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Alexander J. Zaslavski

Turnpike Phenomenon and Symmetric Optimization Problems

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Preface

The book is devoted to the study of symmetric optimization, variational and optimal control problems in infinite dimensional spaces, and turnpike properties of their approximate solutions.

It is a well-known fact that solutions exist for minimization problems on compact metric spaces with lower semicontinuous objective functions. Solutions also exist for minimization problems on metric spaces such that all their bounded closed subsets are compact if objective functions are lower semicontinuous and satisfy a coercivity growth condition. Since in the book we do not use compactness assumptions on spaces where optimization problems are considered, the situation becomes more difficult. In order to overcome this difficulty, we use the so-called generic approach which is applied fruitfully in many areas of analysis (see, for example, [11, 25, 37–40, 42, 62–64, 79, 97, 98, 100, 101, 104, 117, 119] and the references mentioned there).

According to the generic approach, we say that a property holds for a generic (typical) element of a complete metric space (or the property holds generically) if the set of all elements of the metric space possessing this property contains a G_δ everywhere dense subset of the metric space, which is a countable intersection of open everywhere dense sets. In our book [143] we use this approach in order to establish a generic existence of solutions for various classes of minimization problems. In Chapters 4–10 of Zaslavski [143], it is shown that solutions of minimization problems exist generically for different classes of problems which are identified with corresponding spaces of functions with natural complete metrics. Many generic results of this type are collected in Zaslavski [143]. Among them, generic existence results for classes of constrained minimization problems with different type of constraints, for classes of parametric minimization problems, for classes of problems with increasing objective functions, and for classes of vector minimization problems and infinite horizon optimization problems. Any of these classes of problems is identified with a space of functions equipped with a natural complete metric, and it is shown that there exists a G_δ everywhere dense subset of the space of functions such that for any element of this subset the corresponding minimization problem possesses a unique solution and that any minimizing sequence converges to this unique solution. These results are obtained as realizations of variational principles which are extensions or concretization of the variational principle established in Ioffe and Zaslavski [62]. Instead of considering the existence of a solution for a single minimization problem, we investigate it for a class (space) of problems and show that a unique solution exists for most of the problems in the class. It turns out that our results provide a proper explanation of what happens with individual problems in practice. The thing is that because of computational errors and errors in data which always present, instead of solving an individual problem with certain objective (constraint) function we actually solve a similar problem with an approximation of the objective (constraint) function. Probably this new problem possesses desirable properties which hold for most of the problems. This explains, for example, the fact that sometimes in practice we can get results which are better than their theoretical expectations. This happens when an algorithm works under some conditions which hold for "almost all" problems.

Another type of generic existence and well-posedness results is obtained in Mizel and Zaslavski [89] where some class of minimization problems arising in crystallography is studied. Again, like in Zaslavski [143], this class of problems is identified with a space of functions equipped with a natural complete metric. It is shown that there exists a G_δ everywhere dense subset of the space of functions such that for any element of this subset the corresponding minimization problem possesses exactly two different solutions. This happens because the optimization problems in Mizel and Zaslavski [89] are symmetric. Namely, all the objective functions there are even. The result of Mizel and Zaslavski [89] and the importance of the optimization problems arising in crystallography give a strong motivation for our current research on symmetric optimization problems which is presented in this book. In Chapter 2 of this book, we consider several classes of symmetric optimization problems. Any of these classes of problems are identified with a space of functions equipped with a natural complete metric, and it is shown that there exists a G_δ everywhere dense subset of the space of functions such that for any element of this subset the corresponding minimization problem possesses exactly two different solutions and that any minimizing sequence converges to this solution set. These results are obtained as realizations of a variational principle. Some results of Chapter 2 were obtained recently in Zaslavski [171, 174] while some others are new. In Chapter 3, we discuss existence of solutions and well-posedness of for certain classes of parametric minimization problems. Its results are new. In Chapter 4, we present preliminaries which we need in order to study turnpike properties of infinite dimensional optimal control problems. We discuss Banach space valued functions, unbounded operators, C_0 semigroups, evolution equations, and admissible control operators.

In Chapters 5 and 6, we study the turnpike phenomenon for symmetric variational and optimal control problems in infinite dimensional spaces. To have the turnpike property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the objective function and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

The turnpike property was discovered by P. Samuelson in 1948 when he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). It is well known in the economic literature, where it was studied for various models of economic growth. Usually for these models a turnpike is a singleton.

Now it is well known that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems. In our research, using the Baire category (generic) approach, it was shown that the turnpike property holds for a generic (typical) variational problem [134] and for a generic optimal control problem [152].

In this book, we are interested in individual (non-generic) turnpike results for symmetric variational and optimal control problems and in the stability of the turnpike phenomenon under small perturbations of integrands. It turns out that for these problems the turnpike is a singleton $\bar{x}$, such that $(\bar{x}, \bar{u})$ is a global minimizer of the integrand for some $\bar{u}$. It is interesting to note that if $\bar{u}$ is not zero, then the turnpike is not an admissible trajectory (function) for the corresponding problem. This is a new phenomenon which does not occur in our previous study of the turnpike properties for problems without symmetry. In Chapter 5, we study turnpike properties of symmetric variational problems, while optimal control problems of symmetric type are analyzed in Chapter 6. The results of Chapters 5 and 6 are new.

In Chapter 7, we present generic results for symmetric optimization problems arising in crystallography obtained in Mizel and Zaslavski [89], Zaslavski [136]. Chapter 8 contains turnpike results for discrete dispersive dynamical systems obtained in Zaslavski [141, 172, 173, 175].

