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Topics in Global Real Analytic Geometry

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To Alberto Tognoli, in memoriam.

Preface

In the first half of the twentieth century, the theory of complex analytic functions and their zerosets was fully developed. The definition of a holomorphic function has a local nature. Germs of holomorphic functions form a distinguished subring of the ring of germs of continuous functions. From this emerged the notion of an *analytic space*. The definition of a complex analytic set uses *local models* as in the case of complex manifolds. But while local models for manifolds are open sets of $\mathbb{C}^n$, a local model of an analytic space is the zeroset of finitely many analytic functions on an open set of $\mathbb{C}^n$ together with a sheaf of continuous functions, called a *holomorphic* sheaf.

Towards the 1950s, Cartan, Whitney, Bruhat, and others tried to formulate the notion of an analytic space over $\mathbb{R}$. They immediately realized that the real sets satisfying a definition similar to that of a complex analytic space form a category whose elements can have unpleasant behavior. In particular, this category does not share the good properties of complex analytic spaces, such as coherence of their structural sheaves, and the fundamental Cartan's Theorems A and B do not hold in general. In contrast with what happens for instance in the algebraic case, it is not always possible to view a real analytic space as the fixed point set of a suitable *conjugation* on a complex analytic space. Faced with this situation, some doubts arose on the interest of such investigations. For instance, Grothendieck wrote in Cartan [Ca6, Exp. 9, p. 12]:

> Lorsque k est algébriquement clos, il est probablement vrai que tout espace analytique réduit à un point est de la forme qu'on vient d'indiquer, ce qui serait une des variantes du "Nullstellensatz" analytique. Signalons par contre tout de suite que rien de tel n'est vrai si k n'est pas algébriquement clos, par exemple si k est le corps des réels $\mathbb{R}$. Ainsi, le sous-espace analytique de $\mathbb{R}^2$ défini par l'idéal engendré par $x^2 + y^2$ est réduit au point origine, mais son anneau local en ce point n'est pas artinien, mais de dimension de Krull égale à 1. L'intérêt des espaces analytiques, lorsque k n'est pas algébriquement clos, est d'ailleurs douteux.

Ignoring these doubts, Cartan worked to find the obstructions to get a good real category. He proved that his Theorems A and B pass through direct limits. So, since $\mathbb{R}^n$ has in $\mathbb{C}^n$ a fundamental system of open Stein neighborhoods, he proved that every analytic subset of $\mathbb{R}^n$, defined as the zeroset of *global* analytic functions, is the support of a coherent sheaf of ideals. This sheaf of ideals defines a complex analytic subset of a Stein open neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$, hence Theorems A and B hold true. So, he found a good class of real analytic spaces globally defined in $\mathbb{R}^n$ that have a good *complexification*. He wrote in Cartan [Ca2, p. 49]:

> ... la seule notion de sous-ensemble analytique réel (d'une variété analytique-réelle V) qui ne conduise pas à des propriétés pathologiques doit se référer à l'espace complexe ambiant: il faut considérer les sous-ensembles fermés E de V tels qu'il existe une complexification W de V et un sous-ensemble analytique-complexe E' de W, de manière que $E = W \cap E'$. On démontre que ce sont aussi les sous-ensembles de V qui peuvent être définis globalement par un nombre fini d'équations analytiques. La notion de sous-ensemble analytique-réel a ainsi un caractère essentiellement global, contrairement à ce qui avait lieu pour les sous-ensembles analytiques-complexes.

Cartan uses complex notions to describe real properties: for instance, he defines the *complexification* of a germ of real analytic space V_x at a point $x \in \mathbb{R}^n$ and proves that V_x is coherent if and only if the complexification of V_x induces the complexification of V_y on points y close to x.

All these considerations led Cartan to the following characterization of what a good category of real analytic sets should be, proving that for a closed real analytic subset $X \subset \mathbb{R}^n$, the following statements are equivalent:

(1) The set X is the zeroset of finitely many real analytic functions.
(2) There is a coherent idealsheaf in $\mathcal{O}_{\mathbb{R}^n}$ whose zeroset is X.
(3) There is an open neighborhood Ω of $\mathbb{R}^n$ in $\mathbb{C}^n$ and a closed subspace $Y \subset \Omega$ such that $Y \cap \mathbb{R}^n = X$.

So, the notion of *complexification* plays a central role.

Bruhat and Whitney extended the notion of complexification to real analytic manifolds and introduced the name *C-analytic* for the analytic subsets of a real analytic manifold M induced by intersection with M of an analytic subset of the complexification of M. Finally, Tognoli extended the notion of the complexification of a real analytic space admitting a coherent structure, in analogy with Cartan's condition (2).

Tognoli distinguished three types of real analytic spaces: the coherent spaces, whose reduced structure is coherent, those carrying at least one coherent structure (for instance, Whitney and Cartan umbrellas), and those not admitting any coherent structure (the wild examples of Cartan, Bruhat–Cartan, etc.)

Since a C-analytic space X is globally defined, the ring $\mathcal{O}(X)$ of (real) analytic functions on X becomes interesting. One could follow the development of Real Algebraic Geometry and try to imitate its theory. This is easy in the local case but not in the global one. This is because an important step in the algebraic theory is the fact that a non-negative polynomial is a sum of squares of rational functions (Artin's solution of Hilbert's 17th Problem). In the analytic case, this is true for the ring of germs, while in the global case, it is proved, as far as we know, only in some special particular cases, so one cannot expect to get results in analogy with Real Algebraic Geometry.

In this book we follow another path, closer to Cartan's point of view, that is, we deduce results for C-analytic spaces from the properties of their complexification.

In the first two chapters, we mainly recall classical results. More precisely, in Chap. 1, after exposing the main facts on complex analytic spaces, we give the construction of a complexification, showing why it is necessary to pass to C-spaces. We also give some bad examples of real analytic sets in $\mathbb{R}^n$ following Cartan and Bruhat. Chapter 2 describes the construction of irreducible components of complex and real analytic sets. We then see how this notion works when dealing with *normalization* from a local and a global point of view. Concerning *divisors* of a C-space, we try to answer the question of what the conditions are under which a divisor is the divisor of a global analytic function.

As we said, in the local case, the results are very similar to the algebraic ones. For instance, the Nullstellensatz for the ring of complex or real analytic germs is exactly the same as for the ring of complex or real polynomials. This is no longer the case for the ring of global analytic functions. In Chap. 3, we give the proof of the Nullstellensatz for closed ideals in $\mathcal{O}(X)$ where X is a Stein space, following Forster. The primary (infinite) decomposition of a closed ideal allows us to consider irreducible components of a Stein space with *multiplicity* as in the algebraic case.

We remark that in this case, there is a numerical function (namely the primality index) which controls whether Hilbert's Nullstellensatz holds true or not for a closed ideal. It holds if and only if the numerical function is uniformly bounded. We get a somewhat similar result for Hilbert's 17th Problem. If it has a positive solution for $\mathcal{O}(\mathbb{R}^n)$, then the Pythagoras number of the field of meromorphic functions is bounded.[1] Note that in the algebraic case, Hilbert's Problem and the calculation of the Pythagoras number are completely separate problems, while in the analytic case, there is an unexpected relation.

