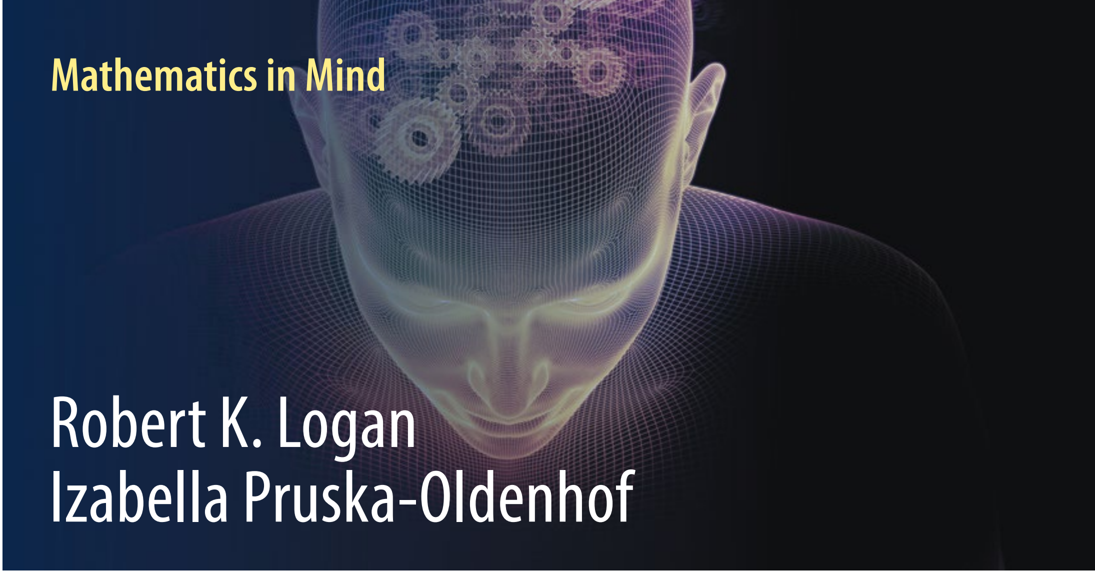

Mathematics in Mind

Robert K. Logan
Izabella Pruska-Oldenhof

A Topology of Mind

Spiral Thought Patterns, the Hyperlinking of Text, Ideas and More

Springer

Mathematics in Mind

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ISSN 2522-5405 ISSN 2522-5413 (electronic)
Mathematics in Mind
ISBN 978-3-030-96435-1 ISBN 978-3-030-96436-8 (eBook)
https://doi.org/10.1007/978-3-030-96436-8

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

What This Book Is About

This book covers many diverse topics related in varying degrees to mathematics in mind. We will examine mathematics in mind from the perspective of the spiral, cyclic, and hyperlinked structures of the human mind in terms of its language, its thoughts, and its various modes of communication.

We report on two studies we have made of the mathematical and topological structures of thought and communication that the human mind creates: one of the spiral and cyclic structures found in human thought, the arts, and social interactions and the other of hyperlinking of separate bodies of information. This book is divided into four parts.

In Part I, we discuss what mathematics in the mind entails and use abductive reasoning to suggest that the human mind, verbal language, and mathematics in mind (in the form of set theory) are interconnected and emerged simultaneously as pre-human hominids made the transition to become fully human, Homo sapiens. We show how the mind capable of conceptualization, verbal language, and mathematical thinking forms a supervenient emergent dynamic system making use of complexity theory. We explore the cognitive origins of mathematical thinking of the human mind and its relation to the emergence of spoken language. We will also examine the emergence of mathematical notation and its impact on mathematics, education, science, accounting, and other applications of mathematics.

We show how spoken language and the ability to conceptualize and thus to plan and to think about things that are not available in the immediate here and now arose through the linking of concepts with percepts and the ability of humans to think in terms of set.

We also show how hyperlinking between spoken words and visual signs led to the origin of the written word and, as we show, arose out of accounting. We examine the importance of mathematical notation for advancing scientific and mathematical thinking while at the same time make a careful distinction between mathematics and science as well as logic and science.