Haifa, Israel — Alexander J. Zaslavski
October 19, 2021

Contents

Chapter 1
Introduction

In this chapter we consider optimization problems on complete metric spaces without compactness assumptions, optimization problems arising in crystallography and symmetric optimization problems in abstract spaces. We also discuss turnpike properties in the calculus of variations. To have the turnpike property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the integrand and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

1.1 Generic Existence of Solutions of Minimization Problems

In our monograph [143] we use the generic approach in order to study the existence of solutions of various optimization problems in Banach spaces and complete metric spaces and show that for most of problems (in the sense of Baire category) all minimizing sequences converge to a unique solution. Here we demonstrate simple applications of this generic approach in optimization.

Let (X, ρ) be a bounded complete metric space. Put

$$\text{diam}\,(X) = \sup\{\rho(x, y) :\ x, y \in X\}$$

and denote by $\mathcal{M}$ the space of all bounded continuous functions $f : X \to R^1$ equipped with a metric

$$d(f_1, f_2) = \sup\{|f_1(x) - f_2(x)| :\ x \in X\},\ \ f_1, f_2 \in \mathcal{M}.$$

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Clearly, $(\mathcal{M}, d)$ is a complete metric space. It is well-known that if the space X is compact, then for every $f \in \mathcal{M}$, the problem

$$\text{minimize } f(x) \text{ subject to } x \in X$$

possesses a solution. Since we do not assume the compactness of X the situation becomes more difficult and less understood. Nevertheless, there exists a subset $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}$ such that for each element of $\mathcal{F}$ the corresponding minimization problem has a unique solution.

Denote by $\mathcal{L}$ the set of all $f \in \mathcal{M}$ for which the problem

$$\text{minimize } f(x) \text{ subject to } x \in X$$

possesses a solution. It turns out that $\mathcal{L}$ is an everywhere dense subset of $\mathcal{M}$.

Indeed, let $f \in \mathcal{M}$ and ϵ be a positive number. There exists $x_\epsilon \in X$ such that

$$f(x_\epsilon) \le f(x) + \epsilon \text{ for all } x \in X.$$

Put

$$f_\epsilon(x) = \max\{f(x),\ f(x_\epsilon)\},\ x \in X.$$

Clearly, $f_\epsilon \in \mathcal{M}$, x_ϵ is a point of minimum of the function f_ϵ and therefore $f_\epsilon \in \mathcal{L}$. It is not difficult to see that for all $x \in X$,

$$f(x) \le f_\epsilon(x) \le f(x) + \epsilon$$

and that $d(f, f_\epsilon) \le \epsilon$. Thus $\mathcal{L}$ is an everywhere dense subset of $\mathcal{M}$.

For any $f \in \mathcal{L}$ let $x_f \in X$ be a point of minimum of the function f such that

$$f(x_f) \le f(x) \text{ for all } x \in X.$$

Let $f \in \mathcal{L}$ and n be a natural number. Define $f_n \in \mathcal{M}$ by

$$f_n(x) = f(x) + (4n)^{-1}(\text{diam}(X) + 1)^{-1}\rho(x, x_f),\ x \in X.$$

It is easy to see that

$$d(f, f_n) \le (4n)^{-1}.$$

Choose a number

$$\delta(f, n) \in (0, (12n^2)^{-1}(\text{diam}(X) + 1)^{-1})$$

and set

$$V(f,n) = \{g \in \mathcal{M} : d(f_n, g) < \delta(f,n)\}.$$

Assume that

$$g \in V(f,n), \ x \in X \text{ and } g(x) \le \inf\{g(z) : z \in X\} + \delta(f,n).$$

It follows from the relations above, the definitions of f_n, $V(f,n)$, x_f and $\delta(f,n)$ that

$$f(x) + (4n)^{-1}(\mathrm{diam}(X)+1)^{-1}\rho(x,x_f)$$

$$= f_n(x) \le g(x) + \delta(f,n) \le g(x_f) + 2\delta(f,n)$$

$$\le f_n(x_f) + 3\delta(f,n) = f(x_f) + 3\delta(f,n) \le f(x) + 3\delta(f,n)$$

and that

$$\rho(x,x_f) \le 3\delta(f,n)4n(\mathrm{diam}(X)+1) < 1/n.$$

Thus for each $g \in V(f,n)$ the following property holds:
if $x \in X$ satisfies $g(x) \le \inf\{g(z) : z \in X\} + \delta(f,n)$, then $\rho(x,x_f) < 1/n$. Set

$$\mathcal{F} = \cap_{n=1}^{\infty} \cup \{V(f,k) : f \in \mathcal{L} \text{ and an integer } k \ge n\}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}$.

Assume that $g \in \mathcal{F}$ and that n is a natural number. By definition of $\mathcal{F}$, there exists a natural number $k \ge n$ and $f \in \mathcal{L}$ such that $g \in V(f,k)$. Combined with the property above this implies that the following property holds:

for each $x \in X$ satisfying $g(x) \le \inf\{g(z) : z \in X\} + \delta(f,k)$ the inequality $\rho(x,x_f) < 1/n$ holds.

Since n is an arbitrary natural number we conclude that any minimizing sequence for the problem

$$\text{minimize } g(z) \text{ subject to } z \in X$$

converges to its unique solution, where g is an arbitrary element of $\mathcal{F}$.

There is also another way to prove the result obtained above.

Let $f \in \mathcal{M}$ and n be a natural number. Choose a positive number

$$\delta(f,n) < (32n^2)^{-1}(\mathrm{diam}(X)+1)^{-1}$$

and choose $x_f \in X$ such that

$$f(x_f) \le \inf\{f(z) : z \in X\} + \delta(f, n).$$

Put

$$f_n(z) = f(z) + (4n)^{-1}(\mathrm{diam}(X) + 1)^{-1}\rho(z, x_f), \ z \in X.$$

Clearly $f_n \in \mathcal{M}$ and

$$d(f, f_n) \le (4n)^{-1}.$$

Set

$$V(f, n) = \{g \in \mathcal{M} : d(f_n, g) < \delta(f, n)\}.$$

Assume that

$$g \in V(f, n), \ x \in X \text{ and } g(x) \le \inf\{g(z) : z \in X\} + \delta(f, n).$$

