the Hahn-Banach theorem, a biorthogonal system $\{x_n; x_n^*\}_{n=1}^\infty$ is bounded if and only if $\{x_n\}_{n=1}^\infty$ is a *uniformly minimal system* (i.e., a minimal system such that $\inf_{m\in\mathbb{N}} \operatorname{dist}\left(\frac{x_m}{\|x_m\|}, \overline{\operatorname{span}}\{x_n\}_{n\in\mathbb{N}, n\neq m}\right) \geq K > 0$). In this case, the system $\{x_n\}_{n=1}^\infty$ is called more precisely K*-uniformly minimal*, and it is clear that

$$\inf_{m\in\mathbb{N}} \operatorname{dist}\left(\frac{x_m}{\|x_m\|}, \overline{\operatorname{span}}\{x_n\}_{n\in\mathbb{N}, n\neq m}\right) = \Big(\sup\{\|x_n\|.\|x_n^*\|; n \in \mathbb{N}\}\Big)^{-1}.$$

Using this remark, it is simple to see that *every separable Banach space X contains an unbounded M-basis.* Indeed, let $\{x_n; x_n^*\}_{n=1}^\infty$ be any M-basis in $X \times X^*$ (its existence is guaranteed by Theorem 1.22). We may assume that $\|x_n\| = 1$ for every n. Let us define, for $n \in \mathbb{N}$,

$$\begin{aligned} v_{2n-1} &:= x_{2n-1}, & v_{2n} &:= x_{2n-1} + \frac{1}{2n} x_{2n}, \\ v_{2n-1}^* &:= x_{2n-1}^* - 2n x_{2n}^*, & v_{2n}^* &:= 2n x_{2n}^*. \end{aligned}$$

It is clear that $\{v_n; v_n^*\}_{n=1}^\infty$ is a biorthogonal system in $X \times X^*$. Moreover,

$$\begin{aligned} &\operatorname{span}\{v_i; 1 \le i \le n\} = \operatorname{span}\{x_i; 1 \le i \le n\}, \\ &\operatorname{span}\{v_i^*; 1 \le i \le n\} = \operatorname{span}\{x_i^*; 1 \le i \le n\}, \ \forall n \in \mathbb{N}, \end{aligned}$$

so $\{v_n; v_n^*\}_{n=1}^\infty$ is an M-basis in $X \times X^*$. Note that, for all $n \in \mathbb{N}$,

$$1 - \frac{1}{2n} \le \|v_{2n}\|.$$

Then

$$\left\| \frac{v_{2n}}{\|v_{2n}\|} - \frac{x_{2n-1}}{\|v_{2n}\|} \right\| = \frac{1}{\|v_{2n}\|} \frac{1}{2n} \le \frac{1}{1 - 1/2n} \frac{1}{2n} = \frac{1}{2n-1},$$

and so

$$\operatorname{dist}\left(\frac{v_{2n}}{\|v_{2n}\|}, \operatorname{span}\{v_i; 1 \le i \le 2n-1\} \right) \le \frac{1}{2n-1}, \ \forall n.$$

It follows that

$$\lim_n \operatorname{dist}\left(\frac{v_{2n}}{\|v_{2n}\|}, \operatorname{span}\{v_i; i \in \mathbb{N},\ i \ne 2n\} \right) = 0,$$

and so the M-basis $\{v_n; v_n^*\}_{n=1}^\infty$ is not bounded.

Lemma 1.25 (Davis and Johnson [DaJo73a]). *Let X be a separable Banach space, and set $m_k := \frac{k(k+1)}{2}$, $k = 0, 1, 2, \ldots$. Then X admits a biorthogonal system $\{x_n; x_n^*\}_{n=1}^\infty$ satisfying:*

(i) $\|x_n\| = \|x_n^*\| = \langle x_n, x_n^* \rangle = 1$ *for all* $n \in \mathbb{N}$.
(ii) *For* $x \in \overline{\operatorname{span}}\{x_n\}_{n=1}^\infty$, $x = \lim_{k\to\infty} \sum_{i=1}^{m_k} \langle x, x_i^* \rangle x_i$.
(iii) $\{x_i\}_{i=m_k+1}^{m_{k+1}}$ *is* $(1 + \frac{1}{k+1})$*-equivalent to the canonical basis of* ℓ_2^{k+1}, $k = 0, 1, 2, \ldots$.
(iv) $\overline{\operatorname{span}}\{x_n; n \in \mathbb{N}\} + \{x_n^*; n \in \mathbb{N}\}_\perp$ *is dense in* X.