In Part II, we examine the spiral and cyclic topological structures of human thought. The spiral structure is a ubiquitous and universal form that one finds in nature in physical structures of whirlpools, tornados, hurricanes, and galaxies throughout the universe. Spirals are also a ubiquitous feature of biological life from the molecular forms such as the double helix of DNA in the genetic makeup of all living things to the patterns of biological forms like the phyllotactic spirals in the arrangement of seeds in a sunflower and of the petals of flowers, the branches of trees, and the shells of mollusks. Spiral structures are also found in various domains of human study and cultural expression including media, the evolution of technology, history, and the thoughts of many scholars such as Marshall McLuhan, Sigmund Freud, and I.A. Richards; certain poets and writers such as Edgar Allan Poe, T.S. Eliot, James Joyce, Gertrude Stein, and Ezra Pound; certain artists such as Wyndham Lewis and the Vorticism Art Movement. We also study the impact of cyclic structures on human thought patterns which are closely related to the spiral. The spiral executes cyclic or rotational motion around a reference point with the radius of rotation either increasing or decreasing. Wave motion is an example of cyclic motion in the physical world. We will examine examples of cyclic structures in the thought patterns of Giambattista Vico's corso and ricorso, the dialectics of G.W.F. and Karl Marx, the eternal return of Friedrich Nietzsche and Mircea Eliade and the notion of the recurring universe of certain cosmologists. As part of our study, we also examine the spiral and cyclic structures in both the abiotic physical world and in the biosphere as a way of demonstrating the universality of these topological structures.

In Part III, we study the role of hyperlinking in human communication, thought, and scholarship. Hyperlinking is basically the linkage of one body of information with another, which came into prominence with the World Wide Web because of the ease with which one body of information can be linked to another. The roots of hyperlinking and hypertexting found in digital media can be traced back to the origin of human communication and its subsequent expression in terms of the origin of the spoken word and its use particularly in epic poetry.

Our choice of the term "hyperlink," which is common in digital media vernacular, is intentional. Firstly, the etymological root of "hyper" is in the Greek word *huper*, which means "over, beyond." Because our interest in this section of our book is the thought process itself, we want to distinguish the linking or connecting processes in the physical world from linking that takes place in mental processes of human thinking, remembering, dreaming, imagining, etc. We are fully aware that the human brain and its dynamic processes are inseparable from the physical world. Secondly, from perspective of neurophysiology, thought processes are electric, electrochemical to be more specific, and rely on electrochemical activation of neurons, which fire neurotransmitters that activate the electrochemical process in other neurons in a domino-like fashion across neuropathways. Digital media, in particular the capacity of accessing and hyperlinking information over computer networks and the Internet, also relies on electricity. With digital technology and the Internet, humans have come close to simulating the dynamics of thinking itself, in particular the capacity for recalling and cross-referencing data—hyperlinking—and through this process turning data into meaningful information. Given that digital media in its operation mimics thinking process of human mind or is an extension of our nervous system (brain included), as Marshall McLuhan would say, it is appropriate to take some liberty with language and use the term hyperlink in connection with human thought and its many other extensions, i.e., other media of communication.

With written communication, a non-digital form of hypertexting arose in handwritten manuscripts in terms of marginalia and illuminations, and later in printed documents including books, magazines, and newspapers. We explore these forms of hyperlinking as well as the digital variety that arose with digital computers, computer networks, and the Internet, through the introduction of which hyperlinking became standard mode of navigating the web, as well as hypermedia associated with nonlinear and interactive media such as the World Wide Web and standalone applications on mobiles for interactive books, videos, music, etc.

In Chapter 11, we focus on the spiral, cyclic, and hyperlinked forms in the arts including literature, poetry, painting, sculpture, architecture, music, and dance. Examples include Petrarchan sonnets, Dante's Divine Comedy, the literature of Proust and James Joyce's Finnegans wake, the art movement of Vorticism, the music of Bach and the fugues of other composers, cinema such as Sorel Etrog's film *Spiral* and Marcel Duchamp's *Anémic Cinéma*, the architectural form of the golden spiral and the spiral staircase, and the dance of Loïe Fuller and Martha Graham. We also consider the linking of information from one domain to another in the general biosphere with organisms other than humans to once again illustrate the universality of this topological structure as we did for the spiral structure.