By the relations above, the definitions of f_n, $V(f, n)$, x_f, and $\delta(f, n)$,

$$f(x) + (4n)^{-1}(\mathrm{diam}(X) + 1)^{-1}\rho(x, x_f) = f_n(x)$$

$$\le g(x) + \delta(f, n) \le g(x_f) + 2\delta(f, n)$$

$$\le f_n(x_f) + 3\delta(f, n) = f(x_f) + 3\delta(f, n) \le f(x) + 4\delta(f, n)$$

and

$$\rho(x, x_f) \le 4\delta(f, n)4n(\mathrm{diam}(X) + 1) < 1/n.$$

Thus for each $g \in V(f, n)$ the following property holds:

if $x \in X$ satisfies $g(x) \le \inf\{g(z) : z \in X\} + \delta(f, n)$, then $\rho(x, x_f) < 1/n$. Set

$$\mathcal{F} = \cap_{k=1}^{\infty} \cup \{V(f, n) : f \in \mathcal{M} \text{ and an integer } n \ge k\}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}$.

Assume that $g \in \mathcal{F}$ and that k is a natural number. By definition of $\mathcal{F}$, there exists a natural number $n \ge k$ and $f \in \mathcal{M}$ such that $g \in V(f, n)$. Combined with the property above this implies that the following property holds:

for each $x \in X$ satisfying $g(x) \le \inf\{g(z) : z \in X\} + \delta(f, n)$ the inequality $\rho(x, x_f) < 1/k$ holds.

Since k is an arbitrary natural number we conclude that any minimizing sequence for the problem

$$\text{minimize } g(z) \text{ subject to } z \in X$$

converges to its unique solution, where g is an arbitrary element of $\mathcal{F}$.

We presented above two proofs of the same generic well-posedness result. In both proofs for a given $f \in \mathcal{M}$ and a positive number ϵ we defined a function $\bar{f} \in \mathcal{M}$, $\bar{x} \in X$ and a positive constant δ such that if $x \in X$ is a δ-approximate solution of the problem $g(z) \to \min$, $z \in X$, where g belongs to a δ-neighborhood of $\bar{f}$, then x belongs to an ϵ-neighborhood of $\bar{x}$.

Many results of this type for various classes of minimization problems are collected in [143]. Here we briefly describe some of them.

In Chapter 4 of [143] we consider problems

$$\text{minimize } f(x) \text{ subject to } x \in X$$

where X is a complete metric space and f belongs to a space of lower semicontinuous functions on X satisfying a certain growth condition. The class of minimization problems is identified with this space of functions. We endow the space of functions with an appropriate metric and show that for most functions f in this metric space the corresponding minimization problem has a unique solution and is well-posed.

In Chapter 4 of [143] we also consider the following class of minimization problems

$$\text{minimize } f(x) \text{ subject to } x \in A$$

studied in [62, 130, 131], where A is a nonempty closed subset of a complete metric space X and f belongs to a space of lower semicontinuous functions on X. This class of problems is identified with a space of pairs (f, A) which is equipped with appropriate complete uniformities. It is shown that for a typical pair (f, A) the corresponding minimization problem has a unique solution and is well-posed.

In Chapter 5 of [143] we continue to consider various classes of minimization problems showing that most problems in these classes are well-posed. In that chapter in order to meet this goal we use a porosity notion. As in Chapter 4 we identify a class of minimization problems with a certain complete metric space of functions, study the set of all functions for which the corresponding minimization problem is well-posed, and show that the complement of this set is not only of the first category but also a σ-porous set.

We now recall the concept of porosity [104, 143].

Let (Y, d) be a complete metric space. We denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous in (Y, d) if there exist $\alpha \in (0, 1]$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there exists $z \in Y$ for which

$$B_d(z, \alpha r) \subset B_d(y, r) \setminus E.$$

A subset of the space Y is called σ-porous in (Y, d) if it is a countable union of porous subsets in (Y, d).

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is smaller than the class of sets which have measure 0 and are of the first category.

To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

In Chapter 5 of [143] we employ the notion of porosity in order to study well-posedness of convex minimization problems and well-posedness for a class of equilibrium problems.

The book [143] contains many other generic existence and well-posedness results established for various classes of optimization problems. In particular, in Chapter 6 of [143] we study a parametric family of the problems

$$\text{minimize } f(b, x) \text{ subject to } x \in X$$

on a complete metric space X with a parameter b which belongs to a Hausdorff compact space $\mathcal{B}$. Here $f(\cdot, \cdot)$ belongs to a space of functions on $\mathcal{B} \times X$ endowed with an appropriate uniform structure. Using the generic approach and the notion of porosity we show that for a typical function $f(\cdot, \cdot)$ the minimization problem has a solution for all parameters $b \in \mathcal{B}$.

Chapter 7 of [143] is devoted to the study of problems

$$\text{minimize } f(x) \text{ subject to } x \in K$$

where K is a closed subset of an ordered Banach space X and f belongs to a space of increasing lower semicontinuous functions on K. Using the generic approach and the notion of porosity we show that for most functions f in this space the corresponding minimization problem has a unique solution.

In Chapter 8 of [143] we study minimization problems with mixed constraints

$$\text{minimize } f(x) \text{ subject to } G(x) = y, \;\; H(x) \le z,$$

where f is a continuous (differentiable) finite valued function defined on a Banach space X, y is an element of a finite dimensional Banach space Y, z is an element of a Banach space Z ordered by a convex closed cone and $G : X \to Y$ and $H : X \to Z$ are continuous (differentiable) mappings. We consider two classes of these problems and show that most of the problems (in the Baire category sense) are well-posed. Our first class of problems is identified with the corresponding complete metric space of quintets (f, G, H, y, z). We show that for a generic quintet (f, G, H, y, z) the corresponding minimization problem has a unique solution and is well-posed. Our second class of problems is identified with the corresponding complete metric space of triples (f, G, H) while y and z are fixed. We show that for a generic triple (f, G, H) the corresponding minimization problem has a unique solution and is well-posed.