Also, in Chap. 3, we give a real Nullstellensatz for the ring $\mathcal{O}(X)$ where X is a C-analytic space. The radical we use is not Risler's real radical, it is the *Łojasiewicz radical*, which is the radical of the convex hull of the given ideal and in general is larger than the real radical of that ideal. This is because we do not know whether a positive semidefinite analytic function is a sum of squares of meromorphic functions. Indeed, if the zeroset Y of the given ideal is such that positive semidefinite functions, having Y as zeroset, *are* sums of squares of meromorphic functions, then the two radicals coincide and we get a result similar to Risler's Nullstellensatz.

Thus, Hilbert's 17th Problem is crucial also in Real Analytic Geometry. Hence, in Chap. 4, we give the state of the art on this problem, whose solution is far from being complete. We also discuss some weaker questions which involve infinite sums of squares.

[1] The Pythagoras number of a ring R is the smallest positive integer p such that all sums of squares in R can be written as sums of p squares. If such p does not exist, then the Pythagoras number is ∞.

When dealing with real objects, inequalities appear immediately, equalities are not enough. Consider, for instance, the orthogonal projection of a circle from $\mathbb{R}^2$ to $\mathbb{R}$.

Łojasiewicz and Hironaka defined a class of subsets of a real analytic manifold, namely *semianalytic sets*, which are locally defined by analytic equalities and inequalities. This class behaves well with respect to topological properties such as closure, taking connected components and so on, but is not stable under proper projections. For this reason, the class of *subanalytic sets* was also introduced. Nevertheless, one can ask whether semianalytic sets defined by finitely many global analytic functions (*global semianalytic sets*) could have better properties. Several authors have investigated these sets by applying the algebraic theory of orders to get topological properties, mainly in dimension less than three. An important result in general dimension is that the closure of a global semianalytic set is locally global. We give a more geometric proof of this result in Chap. 5.

We define an intermediate class of semianalytic sets, which we call *C-semianalytic sets* in analogy with the notion of C-analytic spaces, that is, locally finite unions of global semianalytic sets. We prove that this class is stable under topological operations. Moreover, it is stable under proper invariant holomorphic maps. We do not need to define *C-subanalytic sets* because we can prove that subanalytic sets can be defined by replacing semianalytic sets by C-semianalytic sets.

Several remarkable subsets of a C-analytic set, such as the set of points of a given dimension, or the set where it is not coherent, or the set of local extrema of an analytic function, are C-semianalytic sets.

A theory of irreducibility and of irreducible components, analogous to the one developed for semialgebraic sets, does not hold for C-semianalytic sets. Nevertheless, there is a smaller class where this theory applies. It is the class of *amenable* semianalytic sets, which are locally finite unions of sets of the type $U \cap X$, where U is an open set and X is a C-analytic set.

The last chapter deals with other structures such as the algebra of smooth functions $\mathcal{E}(\mathbb{R}^n)$ and algebras of quasi-analytic functions. In the first section, we compare analytic descriptions of a semianalytic set with smooth descriptions and we see how they change using flat functions. For a global semianalytic set X, we find a minimal closed set $S \subset \overline{X}$, that is, the set of points where X is not locally basic, such that there are finitely many smooth functions that describe X as a basic set and are not flat outside S.

We then prove a Positivstellensatz and a Nullstellensatz in $\mathcal{E}(\mathbb{R}^n)$ for *Łojasiewicz ideals*. In particular, when the ideal is generated by finitely many analytic functions, we prove a modification of Bochnak's conjecture. A similar Nullstellensatz holds in the algebra $\mathcal{C}_M(\mathbb{R}^n)$ of quasi-analytic Denjoy–Carleman functions.

* * *

References in the text are mainly concentrated in a section at the end of each chapter together with some historical notes.

This book is meant for a reader, researcher, or PhD student who feels comfortable with general notions in complex analysis and commutative algebra, for which we refer to some classical texts such as Gunning and Rossi [GuRo], Gunning [Gu], Łojasiewicz [Ło3], and della Sala et al. [SaSarSiTo] for complex analysis and Matsumura [Ma1] and Atiyah and Macdonald [AtMc] for commutative algebra.

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Contents

Chapter 1
The Class of C-Analytic Spaces

In this chapter, following the ideas collected in [Ca3, To1, WhBru], we introduce the class of *real analytic spaces*. This type of space is also called a C-analytic space in the literature. To introduce it we first need to recall the concept of a *complex analytic space*. In what follows all involved topological spaces are assumed to be Hausdorff, paracompact and second-countable.

1.1 Complex Analytic Spaces

A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a subsheaf of the sheaf of germs of continuous functions on X. We recall shortly the notion of a *coherent sheaf*.

Definition 1.1.1 A sheaf $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is called $\mathcal{O}_X$*-coherent* if it is a sheaf of $\mathcal{O}_X$-modules *of finite presentation*, that is, it satisfies the following two conditions.

(i) It is a *finite type sheaf*, that is, for each $x \in X$ there exists an open neighborhood U_x and finitely many sections $\{G_1, \dots, G_k\}$ on U_x generating the fiber $\mathcal{F}_y$ for each $y \in U_x$.
(ii) For each open set $U \subset X$ and for each finite number of sections $H_1, \dots, H_p \in \mathcal{F}(U)$ the *sheaf of relations* among them, that is, the kernel of the homomorphism of sheaves $\sigma : (\mathcal{O}_X^p)_{|U} \to \mathcal{F}_{|U}$ given on each open set $V \subset U$ by

$$\sigma : \mathcal{O}_X(V)^p \to \mathcal{F}(V),\ (A_1, \dots, A_p) \mapsto A_1H_1 + \cdots + A_pH_p$$

is a finite type sheaf.

First of all we define what we mean by a local model of complex analytic space.

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A *local model of complex analytic space* is a pair $(Y, \mathcal{O}_Y)$ comprising a closed subset Y of an open set $\Omega \subset \mathbb{C}^n$ that is the common zeroset of finitely many holomorphic functions $F_1, \dots, F_k$ on Ω and a structure provided by a suitable sheaf $\mathcal{O}_Y$. There are two main ways to define this structure sheaf. Remember that by Oka's Theorem $\mathcal{O}_\Omega$ is a coherent sheaf of $\mathcal{O}_\Omega$-modules, hence a sheaf $\mathcal{F}$ of $\mathcal{O}_\Omega$-modules is coherent if and only if $\mathcal{F}$ is of finite type. In particular, finitely generated sheaves of $\mathcal{O}_\Omega$-modules are coherent.