Proof. Let $(d_n)_{n=0}^\infty$ be a dense sequence in X with $d_0 = 0$. In order to prove the lemma, it is sufficient to define sequences (x_n) in X and (x_n^*) in X^* and finite sets $\emptyset =: S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_{X^*}$ satisfying (i), (iii), and

(v) $x_{m_k+j} \in \left(S_k \cup \{x_i^*\}_{i=1}^{m_k+j-1} \right)_\perp$, $j = 1, 2, \ldots, k+1$, $k = 0, 1, 2, \ldots$,

(vi) $x^*_{m_k+j} \in \left(\{d_i\}_{i=0}^{k} \cup \{x_i\}_{i=1}^{m_k+j-1}\right)^{\perp}$, $j = 1, 2, \ldots, k+1$, $k = 0, 1, 2, \ldots$,

(vii) For each $k = 0, 1, 2, \ldots$ and $x \in \operatorname{span}\{x_i\}_{i=1}^{m_{k+1}}$ there is $x^* \in S_{k+1}$ such that $\|x\| \le \left(1 + \frac{1}{k+1}\right) |\langle x, x^* \rangle|$.

Then $\{x_n; x_n^*\}_{n=1}^{\infty}$ is biorthogonal by (i), (v), and (vi). To get (ii), we can use first (vii) and (v) to obtain that, for any finite sequence (a_i) of scalars,

$$\begin{aligned}\left\|\sum_{i=1}^{m_k} a_i x_i\right\| &\le \left(1 + \frac{1}{k}\right) \max_{x^* \in S_k} \left|\left\langle \sum_{i=1}^{m_k} a_i x_i, x^* \right\rangle\right| \\ &= \left(1 + \frac{1}{k}\right) \max_{x^* \in S_k} \left|\left\langle \sum_{i=1}^{\infty} a_i x_i, x^* \right\rangle\right| \le \left(1 + \frac{1}{k}\right) \left\|\sum_{i=1}^{\infty} a_i x_i\right\|. \end{aligned} \tag{1.1}$$

For every $k \in \mathbb{N}$, let $P_{m_k} : \operatorname{span}\{x_n\}_{n=1}^{\infty} \to \operatorname{span}\{x_n\}_{n=1}^{\infty}$ be the linear projection given by

$$P_{m_k}\left(\sum_{i=1}^{\infty} a_i x_i\right) := \sum_{i=1}^{m_k} a_i x_i,$$

where (a_i) is any eventually zero sequence of scalars. By (1.1), P_{m_k} is a continuous mapping and $\|P_{m_k}\| \le \left(1 + \frac{1}{k}\right)$. It then has a (unique) continuous linear extension (denoted again by P_{m_k}) from $Y := \overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty}$ into itself, and $\|P_{m_k}\| \le \left(1 + \frac{1}{k}\right)$.

Fix $x \in Y$. Observe first that

$$P_{m_k}(x) = \sum_{i=1}^{m_k} \langle x, x_i^* \rangle x_i \text{ for every } k \in \mathbb{N}.$$

To check this, fix $k \in \mathbb{N}$ and take $n \ge m_k$. Then $\{x_n; x_n^* \restriction Y\}_{n=1}^{\infty}$ is a fundamental biorthogonal system in $Y \times Y^*$. By (iii) in Fact 1.5, we can write $x = \sum_{i=1}^{n} \langle x, x_i^* \rangle x_i + z_n$, where $z_n \in \overline{\operatorname{span}}\{x_{n+1}, x_{n+2}, \ldots\} \subset Y$. Then, by the continuity of P_{m_k}, we get $P_{m_k}(z_n) = 0$ and then $P_{m_k}(x) := \sum_{i=1}^{m_k} \langle x, x_i^* \rangle x_i$, as stated.

Second, given $\varepsilon > 0$, we can find $k_0 \in \mathbb{N}$ and $y_{k_0} \in \operatorname{span}\{x_n\}_{n=1}^{m_{k_0}}$ such that $\|x - y_{k_0}\| < \varepsilon$. Then, for $k \ge k_0$,

$$\begin{aligned}\|x - P_{m_k}(x)\| &\le \|x - y_{k_0}\| + \|y_{k_0} - P_{m_k}(y_{k_0})\| + \|P_{m_k}(y_{k_0}) - P_{m_k}(x)\| \\ &= \|x - y_{k_0}\| + \|P_{m_k}(y_{k_0}) - P_{m_k}(x)\| < \varepsilon + \left(1 + \frac{1}{k}\right)\varepsilon < 3\varepsilon.\end{aligned}$$

This proves that $P_{m_k}(x) \to x$ when $k \to \infty$. This is (ii).

Finally, from (vi) we have, for every $k \in \mathbb{N}$, $d_k \in \{x^*_{m_k+1}, x^*_{m_k+2}, \ldots\}_{\perp} = \operatorname{span}\{x_i\}_{i=1}^{m_k} \oplus \{x_n^*; n \in \mathbb{N}\}_{\perp} \subset Y + \{x_n^*; n \in \mathbb{N}\}_{\perp}$ (here we used (ii) in Fact 1.3), so (iv) holds, too.