The nonlinear linking and hyperlinking structures that we consider in this study are rhizomatic. The etymology of rhizome is from the Greek rhizome, which means "mass of roots." Hyperlinks on the web are like a mass of roots that branch out like in an iris flower or turmeric, which has a root system consisting of multiple nodes from which roots and shoots grow. Rhizomatic forms, or forms created by navigating through hyperlinks, have multiple entry and exit points, defy linearity, and are non-hierarchical.

The study of spiral, cyclic, and hyperlinked [rhizomatic] structures in the natural world perhaps represents a deviation from the guidelines of the series Mathematics in Mind to which this book belongs as the main focus of our study is the topology of the human mind, but we introduce these topics because they demonstrate the universality of spiral, cyclic, and hyperlinked structures. It also might help us understand why the language of mathematics is such a powerful and useful tool for describing the many levels of organization throughout our vast and varied universe.

The reader might wonder why we examine spiral structures in both the abiotic and biotic worlds but only touch upon hyperlinked [rhizomatic] structures in the biotic world and do not treat hyperlinking in the abiotic world. The reason is that we believe that information has no meaning as far as the behavior of abiotic matter is concerned as abiotic matter makes no choices as is the case with living organisms. Abiotic matter blindly follows the laws of nature or physics. Information as defined by Douglas MacKay (1969) is "a distinction that makes a difference" and redefined a short time later by Gregory Bates (1973) as follows "what we mean by information—the elementary unit of information—is a difference which makes a difference." Only a living organism can make a distinction or difference that makes a difference because they can make a choice as to how they behave and information is that which helps them make those choices. Abiotic matter cannot make any choices as any change in its state is governed by universal laws and hence it cannot be informed. Living organisms are informed by the information that they interpret and which they transform into a meaningful action or response.

We remind the reader that the origin of the term information comes from the notion of forming the mind. The English word information according to the Oxford English Dictionary (OED) first appears in the written record in 1386 by Chaucer: "Whanne Melibee hadde herd the grete skiles and resons of Dame Prudence, and hire wise informacions and techynges." The word is derived from Latin through French by combining the word inform meaning giving a form to the mind with the ending "ation" denoting a noun of action. The word makes its way into Middle English enforme, informe "give form or shape to," also "form the mind of, teach," from Old French enfourmer, from Latin informare "shape, fashion, describe." These earliest definitions refer to an item of training or molding of the mind. The next notion of information, namely the communication of knowledge appears shortly thereafter in 1450. "Lydg. & Burgh *Secrees* 1695 Ferthere to geve the Enformacioun, Of mustard whyte the seed is profitable."

This does not mean there is no information associated with abiotic matter. But that is not information that abiotic matter possesses but rather it is the information that we humans have about abiotic matter. After all, the natural sciences represent our information about and knowledge of abiotic matter. But the subjects of natural science, abiotic matter, themselves have no interaction with information and do not possess information.

Since only living organisms can be informed because they have choice as to how to realize their purpose or telos to propagate their organization (Kauffman et al. 2007), only they can transform stimuli or data into information and be influenced by information. When natural scientists talk about the exchange of information among different forms of abiotic matter they are speaking metaphorically. What about quantum information you might ask? The atomic states in which information is inscribed has no more information than the ink used to print this book. The atomic states and the ink used to print this book do not possess any information; they are not informed; they are just two different inanimate objects or media used to represent information. The same can be said for the spiral structures or forms and hyperlinks [rhizomes] that will be the subjects of our study. They do not possess information; they merely represent information. There was a folk belief in the Middle Ages that if one could wash the ink of a handwritten manuscript and drink it, one would possess the information of that manuscript. Qubits and ink do not possess information; they merely represent information.

Responding to the Topics and Questions Raised by the Series Editor

In developing our thoughts on a topology of mind, we will also specifically address many of the topics and questions raised by the Series Editor, Marcel Danesi in his call for proposals for this series. The topics and questions that we have selected to address in this study include the following that Professor Danesi suggested. We also have indicated how we plan to address each of these topics and questions:

- The historical context of any topic that involves how mathematical thinking emerged, focusing on archeological and philological evidence.
- Connection between math cognition and symbolism, annotation, and other semiotic processes.
- Interrelationships between mathematical discovery and cultural processes, including technological systems that guide the thrust of cognitive and social evolution.
- Is mathematics an innate faculty or is it forged in cultural-historical context?