Method 1 Let $\mathcal{O}_\Omega$ be the sheaf of germs of holomorphic functions on Ω and let $\mathcal{I}_Y$ be the sheaf of ideals consisting of all holomorphic function germs vanishing on Y. It is a coherent sheaf of ideals (Oka's theorem). Consider now the coherent sheaf $\mathcal{O}_Y = \mathcal{O}_\Omega/\mathcal{I}_Y$. Clearly,

$$Y = \operatorname{supp}(\mathcal{O}_\Omega/\mathcal{I}_Y) = \{y \in \Omega : \mathcal{I}_{Y,y} \neq \mathcal{O}_{\Omega,y}\}$$

and $(Y, \mathcal{O}_Y) = (\operatorname{supp}(\mathcal{O}_\Omega/\mathcal{I}_Y), \mathcal{O}_\Omega/\mathcal{I}_Y)$ is a ringed space. The homomorphism of sheaves $\mathcal{O}_\Omega \to \mathcal{O}_\Omega/\mathcal{I}_Y$ is surjective.

Method 2 Consider the subsheaf $\mathcal{J}$ of $\mathcal{O}_\Omega$ generated by the holomorphic functions $F_1, \dots, F_k$ on Ω and define $\mathcal{O}_Y = \mathcal{O}_\Omega/\mathcal{J}$. As $\mathcal{J} = (F_1, \dots, F_k)\mathcal{O}_\Omega$ is a finitely generated subsheaf of ideals of the coherent sheaf $\mathcal{O}_\Omega$, it is coherent itself, so $\mathcal{O}_Y$ is also a coherent sheaf. Again we have

$$Y = \operatorname{supp}(\mathcal{O}_\Omega/\mathcal{J}) = \{y \in \Omega : \mathcal{J}_y \neq \mathcal{O}_{\Omega,y}\}$$

and $(Y, \mathcal{O}_Y) = (\operatorname{supp}(\mathcal{O}_\Omega/\mathcal{J}), \mathcal{O}_\Omega/\mathcal{J})$ is a ringed space. Again the homomorphism of sheaves $\mathcal{O}_\Omega \to \mathcal{O}_\Omega/\mathcal{J}$ is surjective.

Definitions 1.1.2 A *complex analytic space* is a (Hausdorff, paracompact) topological space X endowed with a sheaf of rings $\mathcal{O}_X$ such that the pair $(X, \mathcal{O}_X)$ is locally isomorphic as a ringed space to a local model endowed with the structure provided in Method 2. If the local models are chosen using the structure provided in Method 1, then we say that $(X, \mathcal{O}_X)$ is a *reduced complex analytic space*.

Forgetting the structure sheaf we get the notion of a *complex analytic set* X. This is a closed subset of an open set $\Omega \subset \mathbb{C}^n$ that admits a local description as the zero set of finitely many holomorphic functions, that is, for each point $x \in \Omega$ there exists an open neighborhood U_x and finitely many holomorphic functions $F_1, \dots, F_k$ on U_x such that

$$X \cap U_x = \{y \in U_x : F_1(y) = 0, \cdots, F_k(y) = 0\}.$$

One can provide X with a structure by considering the sheaf of holomorphic function germs on it. Namely $\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{I}_X$, where $\mathcal{I}_X$ is the sheaf of germs vanishing on X. This structure is often called the *natural structure on* X and it is obtained following Method 1 above. Instead of $\mathcal{I}_X$ we can consider any coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_\Omega$ having X as zeroset and take $\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{J}$. As $\mathcal{J}$ is locally finitely generated, the local models for this structure are those provided by Method 2.

If $(X, \mathcal{O}_X)$ is a non-reduced complex analytic space, then there exists a *reduction morphism* $\rho : (X, \mathcal{O}^r_X) \to (X, \mathcal{O}_X)$, where $\mathcal{O}^r_X$ is the reduced structure on X, which behaves as follows. For each local model $(X \cap U, \mathcal{O}_{X\cap U})$ the sheaf $\mathcal{O}_{X\cap U} = \mathcal{O}_U/\mathcal{J}$ is the quotient of $\mathcal{O}_\Omega$ by a sheaf of ideals $\mathcal{J}$, which is in general properly contained in the sheaf of ideals $\mathcal{I}_{X\cap U}$. Then $\rho_x : \mathcal{O}_{X,x} \to \mathcal{O}^r_{X,x}$ maps each germ $g \in \mathcal{O}_{X,x}$ to its class modulo $\mathcal{I}_{X,x}$ for each $x \in U$.

When we do not mention the structure sheaf $\mathcal{O}_X$ of a complex analytic space we are implicitly considering its reduced structure.

1.1.1 Local Properties

We recall now the main properties of a reduced complex analytic space. We denote by $\mathcal{O}_n$ the local ring of holomorphic function germs at the origin $0 \in \mathbb{C}^n$. As a consequence of the Weierstrass preparation and division theorems one proves that $\mathcal{O}_n$ is an integrally closed, noetherian, factorial domain. In particular, each ideal $\mathfrak{a}$ of $\mathcal{O}_n$ is a finite intersection $\mathfrak{a} = \bigcap_{i=1}^r \mathfrak{q}_i$ of primary ideals $\mathfrak{q}_i$ of $\mathcal{O}_n$ and the ideal of germs $I(\mathcal{Z}(\mathfrak{a}))$ vanishing identically on its zeroset $\mathcal{Z}(\mathfrak{a})$ is exactly its radical $\sqrt{\mathfrak{a}} = \bigcap_{i=1}^r \sqrt{\mathfrak{q}_i}$ (Rückert's Nullstellensatz). The zeroset $X_0 = \mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(\sqrt{\mathfrak{a}})$ is a finite union of *irreducible components* , which are precisely the zerosets of the prime ideals $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, if the decomposition $\sqrt{\mathfrak{a}} = \bigcap_{i=1}^r \sqrt{\mathfrak{q}_i}$ is irredundant, that is, $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ if $i \neq j$.

Recall that the polydisc $\Delta(x, \varepsilon)$ in $\mathbb{C}^n$ of center $x = (x_1, \ldots, x_n)$ and polyradius $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, where each $\varepsilon_i > 0$, is $\Delta(x, \varepsilon) = \prod_{i=1}^n D(x_i, \varepsilon_i)$, where $D(x_i, \varepsilon_i) = \{z \in \mathbb{C} : |z - x_i| < \varepsilon_i\}$ is the disc in $\mathbb{C}$ of center x_i and radius ε_i.

The local properties of the zeroset X_0 of a prime ideal $\mathfrak{p} \subset \mathcal{O}_n$ are described by the following theorem.

Theorem 1.1.3 *There exists a linear change of coordinates and a polydisc* $\Delta(0, \varepsilon) = \Delta_1 \times \Delta_2$, *where* $\Delta_1 \subset \mathbb{C}^d$ *and* $\Delta_2 \subset \mathbb{C}^{n-d}$ *are polydiscs centered at the origin, such that:*

- *Each function germ of a fixed finite subfamily of* $\mathfrak{p}$ *has a representative on* $\Delta(0, \varepsilon)$.
- $A = \mathcal{O}_n/\mathfrak{p}$ *is an integral extension of* $\mathcal{O}_d$, *hence* $\dim A = \dim \mathcal{O}_d = d$.[1]
- *There exists a representative* X *of* X_0, *which is a complex analytic subset of the polydisc* $\Delta(0, \varepsilon)$, *such that outside the (thin) zeroset of a non-zero* $D \in \mathcal{O}_d$,

[1] The *dimension* $\dim(A)$ of a ring A is the supremum over the lengths ℓ of each chain of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_\ell$ of A.

the projection $\pi : X \setminus \mathcal{Z}(D) \to \Delta_1 \setminus \mathcal{Z}(D)$ *is a covering map. In particular* $\dim(X) = d$.