In Chapter 9 of [143] we study classes of vector minimization problems on a complete metric space which are identified with the corresponding complete metric spaces of objective functions. We show that for most (in the sense of Baire category) functions the corresponding vector optimization problem has a solution. Using generic approach we also study other interesting properties of such classes of problems.

In all the results discussed above, which are presented in [143], a class of minimization problems on a complete metric space is identified with the corresponding complete metric space of function and it is shown that a generic (typical) problem has a unique solution which is the limit of any minimizing sequence.

1.2 Optimization Problems Arising in Crystallography

We discuss the structure of minimizers of variational problems considered in [56, 66, 89, 136] which describe step-terraces on surfaces of crystals. It is well-known in surface physics that when a crystalline substance is maintained at a temperature T above its *roughening temperature* T_R then the surface stored energy integrand, usually referred to as *surface tension*, is a smooth function β of the azimuthal angle of orientation θ. Furthermore, β obeys the following:

$$\beta(-\theta) = \beta(\pi - \theta) = \beta(\theta),\ 0 < \beta(\pi/2) \leq \beta(\theta) \leq \beta(0).$$

The classical model is given by

$$J(y) = \int_0^S \beta(\theta)ds$$

where s is arclength and y is a function defined on a fixed interval $[0, L]$ whose graph is the locus under consideration:

$$y \in W^{1,1}(0, L),\ \theta = \arctan y' \in [-\pi/2, \pi/2],$$

while β is a positive π-periodic function which belongs to a space of functions described below. Minimization of J subject to appropriate boundary data is a parametric variational problem. It is closely related to the variational problem defining the Wulff crystal shape as that shape for a domain of prescribed area such that the boundary integral with respect to arclength involving the integrand in J [referred to as the surface tension] attains its minimum value.

For each function $f : X \to R^1$ set $\inf(f) = \inf\{f(x) : x \in X\}$.

Denote by $\mathcal{M}$ the set of all functions $\beta \in C^2(R^1)$ which satisfy the following assumption:

(A)

$$\beta(t) \geq 0 \text{ for every number } t \in R^1,$$

$$\beta(\pi/2) \leq \beta(t) \leq \beta(0) \text{ for every number } t \in R^1,$$

$$\beta(t) = \beta(-t) \text{ for every number } t \in R^1,$$

$$\beta(t+\pi) = \beta(t) \text{ for every number } t \in R^1,$$

$$\beta(0) + \beta''(0) \leq 0.$$

For every pair of functions $\beta_1, \beta_2 \in \mathcal{M}$ put

$$\rho(\beta_1, \beta_2) = \sup\{|\beta_1^{(i)}(t) - \beta_2^{(i)}(t)| : t \in R^1, i = 0, 1, 2\}.$$

It is not difficult to show that the metric space $(\mathcal{M}, \rho)$ is complete.

Denote by $\mathcal{M}_r$ the collection of all functions $\beta \in \mathcal{M}$ which satisfy

$$\beta(t) > 0 \text{ for all } t \in R^1,$$

$$\beta(0) + \beta''(0) < 0.$$

It is clear that the set $\mathcal{M}_r$ is nonempty. The following result was obtained in [89].

Proposition 1.1 *$\mathcal{M}_r$ is an open everywhere dense subset of the metric space $(\mathcal{M}, \rho)$.*

Let $\beta \in \mathcal{M}_r$ be given. Set

$$G_\beta(z) = \beta(\arctan(z))(1+z^2)^{1/2}, \ z \in R^1.$$

Evidently, G_β is a continuous function and

$$G_\beta(z) \to \infty \text{ as } z \to \pm\infty, \ \inf(G_\beta < \beta(0).$$

Note that the final inequality above was shown in [56].

We can rewrite the variational functional J in the form

$$J(y) = \int_0^L G_\beta(y')dx.$$

It was shown in [56] that a function $y \in W^{1,1}(0, L)$ is a minimizer of the functional J if and only if

$$|y'| \in \{z \in R^1 : \; G_\beta(z) = \inf(G_\beta)\} \text{ almost everywhere (a. e.).}$$

In [89] using the Baire category approach, it was shown that for a generic (typical) function β the set

$$\{z \in R^1 : \; G_\beta(z) = \inf(G_\beta)\} = \{z_\beta, -z_\beta\}$$

where z_β is a unique positive number depending only on the function β.

Denote by $\mathcal{F}$ the set of all $\beta \in \mathcal{M}_r$ satisfying the following condition:

(C) There exists a number $z_\beta \in R^1$ such that

$$G_\beta(z) > G_\beta(z_\beta) \text{ for every number } z \in R^1 \setminus \{z_\beta, -z_\beta\}.$$

The following result was obtained in [89].

Theorem 1.2 *$\mathcal{F}$ contains a countable intersection of open everywhere dense subsets of* $(\mathcal{M}, \rho)$.

Therefore, in contrast with the results discussed in the previous section, for a generic (typical) $\beta \in \mathcal{M}_r$ the minimization problem

$$G_\beta(z) \to \min, \; z \in R^1$$

has exactly two different minimizers. This happens because the optimization problem above is symmetric. Namely,

$$G_\beta(-z) = G_\beta(z), \; z \in R^1.$$

The result stated above and the importance of the optimization problems arising in crystallography give a strong motivation for our current research on symmetric optimization problems which is presented in this book. In the next section we discuss some of these symmetric problems.

1.3 Symmetric Optimization Problems

Assume that (X, ρ) is a complete metric space. Denote by $\mathcal{M}_l$ the set of all lower semicontinuous and bounded from below functions $f : X \to R^1$. We equip the set $\mathcal{M}_l$ with the uniformity determined by the following base

$$\mathcal{E}(\epsilon) = \{(f, g) \in \mathcal{M}_l \times \mathcal{M}_l : \; |f(x) - g(x)| \le \epsilon \text{ for all } x \in X\},$$

where $\epsilon > 0$. It is known that this uniformity is metrizable (by a metric d) and complete [143].