It then remains to prove the existence of sequences (x_n) in X and (x_n^*) in X^* such that (i), (iii), (v), (vi), and (vii) hold. This will be done by induction.

To begin, pick x_1 and x_1^* to satisfy (i). Using the compactness of the unit ball of the finite-dimensional space $\operatorname{span}\{x_1\}$ and the Hahn-Banach theorem, pick a finite set S_1 in S_{X^*} to satisfy (vii) for $k = 0$. Assume that, for some $k \in \mathbb{N}$, steps 1 to k already produced $\{x_i; x_i^*\}_{i=1}^{m_k}$ and $\{S_i\}_{i=1}^{k}$. For the next step, set $m := 3k + m_{k+1} + 1$ and use the Dvoretzky theorem (see, for example, [Day73, Thm. IV.2.3]) to get an isomorphism $T : Z \to \ell_2^m$ from an m-dimensional subspace $Z \subset (\{x_i^*\}_{i=1}^{m_k} \cup S_k)_\perp$ onto ℓ_2^m equipped with the $\|\cdot\|_2$-norm such that $\|T\| \leq (1 + 1/k)$ and $\|T^{-1}\| = 1$. We shall define $\{x_i\}_{i=m_k+1}^{m_{k+1}}$ in Z and $\{x_i^*\}_{i=m_k+1}^{m_{k+1}}$ in X^* to satisfy (i), (v), (vi), and

(viii) $\qquad \{Tx_i\}_{i=m_k+1}^{m_{k+1}}$ is orthogonal in ℓ_2^m

by induction. First of all, observe that

$$\dim Z = m > m_k + k \geq \dim \operatorname{span}\{\{x_i\}_{i=1}^{m_k} \cup \{d_i\}_{i=0}^{k}\}.$$

By Lemma 1.19, we can find $x_{m_k+1} \in S_Z$ (so (v) holds for this vector) such that $\operatorname{dist}(x_{m_k+1}, \operatorname{span}\{\{x_i\}_{i=1}^{m_k} \cup \{d_i\}_{i=0}^{k}\}) = 1$. By the Hahn-Banach theorem, we can find $x_{m_k+1}^* \in S_{X^*}$ such that $\langle x_{m_k+1}, x_{m_k+1}^* \rangle = 1$ and (vi) holds for this vector. Assume that, for some $j \in \{2, 3, \ldots, k+1\}$, $\{x_i\}_{i=m_k+1}^{m_k+j-1}$ and $\{x_i^*\}_{i=m_k+1}^{m_k+j-1}$ were already defined to satisfy (i), (v), (vi), and (viii). Let W be the orthogonal complement to $\operatorname{span}\{Tx_i\}_{i=m_k+1}^{m_k+j-1}$ in ℓ_2^m, so $\dim W = m - (j-1)$. Set $G := T^{-1}(W) \cap \left(\operatorname{span}\{x_i^*\}_{i=m_k+1}^{m_k+j-1}\right)_\perp$ and $F := \operatorname{span}\{\{d_i\}_{i=1}^{k} \cup \{x_i\}_{i=1}^{m_k+j-1}\}$. Then $\dim G \geq m - (j-1) - (j-1) = m - 2(j-1) \geq m - 2k = k + m_{k+1} + 1$ and $\dim F \leq k + m_k + j - 1 < k + m_{k+1}$, so $\dim F < \dim G$ and again we can apply Lemma 1.19 to get $x_{m_k+j} \in S_G$ such that $\operatorname{dist}(x_{m_k+j}, F) = 1$. Apply the Hahn-Banach theorem one more time to get $x_{m_k+j}^* \in S_{X^*}$, satisfying (i) and (vi). This finishes the finite induction process and gives $\{x_i\}_{i=m_k+1}^{m_{k+1}}$ in Z and $\{x_i^*\}_{i=m_k+1}^{m_{k+1}}$ in X^*.

Now using the compactness of the unit ball of the finite-dimensional space $\operatorname{span}\{x_i\}_{i=1}^{m_{k+1}}$ and the Hahn-Banach theorem, pick a finite set $S_{k+1} \supset S_k$ in S_{X^*} to satisfy (vii). This completes step $k+1$. Inductively we get $\{x_n; x_n^*\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$. Clearly, they satisfy (i) and (v)–(viii), while (iii) follows from (viii) and the fact that T defined above satisfies $\|T\| \leq (1 + 1/k)$. □

Although the result in the next corollary will be improved in Theorem 1.27, the method of its proof, which can be traced back to Singer [Sing73], will be used often in this book.