- *The difference* $M = X \setminus \mathcal{Z}(D)$ *is a complex analytic manifold defined as the zeroset in* $\Delta(0, \varepsilon)$ *of the representatives of* $n - d$ *elements of* $\mathfrak{p}$ *whose Jacobian matrix has rank* $n - d$ *at each point of* M.
- $M = X \setminus \mathcal{Z}(D)$ *is connected and dense in* X.

The projection $\pi : X \to \Delta_1$ is a branched covering, also called an *analytic cover*.

The description above applies to the irreducible components of the zeroset X_0 of any radical ideal $\mathfrak{a}$ of $\mathcal{O}_n$. In this case the ideal $\mathfrak{a}$ is the intersection of the prime ideals $\mathfrak{p}_i$ associated to the minimal prime ideal $\mathfrak{P}_i = \mathfrak{p}_i/\mathfrak{a}$ of $A = \mathcal{O}_n/\mathfrak{a}$. Then for each minimal prime ideal $\mathfrak{p}_i$ we apply the argument to a suitable representative X_i of $X_{i,0} = \mathcal{Z}(\mathfrak{p}_i)$, which is a complex analytic subset of a polydisc $\Delta(0, \varepsilon)$. This polydisc is the same for all the germs $X_{i,0}$ (after applying a linear change of coordinates that works simultaneously for all the irreducible components of X_0). For each i we obtain a complex analytic manifold $M_i \subset X$, which is dense in X_i. Then $M = \bigcup_i M_i \setminus \bigcup_{i \neq j}(M_i \cap M_j)$ is an open and dense subset of $X = \bigcup_i X_i$. Each connected component of M is dense in an irreducible component X_i of X and each X_i is the closure of a connected component of M. As $\mathfrak{P}_i = \mathfrak{p}_i/\mathfrak{a}$ is a minimal prime ideal of A we get

$$\dim(A) = \max_i\{\dim(A/\mathfrak{P}_i)\} = \max_i\{\dim(\mathcal{O}_n/\mathfrak{p}_i)\} = \max_i\{\dim(X_{i,0})\} = \dim(X_0).$$

Remark 1.1.4 By an *analytic algebra* we mean any ring A isomorphic to $\mathcal{O}_n/\mathfrak{a}$ for some n and some ideal $\mathfrak{a} \subset \mathcal{O}_n$. The description above shows that any analytic set germ is equipped with an analytic algebra, but conversely an analytic algebra determines an analytic set germ, namely the zeroset of the ideal $\mathfrak{a}$ in a neighborhood of $0 \in \mathbb{C}^n$. Moreover, if $f : X_x \to Y_y$ is a holomorphic map between analytic set germs, it induces an algebra homomorphism $f^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ and vice versa. It is easy to prove that f injective implies f^* surjective and f surjective implies f^* injective and vice versa. Hence f is an isomorphism if and only if f^* is an isomorphism.

1.1.1.1 Regular Points of a Reduced Complex Analytic Space

Let $(X, \mathcal{O}_X)$ be a reduced complex analytic space. As the notion of *regular point* has a local nature, we assume $(X, \mathcal{O}_X)$ is a reduced local model. Thus, X is a closed subset of an open set $\Omega \subset \mathbb{C}^n$ and $\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{I}_X$, where $\mathcal{I}_X$ is the sheaf of ideals of all holomorphic germs vanishing on X. For each $x \in X$ let $F_1, \dots, F_\ell$ be generators of $\mathcal{I}_{X,x}$. We write

$$r_x = \operatorname{rk}\left(\frac{\partial(F_1, \dots, F_\ell)}{\partial(\mathrm{x}_1, \dots, \mathrm{x}_n)}(x)\right) = \operatorname{rk}\left(\frac{\partial F_i}{\partial \mathrm{x}_j}(x)\right)_{\substack{1 \le i \le \ell \\ 1 \le j \le n}} \le \min\{n, l\}.$$

It is straightforward to show that changing the set of generators, the value r_x does not change, that is, it depends only on $\mathcal{I}_{X,x}$. Indeed, if $H_1, \ldots, H_s$ is another system of generators, it is enough to prove the rank of the Jacobian of $F_1, \ldots F_\ell, H_i$ is the same as the rank of the Jacobian of $F_1, \ldots F_\ell$. Put $F_{\ell+1} = H_i = G_1 F_1 + \cdots + G_\ell F_\ell$ where $G_i \in \mathcal{O}(\Omega)$. Thus, if $x \in X$, we have $F_1(x) = \cdots = F_\ell(x) = 0$ and we conclude

$$\operatorname{rk}\left(\frac{\partial F_i}{\partial \mathrm{x}_j}(x)\right)_{\substack{1\le i\le \ell+1\\1\le j\le n}} = \operatorname{rk}\left(\left(\frac{\partial F_i}{\partial \mathrm{x}_j}(x)\right)_{\substack{1\le i\le \ell\\1\le j\le n}}\left(\sum_{i=1}^{\ell} G_i \frac{\partial F_i}{\partial \mathrm{x}_j}(x) + \sum_{i=1}^{\ell}\frac{\partial G_i}{\partial \mathrm{x}_j}(x) F_i(x)\right)_{1\le j\le n}\right)$$

$$= \operatorname{rk}\left(\left(\frac{\partial F_i}{\partial \mathrm{x}_j}(x)\right)_{\substack{1\le i\le \ell\\1\le j\le n}}, \left(\sum_{i=1}^{\ell} G_i \frac{\partial F_i}{\partial \mathrm{x}_j}(x)\right)_{1\le j\le n}\right) = \operatorname{rk}\left(\frac{\partial F_i}{\partial \mathrm{x}_j}(x)\right)_{\substack{1\le i\le \ell\\1\le j\le n}} = r_x.$$

Definition 1.1.5 The *rank* of the sheaf of ideals $\mathcal{I}_X$ at a point $x \in X$ is

$$r(x) = \min_{W_x}\{\max_{y\in W_x}\{r_y\}\} \le n,$$

where W_x runs over all the open neighborhoods of x in X.

As the sheaf of ideals $\mathcal{I}_X$ is coherent, $r_x \le r_y$ for each y in a small neighborhood of x. In addition, $\{x \in X : r_x = r(x)\}$ *is an open subset of* X. Indeed, pick a point $x \in X$ such that $r_x = r(x)$. Let $W_x \subset X$ be an open neighborhood of x such that $r(x) = \max_{y\in W_x}\{r_y\}$. As $r_x = r(x)$ and the sheaf of ideals $\mathcal{I}_X$ is coherent, we may assume that $r_y = r(x)$ for all $y \in W_x$, so $r_y = r(y)$ for all $y \in W_x$, so $W_x \subset \{x \in X : r_x = r(x)\}$ and this set is open.

Definition 1.1.6 The point $x \in X$ is *regular* if $r_x = r(x)$.