Denote by $\mathcal{M}_c$ the set of all continuous functions $f \in \mathcal{M}_l$. It is not difficult to see that $\mathcal{M}_c$ is a closed subset of $\mathcal{M}_l$.

For each $f \in \mathcal{M}_l$ set

$$\inf(f) = \inf\{f(x) : \ x \in X\}.$$

Consider a minimization problem

$$f(x) \to \min, \ x \in X,$$

where $f \in \mathcal{M}_l$.

Assume that a mapping $T : X \to X$ is continuous and the mapping $T^2 = T \circ T$ is an identity mapping in X:

$$T^2(x) = x \text{ for all } x \in X.$$

Let $\mathcal{A}_T$ be either the set of all $f \in \mathcal{M}_l$ such that

$$f(T(x)) = f(x) \text{ for all } x \in X$$

or the set of all $f \in \mathcal{M}_c$ satisfying the equation above.

Clearly, $\mathcal{A}_T$ is a closed subset of $\mathcal{M}_l$. It is equipped with the relative topology induced by the metric d.

Given $f \in \mathcal{A}_T$ we say that the problem of minimization of f on X is well-posed with respect to data in $\mathcal{M}_l$ if the following assertions hold:

(1) there exists $x_f \in X$ such that

$$\{x \in X : \ f(x) = \inf(f)\} = \{x_f, \ T(x_f)\}.$$

(2) For each $\epsilon > 0$ there is a neighborhood $\mathcal{V}$ of f in $\mathcal{M}_l$ and $\delta > 0$ such that for each $g \in \mathcal{V}$ if $z \in X$ satisfies $g(z) \le \inf(g) + \delta$, then

$$|g(z) - f(x_f)| \le \epsilon$$

and

$$\min\{\rho(z, \{x_f, T(x_f)\}), \ \rho(T(z), \{x_f, T(x_f)\})\} \le \epsilon.$$

In [171] it was shown that there exists a set $\mathcal{F} \subset \mathcal{A}_T$ which is a countable intersection of open and everywhere dense sets in $\mathcal{A}_T$ such that for each $f \in \mathcal{F}$, the minimization problem of f on X is well-posed with respect to $\mathcal{M}_l$.

This result was also obtained in [174] using a general variational principle for symmetric optimization problems. More precisely, the following result was obtained in [174].

Theorem 1.3 *There exists an everywhere dense set $\mathcal{B} \subset \mathcal{A}_T$ which is a countable intersection of open subsets of $\mathcal{A}_T$ such that for any $f \in \mathcal{B}$ the minimization problem of f on X is well-posed with respect to $\mathcal{M}_l$.*

This result as well as many other generic well-posedness results for symmetric problems are collected in Chapter 2. Here we present one of them (see Theorem 2.31) related to the theorem above.

Denote by $\mathrm{Fix}(T)$ the set of all $x \in X$ such that

$$T(x) = x.$$

Since the mapping T is continuous the set $\mathrm{Fix}(T)$ is closed. Note that it can be empty. We assume that the set

$$X \setminus \mathrm{Fix}(T) \text{ is everywhere dense.}$$

Denote by $\mathcal{M}_f$ the set of all $f \in \mathcal{M}_l$ which are continuous at any point of $\mathrm{Fix}(T)$. If $\mathrm{Fix}(T) = \emptyset$, then $\mathcal{M}_f = \mathcal{M}_l$. Clearly, $\mathcal{M}_f$ is a closed subset of $\mathcal{M}_l$.

Now assume that $\mathcal{A}_T$ is either the set of all $f \in \mathcal{M}_f$ such that

$$f(T(x)) = f(x) \text{ for all } x \in X$$

or the set of all $f \in \mathcal{M}_c$ satisfying the equation above.

Clearly, $\mathcal{A}_T$ is a closed subset of $\mathcal{M}_l$. It is equipped with the relative topology.

The following result is proved in Chapter 2.

Theorem 1.4 *There exists an everywhere dense set $\mathcal{B} \subset \mathcal{A}_T$ which is a countable intersection of open subsets of $\mathcal{A}_T$ such that for any $f \in \mathcal{B}$ the minimization problem of f on X is well-posed with respect to $\mathcal{A}$ and has exactly two different minimizers.*

1.4 Turnpike Property for Variational Problems

The study of the existence and the structure of solutions of optimal control problems and dynamic games defined on infinite intervals and on sufficiently large intervals has been a rapidly growing area of research [7–9, 20, 21, 29, 41, 44–46, 65, 69, 70, 83, 95, 134, 138, 142, 145, 151, 155, 160, 161, 164, 166, 169] which has various applications in engineering [5, 29, 76, 134], in models of economic growth [6, 10, 27–30, 43, 48, 60, 61, 68, 75, 82, 88, 96, 108, 110, 111, 118, 134, 149, 156, 165, 166], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [12, 104, 122], in model predictive control [36, 51], and in the theory of thermodynamical equilibrium for materials [34, 77, 85–87]. Discrete-time problems optimal control problems were considered in [9, 13, 14, 24, 47, 57, 123, 124, 129, 133, 140, 144, 146, 149, 153, 154, 157, 162, 163, 165], finite dimensional continuous-time problems were analyzed in [20, 22, 23, 26, 31, 75, 78, 81, 84, 99, 125, 127, 132, 135, 152, 167, 168], infinite dimensional optimal control was studied in [29, 30, 52–54, 91, 93, 94, 102, 109, 113, 114, 126, 128, 143, 170] while solutions of dynamic games were discussed in [19, 49, 50, 55, 58, 72, 103, 147, 150, 158, 159]. Sufficient and necessary conditions for the turnpike phenomenon were obtained in [132, 133, 135, 153] for finite dimensional variational problems and for discrete-time optimal control problems in compact metric space.