Corollary 1.26. *Let X be a separable Banach space. Then, for every $\varepsilon > 0$, there exists*

(i) *a $(1+\varepsilon)$-bounded fundamental biorthogonal system in $X \times X^*$; and*
(ii) *a $(1+\varepsilon)$-bounded total biorthogonal system in $X \times X^*$.*

Proof. (i) Let $\{x_n; x_n^*\}_{n=1}^{\infty}$ be the biorthogonal system constructed in Lemma 1.25. Fix a sequence (y_n) dense in the unit ball of $\{x_n^*; n \in \mathbb{N}\}_{\perp}$. Fix $\varepsilon > 0$.

Let us denote $B_0 := \{1\}$ and $B_k := \{m_k + 1, \ldots, m_{k+1}\}$, where $m_k := k(k+1)/2$, $k \in \mathbb{N}$. Arrange the sequence (n) in a matrix by putting consecutive blocks B_n along the inverse diagonals (so B_0 goes to position (1, 1), B_1 to (1, 2), B_2 to (2, 1), B_3 to (1, 3), B_4 to (2, 2), B_5 to (3, 1), and so on).

Fix a row $n \in \mathbb{N}$. Then, for every $N \in \mathbb{N}$, it is possible to find a block $B_{k(n,N)}$ in this row and positive real numbers $a_{n,i}$, $i \in B_{k(n,N)}$ such that

$$\left(\sum_{i \in B_{k(n,N)}} a_i^2\right)^{1/2} \leq 1, \quad \sum_{i \in B_{k(n,N)}} a_i \geq N.$$

For i in this row and not in $\bigcup_{N \in \mathbb{N}} B_{k(n,N)}$, put $a_i = 1$. Put $w_i := x_i - \varepsilon y_n$ for every i in this row n. Do this for every row n. We obtain a system $\{w_i; x_i^*\}_{i=1}^{\infty}$, clearly a $(1+\varepsilon)$-bounded biorthogonal system in $X \times X^*$.

We claim that it is fundamental. Indeed, let $x^* \in \{w_i; i \in \mathbb{N}\}^{\perp}$, $\|x^*\| = 1$. Then $\langle x_i, x^*\rangle = \varepsilon\langle y_n, x^*\rangle$ for all i in row n, for all $n \in \mathbb{N}$. Fix again a row n. Given $N \in \mathbb{N}$, find $B_{k(n,N)}$ as above. Then

$$\begin{aligned} 2 > 2\left(\sum_{i \in B_{k(n,N)}} a_i^2\right)^{1/2} \\ \geq \left\|\sum_{i \in B_{k(n,N)}} a_i x_i\right\| \geq \left|\left\langle \sum_{i \in B_{k(n,N)}} a_i x_i, x^*\right\rangle\right| \\ = \varepsilon\langle y_n, x^*\rangle \sum_{i \in B_{k(n,N)}} a_i \geq \varepsilon\langle y_n, x^*\rangle N. \end{aligned}$$

(The second inequality follows from the 2-equivalence of the block with the ℓ_2-basis; see (iii) in Lemma 1.25.) This is true for every $N \in \mathbb{N}$, so $\langle y_n, x^*\rangle = 0$. As this happens for every $n \in \mathbb{N}$, it follows that $x^* \in \{y_n : n \in \mathbb{N}\}^{\perp}$, so $x^* \in \{x_n; n \in \mathbb{N}\}^{\perp}$. Moreover, $x^* \in \{x_n^*\}_{\perp}^{\perp}$. From property (iv) in Lemma 1.25, we obtain $x^* = 0$.

(ii) Start again from the system $\{x_n; x_n^*\}_{n=1}^{\infty}$ given in Lemma 1.25. Fix $\varepsilon > 0$. Note first that, due to (iv) in this lemma, $x_n^* \xrightarrow{w^*} 0$. Let $\{z_n : n \in \mathbb{N}\}$ be a w^*-dense subset of the unit ball of $\{x_n; n \in \mathbb{N}\}^{\perp}$. Use the matrix of indices defined in part (i) of the proof. For $n \in \mathbb{N}$, let us define

$$w_i^* := x_i^* - \varepsilon z_n^* \text{ if } i \text{ is in row } n.$$

Obviously, $\{x_n; w_n^*\}_{n=1}^{\infty}$ is a $(1+\varepsilon)$-bounded biorthogonal system in $X \times X^*$. We claim that it is total. Indeed, let $x \in \{w_n^*; n \in \mathbb{N}\}_{\perp}$. Fix $n \in \mathbb{N}$. Then $\langle x, x_i^*\rangle = \varepsilon\langle x, z_n^*\rangle$ if i belongs to row n. Let $i \to \infty$ in this row. We get $\langle x, z_n^*\rangle = 0$. This holds for every $n \in \mathbb{N}$, so $x \in \overline{\operatorname{span}}\{x_n; n \in \mathbb{N}\}$. Moreover, $\langle x, x_n^*\rangle = 0$ for all $n \in \mathbb{N}$. From (ii) in Lemma 1.25, we get $x = 0$. □