We denote by $\mathrm{Reg}(X)$ the set of regular points of X and $\mathrm{Sing}(X) = X \setminus \mathrm{Reg}(X)$ the set of *singular points* of X. As $\mathrm{Reg}(X)$ is an open subset of X, the set $\mathrm{Sing}(X)$ is closed in X.

Let us prove next that each connected component N of $\mathrm{Reg}(X)$ is a complex analytic manifold. More precisely,

Proposition 1.1.7 *Let $x_0 \in \mathrm{Reg}(X)$ be a regular point of X. Then there exists an open neighborhood U of x_0 in Ω such that $X \cap U$ is a complex manifold of dimension $n - r(x_0)$.*

Proof Fix $r = r(x_0)$ and sections $F_1, \ldots, F_r$ of $\mathcal{I}_X$ in a neighborhood U of x_0 such that their Jacobian matrix has rank r at each point of $X \cap U$. The set $M = \{F_1 = 0, \ldots, F_r = 0\} \subset U$ is by the Implicit Function Theorem a complex manifold of dimension $n - r$, which in addition contains $X \cap U$. As for the converse we show that there exists a perhaps smaller neighborhood $W \subset U$ of x_0 such that $X \cap W = M \cap W$, or equivalently, $X_{x_0} = M_{x_0}$. To that end, it is enough to show that each function germ $H \in \mathcal{I}_{X,x_0}$ vanishes on M_{x_0}.

As the holomorphic functions $F_1, \dots, F_r$ have Jacobian matrix of rank r, we can complete this collection with linear functions $L_{r+1}, \dots, L_n$ depending on the variables $(\mathtt{z}_1, \dots, \mathtt{z}_n)$ in such a way that

$$\begin{aligned}
\mathtt{y}_1 &= F_1(\mathtt{z}_1, \dots, \mathtt{z}_n)\\
&\vdots\\
\mathtt{y}_r &= F_r(\mathtt{z}_1, \dots, \mathtt{z}_n)\\
\mathtt{y}_{r+1} &= L_{r+1}(\mathtt{z}_1, \dots, \mathtt{z}_n)\\
&\vdots\\
\mathtt{y}_n &= L_n(\mathtt{z}_1, \dots, \mathtt{z}_n)
\end{aligned}$$

provide a holomorphic system of coordinates on an open neighborhood $V \subset U$ of x_0, that maps $M \cap V$ onto an open subset of the linear subspace $\{\mathtt{y}_1 = 0, \dots, \mathtt{y}_r = 0\}$. We may assume that $M \cap V$ is connected. In this way, we can define

$$\varphi : V \to V' = \varphi(V),\ z = (\mathtt{z}_1, \dots, \mathtt{z}_n) \mapsto (F_1(z), \dots, F_r(z), L_{r+1}(z), \dots, L_n(z))$$

and consider its inverse $\psi = (\psi_1, \dots, \psi_n) : V' \to V$. Observe that

$$\psi(\{\mathtt{y}_1 = 0, \dots, \mathtt{y}_r = 0\} \cap V') = M \cap V.$$

In addition, $F_i' = F_i \circ \psi = \mathtt{y}_i$ for $i = 1, \dots, r$. To prove that $H_{|M\cap V}$ is identically zero, we show that the restriction of $H' = H \circ \psi$ to $\{\mathtt{y}_1 = 0, \dots, \mathtt{y}_r = 0\} \cap V'$, which is a holomorphic function of the last $n - r$ coordinates, is identically zero. Write $y_0 = \varphi(x_0)$. As $\{\mathtt{y}_1 = 0, \dots, \mathtt{y}_r = 0\} \cap V'$ is connected, we have to show

$$\frac{\partial^\alpha H'}{\partial \mathtt{y}_{r+1}^{\alpha_1} \dots \partial \mathtt{y}_n^{\alpha_n}}(y_0) = 0 \quad \text{for each multi-index} \quad \alpha = (\alpha_1, \dots \alpha_{n-r}) \in (\mathbb{N})^{n-r}.$$

For $i = r + 1, \dots, n$ one has

$$\det \begin{vmatrix} \frac{\partial F_1'}{\partial \mathtt{y}_1} & \cdots & \frac{\partial F_1'}{\partial \mathtt{y}_r} & \frac{\partial F_1'}{\partial \mathtt{y}_i} \\ \vdots & \ddots & \vdots & \\ \frac{\partial F_r'}{\partial \mathtt{y}_1} & \cdots & \frac{\partial F_r'}{\partial \mathtt{y}_r} & \frac{\partial F_r'}{\partial \mathtt{y}_i} \\ \frac{\partial H'}{\partial \mathtt{y}_1} & \cdots & \frac{\partial H'}{\partial \mathtt{y}_r} & \frac{\partial H'}{\partial \mathtt{y}_i} \end{vmatrix} = \det \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{\partial H'}{\partial \mathtt{y}_1} & \frac{\partial H'}{\partial \mathtt{y}_2} & \cdots & \frac{\partial H'}{\partial \mathtt{y}_r} & \frac{\partial H'}{\partial \mathtt{y}_i} \end{vmatrix} = \frac{\partial H'}{\partial \mathtt{y}_i}. \tag{1.1.1}$$

Next, define

$$D_s(H) = \det \left| \frac{\partial(F_1, \ldots, F_r, H)}{\partial(z_{s_1}, \ldots, z_{s_{r+1}})} \right|$$

for each $s = (s_1, \ldots, s_{r+1}) \in \{1, \ldots, n\}^{r+1}$ such that $1 \le s_1 < \cdots < s_{r+1} \le n$. As the rank of $\mathcal{I}_X$ is r, we deduce that $D_s(H)$ vanishes on $\operatorname{Reg}(X) \cap U$. Consequently, as $\operatorname{Reg}(X)$ is dense in X_{x_0}, we have $D_s(H) \in \mathcal{I}_{X,x_0}$. Thus, $D_s(H) \circ \psi \in \mathcal{I}_{X',y_0}$ where $X' = \varphi(X \cap V)$ and $\mathcal{I}'_X$ is the sheaf of ideals on V consisting of all holomorphic function germs vanishing on $X' \cap V$.