In this section, which is based on [132], we discuss the structure of approximate solutions of variational problems with continuous integrands $f : [0, \infty) \times R^n \times R^n \to R^1$ which belong to a complete metric space of functions. We do not impose any convexity assumption. The main result of this section, obtained in [132], deals with the turnpike property of variational problems.

We consider the variational problems

$$\int_{T_1}^{T_2} f(t, z(t), z'(t))dt \to \min, \ z(T_1) = x, \ z(T_2) = y, \qquad (P)$$

$$z : \ [T_1, T_2] \to R^n \text{ is an absolutely continuous function (a. c.),}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$ and $f : [0, \infty) \times R^n \times R^n \to R^1$ belongs to a space of integrands described below.

It is well-known that the solutions of the problems (P) exist for integrands f which satisfy two fundamental hypotheses concerning the behavior of the integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex [33, 112]. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals which are crucial in most of the existence proofs, although there are some interesting theorems without convexity [32, 90, 92]. For integrands f which do not satisfy the convexity assumption the existence of solutions of the problems (P) is not guaranteed and in this situation we consider *δ-approximate solutions*.

Let $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, $f : [0, \infty) \times R^n \times R^n \to R^1$ be an integrand and let δ be a positive number. We say that an absolutely continuous (a.c.) function $u : [T_1, T_2] \to R^n$ satisfying $u(T_1) = x$, $u(T_2) = y$ is a δ-approximate solution of the problem (P) if

$$\int_{T_1}^{T_2} f(t, u(t), u'(t))dt \leq \int_{T_1}^{T_2} f(t, z(t), z'(t))dt + \delta$$

for each a.c. function $z : \ [T_1, T_2] \to R^n$ satisfying $z(T_1) = x, \ z(T_2) = y$.

The main result of [132] deals with the turnpike property of the variational problems (P). As usual, to have this property means, roughly speaking, that the approximate solutions of the problems (P) are determined mainly by the integrand and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

In the classical turnpike theory, it was assumed that a cost function (integrand) is convex. The convexity of the cost function played a crucial role there. In [132] we get rid of convexity of integrands and establish necessary and sufficient conditions for the turnpike property for a space of nonconvex integrands $\mathcal{M}$ described below.

Let us now define the space of integrands. Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant and let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$. Denote by $\mathcal{M}$ the set of all continuous functions $f : [0,\infty) \times R^n \times R^n \to R^1$ which satisfy the following assumptions:

- A(i) the function f is bounded on $[0,\infty) \times E$ for any bounded set $E \subset R^n \times R^n$;
- A(ii) $f(t,x,u) \ge \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(t,x,u) \in [0,\infty) \times R^n \times R^n$;
- A(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(t,x_1,u) - f(t,x_2,u)| \le \epsilon \max\{f(t,x_1,u), f(t,x_2,u)\}$$

for each $t \in [0,\infty)$ and each $u, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \le M,\ i = 1,2,\ |u| \ge \Gamma, \quad |x_1 - x_2| \le \delta;$$

- A(iv) for each $M, \epsilon > 0$ there exists $\delta > 0$ such that $|f(t,x_1,u_1) - f(t,x_2,u_2)| \le \epsilon$ for each $t \in [0,\infty)$ and each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i|, |u_i| \le M,\ i = 1,2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \le \delta.$$

It is easy to show that an integrand $f = f(t,x,u) \in C^1([0,\infty) \times R^n \times R^n)$ belongs to $\mathcal{M}$ if f satisfies assumption A(ii), and if $\sup\{|f(t,0,0)| : t \in [0,\infty)\} < \infty$ and also there exists an increasing function $\psi_0 : [0,\infty) \to [0,\infty)$ such that

$$\sup\{|\partial f/\partial x(t,x,u)|,\ |\partial f/\partial u(t,x,u)|\} \le \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $t \in [0,\infty)$ and each $x, u \in R^n$.

For the set $\mathcal{M}$ we consider the uniformity which is determined by the following base:

$$E(N,\epsilon,\lambda) = \{(f,g) \in \mathcal{M} \times \mathcal{M} : |f(t,x,u) - g(t,x,u)| \le \epsilon$$

$$\text{for each } t \in [0,\infty) \text{ and each } x, u \in R^n \text{ satisfying } |x|, |u| \le N$$

$$\text{and } (|f(t,x,u)|+1)(|g(t,x,u)|+1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for each } t \in [0,\infty) \text{ and each } x, u \in R^n \text{ satisfying } |x| \le N\},$$

where $N > 0, \epsilon > 0, \lambda > 1$.

It is not difficult to show that the space $\mathcal{M}$ with this uniformity is metrizable (by a metric ρ_w). It is known (see [132]) that the metric space $(\mathcal{M}, \rho_w)$ is complete. The metric ρ_w induces in $\mathcal{M}$ a topology.

We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(t, x(t), x'(t))dt$$

where $f \in \mathcal{M}$, $0 \le T_1 < T_2 < \infty$ and $x : [T_1, T_2] \to R^n$ is an a.c. function.

For $f \in \mathcal{M}$, $y, z \in R^n$ and numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ we set

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) : x : [T_1, T_2] \to R^n$$

$$\text{is an a.c. function satisfying } x(T_1) = y, \ x(T_2) = z\}.$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < \infty$ for each $f \in \mathcal{M}$, each $y, z \in R^n$ and all numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathcal{M}$. A locally absolutely continuous (a.c.) function $x : [0, \infty) \to R^n$ is called an (f)-*good function* [134] if for any a.c function $y : [0, \infty) \to R^n$ there is a number M_y such that

$$I^f(0, T, y) \ge M_y + I^f(0, T, x) \text{ for each } T \in (0, \infty).$$

The following result was proved in [132].