We will make use of the archeological evidence of Denise Schmandt-Besserat (1978) who showed how mathematical notation for numbers first emerged in Mesopotamia when clay tokens used as receipts for tributes given to priests were embedded in clay tablets to reveal the first examples of mathematical notation and thinking. The first mathematicians were accountants. Schmandt-Besserat (1978) also showed that the emergence of mathematical notation occurred simultaneously with the emergence of writing and that symbols representing numbers and words appear together on the same tablets. This story will reveal the connection between math cognition and the semiotic processes of symbolism and annotation and at the same time shows the interrelationships between mathematical discovery and cultural processes, including technological systems that guide the thrust of cognitive and social evolution. The close connection of mathematics and writing has implication for mathematics education, especially for those who learn arithmetic first.

The story of the origin of mathematical notation and writing will also allow us to conclude that mathematics, other than simple enumeration, is forged in the cultural-historical context and is not an innate faculty, given that mathematics thrives in cultures that possess written language.

The next batch of questions we will address and how we will address them are the following:

- Other thematic areas that have implications for the study of math and mind, including ideas from disciplines such as philosophy, linguistics, and so on.
- Is mathematics a unique type of human conceptual system, sustained by specific and localized neural structures, or does it share neural systems with other faculties such as language and drawing?
- What structures, if any, do mathematics and language share?

These next three items will be addressed when we show how the emergence of spoken language led to the kind of conceptualization that makes mathematical thinking possible. Here, we will make use of Logan's (2007) study *The Extended Mind: The Emergence of Language, the Human Mind and Culture* where it is shown that the emergence of spoken language allowed humans to create the conceptualizations necessary for mathematical thinking beginning with simple enumeration. Before humans had language, the brain was basically a percept processor that did not have access to concepts. Our first words were also our first concepts. The word water was a concept that united all of our percepts with the water we drink, cook with, wash with, falls as rain, and which we find in rivers, lakes, and oceans. Without language we cannot conceive of quantities, the basic atoms of mathematics. With verbal language, the hominid brain bifurcated into the human mind fully capable of the conceptualization that made mathematical thinking possible. This idea is represented by the mathematical formula: mind = brain + language.

Spoken language gave rise to written language which allowed more complex forms of mathematical thinking beyond enumeration to develop. This establishes that mathematics shares neural faculties such as language and graphic representation as mathematical notation arose out of the construction of three-dimensional clay tokens followed by them being embedded in clay tablets as explained by Denise Schmandt-Besserat (1978). Each token represented a different agricultural commodity. The tokens for the large and small measures of grain came to represent the numbers 10 and 1, respectively. To distinguish the visual signs representing the numbers 10 and 1 from the signs representing the words for the large and small measures of grain, the signs representing words were not created by pushing the clay tokens into the wet clay, but rather the shape that the tokens representing agricultural commodities were drawn using a stylus work on the wet clay. The signs representing the numbers 10 and 1 continued to be created by pushing the tokens into the wet clay. As a result, the first notational system for representing words and numerals emerged at the same time some 5000 years ago with a separate notation for words created by sketching with a stylus and numerals created by imprinting the tokens into the wet clay.

How This Book Came About as the Result of Many Collaborations

This book emerged from three sources:

1. A collaboration dating back to 2011 on spirals and cyclic processes initiated by Izabella Pruska-Oldenhof's observation of the existence of a spiral structure in the thinking of Marshall McLuhan and a number of other thinkers, an insight that she shared with Bob Logan.
2. A collaboration that was initiated by the University of Toronto undergraduate student Emma Findlay-White, during an independent study supervised by Bob

Logan. This study revealed that hyperlinking is not just a recent phenomenon that emerged with the Internet and hypertext but that it represents a way in which the human mind engages in and organizes its thoughts and communications dating back to the oral composition of epic poetry such as Homer and to the organization of the first printed form of the Talmud by Joshua Solomon Soncino in 1483 more than 500 years before the appearance of the World Wide Web. We also developed the idea that the very origin of spoken language can be thought of as a result of the hyperlinking of concepts with percepts and the origin of writing and mathematical notation with the hyperlinking of visual signs with elements of spoken language. Emma's research during her independent study was a significant inspiration for parts of this book and her collaborations and contributions are hereby acknowledged.