We have

$$\frac{\partial F'_j}{\partial y_\ell} = \sum_{k=1}^{n} \left(\frac{\partial F_j}{\partial z_k} \circ \psi \right) \cdot \frac{\partial \psi_k}{\partial y_\ell} \quad \text{and} \quad \frac{\partial H'}{\partial y_\ell} = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial z_k} \circ \psi \right) \cdot \frac{\partial \psi_k}{\partial y_\ell}$$

for $j = 1, \ldots, r$ and $\ell = 1, \ldots, n$. Thus, for $i = r+1, \ldots, n$ we have

$$\begin{pmatrix} \frac{\partial F'_1}{\partial y_1} & \cdots & \frac{\partial F'_1}{\partial y_r} & \frac{\partial F'_1}{\partial y_i} \\ \vdots & \ddots & \vdots & \\ \frac{\partial F'_r}{\partial y_1} & \cdots & \frac{\partial F'_r}{\partial y_r} & \frac{\partial F'_r}{\partial y_i} \\ \frac{\partial H'}{\partial y_1} & \cdots & \frac{\partial H'}{\partial y_r} & \frac{\partial H'}{\partial y_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_r}{\partial z_1} & \cdots & \frac{\partial F_r}{\partial z_n} \\ \frac{\partial H}{\partial z_1} & \cdots & \frac{\partial H}{\partial z_n} \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_1}{\partial y_1} & \cdots & \frac{\partial \psi_1}{\partial y_r} & \frac{\partial \psi_1}{\partial y_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \psi_n}{\partial y_1} & \cdots & \frac{\partial \psi_n}{\partial y_r} & \frac{\partial \psi_n}{\partial y_i} \end{pmatrix}. \tag{1.1.2}$$

Hence, using (1.1.1), (1.1.2) and the Binet–Cauchy formula for the determinant of the product of two rectangular matrices of transposed shapes, we deduce

$$\frac{\partial H'}{\partial y_i} = \sum_{\substack{s=(s_1,\ldots,s_{r+1}) \\ s_1 < s_2 < \ldots < s_{r+1}}} (D_s(H) \circ \psi) \det \left| \frac{\partial(\psi_{s_1}, \ldots, \psi_{s_{r+1}})}{\partial(y_1, \ldots, y_r, y_i)} \right| \in \mathcal{I}_{X',y_0}.$$

This implies that if $H \in \mathcal{I}_{X,x_0}$, then $\dfrac{\partial H'}{\partial y_i} \in \mathcal{I}_{X',y_0}$ for $i = r+1, \ldots, n$. Consequently, $\dfrac{\partial H'}{\partial y_i} \circ \varphi \in \mathcal{I}_{X,x_0}$. Then we can apply recursively the same trick to $\dfrac{\partial H'}{\partial y_i} \circ \varphi$ for $i = r+1, \ldots, n$, to deduce

$$\frac{\partial^2 H'}{\partial y_i \partial y_s} \in \mathcal{I}_{X',y_0}$$

for $r+1 \le i, s \le n$, and so on. Thus, all derivatives of H' of all orders vanish at y_0, so H' is the zero function on $M \cap V$ and we conclude $X_{x_0} = M_{x_0}$, as required. □

Remarks 1.1.8 Let N be a connected component of $\operatorname{Reg}(X)$.

(i) If $x, y \in N$, then $r(x) = r(y)$. Indeed, by Proposition 1.1.7 the number $r(x)$ is locally constant in $\operatorname{Reg}(X)$. Hence it is constant on the connected components of $\operatorname{Reg}(X)$.
(ii) N is a connected complex analytic manifold of dimension $n - r(x_0)$, where x_0 is any of the points of N.
(iii) The closure in X of a connected component of $\operatorname{Reg}(X)$ is an irreducible subset of X. This can be proved by the same argument used in Rückert's Nullstellensatz.

1.1.1.2 Zariski's Tangent Space

We now approach regular points from another point of view. This requires the introduction of Zariski's tangent space. Let $(X, \mathcal{O}_X)$ denote a reduced complex analytic space.

Definition 1.1.9 Let $x \in X$ be a point and $F_1, \ldots, F_k$ be generators of the ideal $\mathcal{I}_{X,x}$. *Zariski's tangent space* $T_x X$ of X at x is defined by

$$T_x X = \ker(J(F_1, \ldots, F_k)(x)),$$

where $J(F_1, \ldots, F_k)(x)$ is the Jacobian matrix of $F_1, \ldots, F_k$ at the point x.

By definition $\dim(T_x X) = n - r_x$ and it has minimal dimension when $x \in \operatorname{Reg}(X)$, that is, when $J(F_1, \ldots, F_k)(x)$ has rank $r(x)$.

Lemma 1.1.10 *Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$ and recall that $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{C}$. Then the $\mathcal{O}_{X,x}/\mathfrak{m}_x$-linear space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is isomorphic to $T_x X$ as $\mathbb{C}$-linear space.*

Proof Denote by $\mathfrak{M}_x$ the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,x}$. Observe that

$$\mathfrak{m}_x/\mathfrak{m}_x^2 = (\mathfrak{M}_x/\mathcal{I}_{X,x}) \Big/ ((\mathfrak{M}_x^2 + \mathcal{I}_{X,x})/\mathcal{I}_{X,x}) \cong \mathfrak{M}_x/(\mathfrak{M}_x^2 + \mathcal{I}_{X,x}).$$

Let $\mathbb{C}^{n*}$ be the dual linear space of $\mathbb{C}^n$ and consider the linear map

$$L : \mathfrak{M}_x \to \mathbb{C}^{n*}, \ f \mapsto L(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(x) u_i.$$

Observe that L is surjective and $\ker(L) = \mathfrak{M}_x^2$. Consequently, L induces an isomorphism $[L] : \mathfrak{M}_x/\mathfrak{M}_x^2 \to \mathbb{C}^{n*}, \ F + \mathfrak{M}_x^2 \mapsto L(F)$.

Let $F_1, \ldots, F_k$ be a system of generators of $\mathcal{I}_{X,x}$. Then we get

$$T_x X = \ker(L(F_1)) \cap \cdots \cap \ker(L(F_k))$$

and we can identify the dual space T_xX^* with the quotient $\mathbb{C}^{n*}/\langle L(F_1), \dots, L(F_k)\rangle$, where $\langle L(F_1), \dots, L(F_k)\rangle$ denotes the subspace spanned by $L(F_1), \dots, L(F_k)$.

Indeed, consider the linear map

$$\Gamma : \mathbb{C}^{n*} \to T_xX^*, \ H \mapsto H_{|T_xX}.$$

As each linear form $G : T_xX \to \mathbb{C}$ is the restriction of a linear form $H : \mathbb{C}^n \to \mathbb{C}$, the previous linear map is surjective. Consequently, $T_xX^* \cong \mathbb{C}^{n*}/\ker(\Gamma)$. As $T_xX = \ker(L(F_1)) \cap \cdots \cap \ker(L(F_k))$, we conclude that $\ker(\Gamma) = \langle L(F_1), \dots, L(F_k)\rangle$.

We have the following commutative diagram

$$\begin{array}{ccccc}
\mathfrak{M}_x & \xrightarrow{L} & \mathbb{C}^{n*} & \xrightarrow{\Gamma} & T_xX^* \\
& \searrow & \updownarrow [L] & \searrow & \updownarrow [\Gamma] \\
& & \mathfrak{M}_x/\mathfrak{M}_x^2 & & \mathbb{C}^{n*}/\langle L(F_1), \dots, L(F_k).\rangle
\end{array}$$

Thus,

$$L^{-1}(\langle L(F_1), \dots, L(F_k)\rangle) = \langle F_1, \dots, F_k\rangle + \ker(L) = \mathcal{I}_{X,x} + \mathfrak{M}_x^2,$$

and we conclude

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \mathfrak{M}_x/(\mathfrak{M}_x^2 + \mathcal{I}_{X,x}) \cong \mathfrak{M}_x/\ker(L) \cong \mathbb{C}^{n*}/\langle L(F_1), \dots, L(F_k)\rangle \cong T_xX^*,$$

as required. □

As a consequence we get a characterization of a regular point $x \in X$ in terms of the algebraic properties of the ring $\mathcal{O}_{X,x}$. Recall that a local noetherian ring $(A, \mathfrak{m})$ is called *regular* if $\dim(A) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$, where $\kappa = A/\mathfrak{m}$ is the *residue field of* A.