Proposition 1.5 *Let $f \in \mathcal{M}$ and let $x : [0, \infty) \to R^n$ be a bounded a.c. function. Then the function x is (f)-good if and only if there is $M > 0$ such that*

$$I^f(0, T, x) \le U^f(0, T, x(0), x(T)) + M \textit{ for any } T > 0.$$

Let us now give the precise definition of the turnpike property.

Assume that $f \in \mathcal{M}$. We say that f has the turnpike property, or briefly TP, if there exists a bounded continuous function $X_f : [0, \infty) \to R^n$ which satisfies the following condition:

For each $K, \epsilon > 0$ there exist constants $\delta, L > 0$ such that for each $x, y \in R^n$ satisfying $|x|, |y| \le K$, each $T_1 \ge 0$, $T_2 \ge T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$v(T_1) = x, \ v(T_2) = y, \ I^f(T_1, T_2, v) \le U^f(T_1, T_2, x, y) + \delta$$

the inequality $|v(t) - X_f(t)| \le \epsilon$ holds for all $t \in [T_1 + L, T_2 - L]$.

The function X_f is called the turnpike of f.

Assume that $f \in \mathcal{M}$ and $X : [0, \infty) \to R^n$ is a bounded continuous function. How to verify if the integrand f has TP and X is its turnpike? In [132] we introduced two properties (P1) and (P2) and show that f has TP if and only if f possesses the properties (P1) and (P2). The property (P2) means that all (f)-good functions have the same asymptotic behavior while the property (P1) means that if an a.c. function $v : [0, T] \to R^n$ is an approximate solution and T is large enough, then there is $\tau \in [0, T]$ such that $v(\tau)$ is close to $X(\tau)$.

The next theorem is the main result [132].

Theorem 1.6 *Let $f \in \mathcal{M}$ and $X_f : [0, \infty) \to R^n$ be a bounded absolutely continuous function. Then f has the turnpike property with X_f being the turnpike if and only if the following two properties hold:*

(P1) For each $K, \epsilon > 0$ there exist $\gamma, l > 0$ such that for each $T \geq 0$ and each a.c. function $w : [T, T + l] \to R^n$ which satisfies

$$|w(T)|, |w(T+l)| \leq K,\ I^f(T, T+l, w) \leq U^f(T, T+l, w(T), w(T+l))+\gamma$$

there is $\tau \in [T, T + l]$ for which $|X_f(\tau) - v(\tau)| \leq \epsilon$.

(P2) For each (f)-good function $v : [0, \infty) \to R^n$,

$$|v(t) - X_f(t)| \to 0 \text{ as } t \to \infty.$$

In [132] we proved the following theorem which is an extension of Theorem 1.6.

Theorem 1.7 *Let $f \in \mathcal{M}$, $X_f : [0, \infty) \to R^n$ be an (f)-good function. Assume that the properties (P1), (P2) hold. Then for each $K, \epsilon > 0$ there exist $\delta, L > 0$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies*

$$|v(T_1)|, |v(T_2)| \leq K,\ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta$$

the inequality $|v(t) - X_f(t)| \leq \epsilon$ holds for all $t \in [T_1 + L, T_2 - L]$.

In our recent [170] book we study sufficient and necessary conditions for the turnpike phenomenon, using the approach developed in [132, 133, 135, 153], for discrete-time optimal control problems in metric spaces, which are not necessarily compact, and for continuous-time infinite dimensional optimal control problems. Its main results have Theorem 1.6 as their prototype.

In the present book we study the turnpike phenomenon for problems with symmetric integrands. It turns out that for these problems the turnpike is a singleton which is a global minimizer of the integrand.

Since the discovery of the turnpike phenomenon by Paul Samuelson in 1948, different versions of the turnpike property were considered in the literature. In this book as well as in [132, 133, 135, 153, 170], we study the turnpike property introduced and used in our previous research [127, 134, 152, 153, 164]. This turnpike property differs from other versions and has important features. Our turnpike property is a property of approximate solutions. As it was shown in [127, 152], our turnpike property holds for most problems belonging to large classes of variational and optimal control problems.

1.5 Variational Problems with Symmetric Integrands

Assume that $f : R^n \times R^n \to R^1$ is a bounded from below borelian function such that

$$f(x, y) = f(x, -y) \text{ for all } x, y \in R^n,$$

there exists $(\bar{x}, \bar{y}) \in R^n \times R^n$ such that

$$f(\bar{x}, \bar{y}) = \inf(f) := \inf\{f(\xi, \eta) : \ \xi, \eta \in R^n\},$$

$$\{(x, y) \in R^n \times R^n : \ f(x, y) = \inf(f)\} = \{(\bar{x}, \bar{y}), \ (\bar{x}, -\bar{y})\}$$

and that the following assumptions hold:

(A1) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $(x, y) \in R^n \times R^n$ satisfying

$$f(x, y) \le \inf(f) + \delta$$

the inequalities

$$|x - \bar{x}| \le \epsilon$$

and

$$\min\{|y - \bar{y}|, \ |y + \bar{y}|\} \le \epsilon$$

hold;

(A2) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $(x, y) \in R^n \times R^n$ satisfying

$$|x - \bar{x}| \le \delta, \ |y - \bar{y}| \le \delta$$

the inequality

$$f(x, y) \le f(\bar{x}, \bar{y}) + \epsilon$$

is true.

Assumption (A2) means that the function f is continuous at the point $(\bar{x}, \bar{y})$ while assumption (A1) means that the minimization problem

$$f(x, y) \to \min, \ x, y \in R^n$$

is well-posed. According to the results of Chapter 2 this well-posedness property holds for most symmetric objective functions.