Consider the local noetherian ring $A = \mathcal{O}_{\mathbb{C}^n,x}/\mathcal{I}_{X,x}$ and assume that $\mathcal{I}_{X,x}$ is a prime ideal. We know that, up to a linear change of coordinates, A is an integral extension of the local ring $\mathcal{O}_d$ of holomorphic germs in d variables. Thus, $\dim(A) = d$.

Lemma 1.1.11 *Under the hypothesis above one has* $d = n - r(x)$ *and there exists an open neighborhood* $X' \subset X$ *of* x *such that* $r(y) = r(x)$ *for each* $y \in X'$. *In particular,* $d = n - r(x) \le n - r_x = \dim(\mathfrak{m}_x/\mathfrak{m}_x^2)$.

Proof The germ X_x has, after a linear change of coordinates, a representative $X' = X \cap \Delta(x, \varepsilon)$ in a polydisc $\Delta(x, \varepsilon)$ centered at x and an open subset $M \subset X'$, which is connected and dense in X such that the projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$ onto the first d coordinates induces a covering from M to an open subset of $\mathbb{C}^d$. Thus, $\dim(M) = d$ and M is defined by representatives of $n - d$ elements in $\mathcal{I}_{X,x}$ whose Jacobian matrix has rank $n-d$ at each point of M. Observe that $M \subset \text{Reg}(X)$ because at each point $z \in M$ one has $r_z = n-d$. Pick a point $y \in X' \setminus M$. As M is dense in X', there exists a $z \in M$ close to y, so $r_y \leq r_z = n-d$. Thus, $r(x) = n-d$ because for each point $y \in X$ close to x, we have $r_y \leq n-d$ and as M is dense in X', there exists points $z \in M$ close to x and at these points $r_z = n-d$. We have already proved in addition that if $y \in X'$, then $r(y) = r(x)$, as required. □

Theorem 1.1.12 *Let $(X, \mathcal{O}_X)$ be a reduced complex analytic space. Then a point $x \in X$ is regular if and only if the ring $\mathcal{O}_{X,x}$ is a local regular ring.*

Proof We may assume that $(X, \mathcal{O}_X)$ is a reduced local model. If x is regular, then $r_x = r(x)$, so $\dim(T_xX) = n - r_x = n - r(x) = \dim(\mathcal{O}_{X,x})$, that is, the ring $\mathcal{O}_{X,x}$ is regular. Conversely, if $\mathcal{O}_{X,x}$ is regular, then $n - r(x) = \dim(\mathcal{O}_{X,x}) = \dim_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(T_xX) = n - r_x$, so $r_x = r(x)$ and the point x is regular. □

Corollary 1.1.13 *The subset* $\text{Sing}(X)$ *of a reduced complex analytic space* $(X, \mathcal{O}_X)$ *is a complex analytic subset of X.*

Proof Pick a point $x \in X$. We distinguish two cases.

(1) Suppose first that the germ X_x is irreducible and define $r = r(x)$. There exist an open neighborhood U and sections $F_1, \dots, F_r$ of the sheaf $\mathcal{I}_X$ defined on the open set U such that they generate $\mathcal{I}_{X,y}$ for each $y \in U$. After shrinking U, we may assume that $r(y) = r(x)$ for each point $y \in U$. Then, $y \in U$ is a singular point if and only if $r_y < r(x) = r$. Thus, $\text{Sing}(X) \cap U$ is the set of the points $y \in X$ at which all minors of order r of the Jacobian matrix

$$\Big(\frac{\partial F_i}{\partial \mathrm{x}_j}(x)\Big)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n}}$$

are zero. Thus, $\text{Sing}(X) \cap U$ is a complex analytic subset of U.

(2) Suppose next that the germ X_x is reducible. Then, there are an open neighborhood U and complex analytic subsets $X_1, \dots, X_s$ of U such that $X \cap U = X_1 \cup \dots \cup X_s$, each germ $X_{i,x}$ is irreducible and X_i contains a connected and dense complex analytic manifold of the same dimension as X_i. Using the fact the ring $\mathcal{O}_{X,y}$ is not an integral domain if the germ X_y is reducible, we conclude that

$$\text{Sing}(X \cap U) = \textstyle\bigcup_{i=1}^s \text{Sing}(X_i) \cup \bigcup_{i \neq j}(X_i \cap X_j).$$

Using (1) and the fact that each X_i has a finite system of holomorphic equations in U (shrinking U if necessary), we conclude that $\text{Sing}(X \cap U)$ is a complex analytic subset of U, as required.

□

For an arbitrary complex analytic space $(X, \mathcal{O}_X)$, not necessarily reduced, we say that a point $x \in X$ is *regular* if the ring $\mathcal{O}_{X,x}$ is a local regular ring. Otherwise, we say that the point x is a *singular point of* X. We again denote by $\mathrm{Reg}(X)$ the set of regular points of X and $\mathrm{Sing}(X)$ the set of singular points of X.

1.1.2 Stein Spaces

Let $(X, \mathcal{O}_X)$ be a complex analytic space. Denote by $\mathcal{O}(X)$ the algebra of its holomorphic functions. This algebra can be very small. For instance, if X is compact, like the projective space $\mathbb{P}^n(\mathbb{C})$, the maximum principle shows that $\mathcal{O}(X)$ reduces to the set $\mathbb{C}$ of constant functions. Conversely, if X is a closed analytic subset of $\mathbb{C}^n$, it has a lot of holomorphic functions. We now give a list of desirable properties that analytic subsets of $\mathbb{C}^n$ possess. The first one is *to provide local coordinates*.

Definition 1.1.14 Let $(X, \mathcal{O}_X)$ be a reduced complex analytic space and let $x \in X$. We say that finitely many holomorphic functions $F_1, \ldots, F_n$ on an open neighborhood $U \subset X$ of x *provide local coordinates* if they define a closed embedding $F = (F_1, \ldots, F_n) : U \to \Omega \subset \mathbb{C}^n$, where Ω is an open subset. Observe that F induces an isomorphism between $(U, \mathcal{O}_{|U})$ and a local model $(Y = F(U), \mathcal{O}_\Omega/\mathcal{I}_Y)$ in Ω.

Here is the announced list of properties of a closed analytic subset $X \subset \mathbb{C}^n$.

(i) For any unbounded sequence of points $\{x_m\}_m$ in X there exists a holomorphic function f on X such that $\lim_{m\to\infty} |f(x_m)| = \infty$.[2]
(ii) Holomorphic functions on X separate points and provide local coordinates at each point $x \in X$.
(iii) X is not compact unless X is finite.

These three properties characterize a larger class of analytic spaces.

Definition 1.1.15 A complex analytic space $(X, \mathcal{O}_X)$ is a *Stein space* if it satisfies conditions (i), (ii), (iii) above.

Among the most important results concerning Stein spaces we point out Cartan's Theorems A and B.

Theorem 1.1.16 *Let $(X, \mathcal{O}_X)$ be a Stein space. Then each coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X satisfies the following properties.*

(A) *Each fiber $\mathcal{F}_x$ is generated by global sections of $\mathcal{F}$.*
(B) *$H^q(X, \mathcal{F}) = 0$ for each $q > 0$.*

[2] By an unbounded sequence we mean an infinite sequence which intersects all compact sets in a finite number of points.

In particular, when $(X, \mathcal{O}_X)$ is a complex analytic subspace of $\mathbb{C}^n$, one gets the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{C}^n} \to \mathcal{O}_X \to 0,$$

where each involved sheaf is by Oka's theorem a coherent $\mathcal{O}_{\mathbb{C}^n}$-module. This implies using Cartan's theorem B that each holomorphic function on X is the restriction to X of a holomorphic function on $\mathbb{C}^n$.

Next we give a characterization of open Stein subsets of $\mathbb{C}^n$.

Theorem 1.1.17 (Characterization of Open Stein Sets) *Let Ω be a connected open subset of C^n. The following are equivalent.*

(i) *Ω is a Stein manifold.*
(ii) *Ω is holomorphically convex.*
(iii) *Ω is a holomorphy domain.*[3]

A very relevant example of a Stein open subset of $\mathbb{C}^n$ is a polydisc. Note that one can choose local models as subspaces of a polydisc. As a consequence of Theorem B, a closed subspace of a Stein space is also Stein, so we get that any complex analytic space is locally Stein. In particular, closed subspaces of $\mathbb{C}^n$ are Stein spaces. The converse is almost true: any Stein space can be embedded in $\mathbb{C}^n$ as a closed analytic subspace for some n large enough under the additional hypothesis that $\sup_{x \in X} \dim(T_x X) < \infty$. More precisely

Theorem 1.1.18 (Narasimhan) *Any Stein space $(X, \mathcal{O}_X)$ of dimension n admits a one-to-one proper holomorphic map into $\mathbb{C}^{2n+1}$, that is, a holomorphic embedding at each regular point of X. Assume in addition that for each point $x \in X$ there exists an open neighborhood in X that can be holomorphically embedded as a closed analytic subset of an open subset of $\mathbb{C}^N$ (with the analytic structure induced by $\mathbb{C}^N$), where $N > n$ is fixed. Then, there exists a one-to-one proper map $\varphi : X \to \mathbb{C}^{N+n}$ whose image (with the induced analytic structure provided by $\mathbb{C}^{N+n}$) is isomorphic to X by means of φ.*

In both cases above, the set of embeddings is dense in the space of holomorphic maps into $\mathbb{C}^m$ (where $m = 2n + 1$ in the first case and $m = N + n$ in the second case) if we endow such space with the compact-open topology.

As an application of Cartan's Theorem B we recall an argument due to Grauert.

[3] A connected open set $U \subset \mathbb{C}^n$ is *holomorphically convex* if the holomorphic envelope of a compact subset of U is compact. It is a *holomorphy domain* if there do not exist non-empty open sets $\Omega \subset U$ and $V \subset \mathbb{C}^n$ connected and not included in U such that $\Omega \subset U \cap V$ and there are holomorphic functions $f \in \mathcal{O}(U)$, $g \in \mathcal{O}(V)$, such that the restriction of g to Ω coincides with the restriction of f to Ω. Roughly speaking, a holomorphy domain is a set which is maximal in the sense that there exists a holomorphic function on this set which cannot be extended to a bigger set.

Proposition 1.1.19 *A Stein subspace of a Stein space has finitely many global holomorphic equations. More precisely, if n is the dimension of the ambient Stein space, it is the zeroset of at most $n + 1$ global holomorphic equations.*

Proof If $Y \subset X$ is a closed subspace of a Stein space X, we take first a non-identically zero holomorphic function F_1 on X vanishing identically on Y, so $\dim(\{F_1 = 0\}) = \dim(X) - 1$. It exists by Theorem B, indeed as Y is a Stein subspace of the Stein space $(X, \mathcal{O}_X)$, pick a point $p \in X \setminus Y$, and as $Y \cup \{p\}$ is a Stein subspace of the Stein space $(X, \mathcal{O}_X)$, we can consider a holomorphic function F_1 on X that takes values 0 on Y and 1 on p. If the zeroset of F_1 is Y, we are done. Otherwise, we decompose the zeroset $\{F_1 = 0\}$ as the union of its irreducible components. We pick a point p_Z in each irreducible component Z of $\{F_1 = 0\}$ that does not lie inside Y. Then, there exists a holomorphic function F_2 on X that vanishes identically on Y and takes the value 1 at each point p_Z. Now, the common zeroset of F_1, F_2 outside Y has strictly smaller dimension than the dimension of $\{F_1 = 0\} \setminus Y$. We repeat the same trick until we find holomorphic functions $F_3, \dots, F_k$ such that $Y \subset \{F_1 = 0, \dots, F_k = 0\}$ and

$$\dim(\{F_1 = 0, \dots, F_k = 0\} \setminus Y) \leq 0.$$

Observe that $\{F_1 = 0, \dots, F_k = 0\} = Y \cup D$, where D is a (possibly empty) discrete set. If F_{k+1} is a holomorphic function on X vanishing identically on Y and taking value 1 at each isolated point of the discrete set D, we describe Y as the common zeroset of $F_1, \dots, F_{k+1}$. By construction $k + 1 \leq n + 1$. □

1.1.2.1 Cartan's Theorems A and B and Direct Limits

The next result plays a fundamental role in this framework, because it implies Theorems A and B for a large class of real analytic spaces, that will be introduced later.

Theorem 1.1.20 *Let Z be a closed subset of a complex analytic space $(X, \mathcal{O}_X)$ and define $\mathcal{O}_Z = \mathcal{O}_{|Z}$. Suppose that Z has a fundamental system of open Stein neighborhoods in X. Then Theorems A and B hold for Z, that is, for each $\mathcal{O}_Z$-coherent sheaf of modules $\mathcal{F}$ on Z we have:*

(A) *$\mathcal{F}_x$ is generated (as $\mathcal{O}_{Z,x}$-module) by the image of natural map $H^0(Z, \mathcal{F}) \to \mathcal{F}_x$ for each $x \in Z$, that is, $\mathcal{F}_x$ is generated by global sections of $\mathcal{F}$.*
(B) *$H^q(Z, \mathcal{F}) = 0$ for each $q > 0$.*

In what follows, given a complex analytic space $(X, \mathcal{O}_X)$ and a closed subset $C \subset X$, the sheaves of $\mathcal{O}_{|C}$-modules will be called *analytic sheaves on C*. First of all we need the following proposition.

Proposition 1.1.21 *Let C be a closed subset of a complex analytic space $(X, \mathcal{O}_X)$. Let $\mathcal{G}$ be a coherent analytic sheaf on C. Then there exists a triple $(U, \mathcal{F}, \varphi)$, where*