Let $a > 0$ and $\psi : [0, \infty) \to [0, \infty)$ be an increasing function satisfying

$$\lim_{t \to \infty} \psi(t) = \infty.$$

Assume that the following assumption holds:

(A3) the function f is bounded on all bounded sets and for each $(x, u) \in R^{\times} R^n$,

$$f(x, u) \geq \psi(|u|)|u| - a.$$

For each pair of nonnegative numbers $T_1 < T_2$ and each $y, z \in R^n$ we denote by $W^{1,1}(T_1, T_2)$ the set of all a. c. functions $x : [T_1, T_2] : R^n$, consider the problems

$$\int_{T_1}^{T_2} f(x(t), x'(t))dt \to \min, \qquad (P_{T_1,T_2})$$

$$x \in W^{1,1}(T_1, T_2),$$

$$\int_{T_1}^{T_2} f(x(t), x'(t))dt \to \min, \qquad (P_{T_1,T_2,y})$$

$$x \in W^{1,1}(T_1, T_2), \ x(T_1) = y,$$

$$\int_{T_1}^{T_2} f(x(t), x'(t))dt \to \min, \qquad (P_{T_1,T_2,y,z})$$

$$x \in W^{1,1}(T_1, T_2), \ x(T_1) = y, \ x(T_2) = z$$

and define

$$U(T_1, T_2) = \inf\{\int_{T_1}^{T_2} f(x(t), x'(t))dt : \ x \in W^{1,1}(T_1, T_2)\},$$

$$U(T_1, T_2, y) = \inf\{\int_{T_1}^{T_2} f(x(t), x'(t))dt : \ x \in W^{1,1}(T_1, T_2), \ x(T_1) = y\},$$

$$U(T_1, T_2, y, z) = \inf\{\int_{T_1}^{T_2} f(x(t), x'(t))dt :$$

$$x \in W^{1,1}(T_1, T_2),\ x(T_1) = y,\ x(T_2) = z\}.$$

There are two cases: $\bar{y} = 0$; $\bar{y} \neq 0$. If $\bar{y} = 0$, then for each $T_2 > T_1 \geq 0$, the function $x(t) = \bar{x}$, $t \in [T_1, T_2]$ is a solution of the problems (P_{T_1,T_2}), $(P_{T_1,T_2,\bar{x}})$, $(P_{T_1,T_2,\bar{x},\bar{x}})$.

For each pair of numbers $T_1 < T_2$ and each $x \in W^{1,1}(T_1, T_2)$ set

$$I(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))dt.$$

In Chapter 5 we prove the following result.

Theorem 1.8 *Let $T > 0$. Then*

$$U(0, T) = U(0, T, \bar{x}) = U(0, T, \bar{x}, \bar{x}) = Tf(\bar{x}, \bar{y}).$$

Moreover, for each $\epsilon > 0$ there exists $x \in W^{1,1}(0, T)$ such that

$$x(0) = x(T) = \bar{x},$$

$$I(0, T, x) \leq Tf(\bar{x}, \bar{y}) + \epsilon,$$

$$|x(t) - \bar{x}| \leq \epsilon,\ t \in [0, T],$$

$$x'(t) \in \{\bar{y},\ -\bar{y}\},\ t \in [0, T]\ a.\ e..$$

Note that if $\bar{y} \neq 0$, then for every positive T problems $(P_{0,T})$, $(P_{0,T,\bar{x}})$ and $(P_{0,T,\bar{x},\bar{x}})$ have no minimizers.

The following useful result is also obtained in Chapter 5.

Theorem 1.9 *Let $L_0, M_0 > 0$. Then there exist $M_1 > 0$ such that for each $T > L_0$ and each $y, z \in R^n$ satisfying $|y|, |z| \leq M_0$ the inequality*

$$U(0, T, y, z) \leq Tf(\bar{x}, \bar{y}) + M_1$$

holds.

We denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue measurable set $\Omega \subset R^1$

The next theorem is the first turnpike result of Chapter 5. It shows that for approximate solutions x of our variational problems on intervals $[0, T]$, where T is sufficiently large and given values $x(0), x(T)$ at the end points belong to a given bounded set C, the Lebesgue measure of all points $t \in [0, T]$ such that $(x(t), x'(t))$

does not belong to an ϵ-neighborhood of the set $\{(\bar{x}, \bar{y}),\ (\bar{x}, -\bar{y})\}$ does not exceed a constant L which depends only on ϵ and the set C and does not depend on $T, x(0), x(T)$. In the literature this property is known as the weak turnpike property.

Theorem 1.10 *Let $\epsilon \in (0, 1)$ and $L_0, M_0, M_1 > 0$. Then there exists $L_1 > L_0$ such that for each $T > L_1$ and each $x \in W^{1,1}(0, T)$ such that*

$$|x(0)| \le M_0$$

and at least one of the following conditions holds:

(a)

$$|x(T)| \le M_0,\ \ I^f(0, T, x) \le U(0, T, x(0), x(T)) + M_1;$$

(b)

$$I^f(0, T, x) \le U(0, T, x(0)) + M_1$$

the inequality

$$mes(\{t \in [0, T] :\ \max\{|x(t) - \bar{x}|,$$

$$\min\{|x'(t) - \bar{y}|,\ |x'(t) + \bar{y}|\}\} > \epsilon\}) \le L_1.$$

The following theorem is also proved in Chapter 5. It shows that for approximate solutions x of our variational problems on intervals $[0, T]$, where T is sufficiently large and given values $x(0), x(T)$ at the end points belong to a given bounded set C, the set of all points $t \in [0, T]$ such that $x(t)$ does not belong to an ϵ-neighborhood of $\bar{x}$ is contained in the union of two intervals, where the first interval contains 0, the second one contains T and their lengths do not exceed a constant L which depends only on ϵ and the set C and does not depend on $T, x(0), x(T)$. In the literature this property is known as the turnpike property.

Theorem 1.11 *Let $\epsilon \in (0, 1]$. Then there exist $L, \delta > 0$ such that for each $T > 2L$ and each $u \in W^{1,1}(0, T)$ such that*

$$|u(0)| \le M$$

and at least one of the following conditions holds:

$$|u(T)| \le M,\ \ I(0, T, u) \le U(0, T, u(0), u(T)) + \delta;$$

$$I(0, T, x) \le U(0, T, u(0)) + \delta$$