3. Some earlier studies of Robert K. Logan published in the journal *Semiotic* and three of his books:

   - *Semiotica* 125-1/3 (Logan 1999) "The Social, Economic and Educational Impacts of Notational Systems"
   - *Semiotica* 155-1 (Logan and Schumann 2005) "The Symbolosphere, Conceptualization, Language and Neo-Duality" co-authored with John Schumann
   - *Semiotica* 160 (Logan 2006) "Neo-dualism and the Bifurcation of the Symbolosphere into the Mediasphere and the Human Mind"
   - *The Alphabet Effect*: Logan (2004a)
   - *The Sixth Language: Learning a Living in the Internet Age* (Logan 2004b)
   - *The Extended Mind: The Emergence of Language, the Human Mind and Culture* (Logan 2007)

The material in this book draws upon a number of other previous collaborations including a collaboration of Logan (2004a) with Marshall McLuhan in which a link was found between phonetic writing and the alphabet in particular with codified law, monotheism, abstract science, deductive logic, and hence axiomatic geometry.

A second collaboration between Logan and Denise Schmandt-Besserat, who discovered the link between the written word and mathematical notation through the interpretations of her archeological finds in the Near East. This collaboration led to a better understanding of mathematics learning and the evolution of language from speech to writing and mathematics, to science, to computing, and to the Internet as described by Logan (2004b) in the book *The Sixth Language: Learning a Living in the Internet Age*.

This study also draws upon the ideas developed in *The Extended Mind: The Emergence of Language, the Human Mind and Culture* (Logan 2007) which developed as a result of Logan's interaction with two communities, one with linguists and the other with emergent dynamicists or complexity theorists and therefore draws upon the ideas and suggestions of the following scholars Morten Christiansen (1994) and Terrence Deacon (1997, 2012) from linguistics and emergent dynamics; and Ilya Prigogine (1980) and Stuart Kaufmann (1995) from complexity theory and emergent dynamics.

This book, like all the other books that have ever been written, is not complete but at some point, one has to report one's findings. We invite you, the reader to fill in the gaps, the things that we missed, because we believe that mathematics and specifically spirals and hyperlinking are a fundamental part of human cognition. We invite you, the reader, to fill in the gaps and if you would be so kind to share them with us at the following e-mail addresses i2pruska@ryerson.ca and logan@physics.utoronto.ca.

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Izabella Pruska-Oldenhof

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Acknowledgments

We want to thank Professor Marcel Danesi for the invitation to publish our results in this Springer series Mathematics in Mind.

Contents

Prologue to Part I
Mathematics in Mind

We will develop the hypothesis in Part I—Mathematics in Mind that the human mind is intrinsically verbal and mathematical and that language and mathematical thinking co-emerged at the dawn of the emergence of Homo sapiens. Language and mathematical thinking are the difference that separates human beings from the rest of living beings including our most recent ancestors who also belong to the genus Homo ranging from Homo habilis and Homo erectus to Homo Neaderthalensis and Homo Denisova (the jury is out as to whether denisovans are a sub-species of sapiens or a separate species).

In Chap. 1 we scope out the nature of mathematics in the human mind, which we show involves much more than just the formal study of mathematics in schools and the academic discipline of professional mathematicians. All human thought is at a certain level mathematical. We describe the different ways that mathematics enters human thought and language.

In Chap. 2 we will develop the hypothesis as an abduction that (1) verbal language, which permitted conceptualization, and (2) a primitive form of set theory or the ability to classify percepts or experiences, co-emerged creating the human mind and the fully human species Homo sapiens. In other we will argue that the ability to speak and mathematics in the mind in the form of set theory originated simultaneously. We also argue that language is not just the medium to communicate thought but that it is the medium in which thought emerges. The association of a word as a concept with a set of percepts related to that word/concept has a mathematical structure to it. Finally, we describe the emergence of grammar.

In Chap. 3 we study the impact of language both oral and written on the formulation of thought. We then examine the impact of the emergence of writing and mathematical notation on mathematical thinking. We will show that writing to express human thoughts and mathematical notation to represent quantities arose simultaneously. We trace the historic origin of the concept of zero and the following mathematical concepts that followed in its wake including the place number system, negative numbers, infinity, and algebra. In particular we examine the effect of phonetic writing systems including the alphabet and their effect on and/or relation to codified law, monotheism, deductive logic and abstract science. We describe the evolution of language from speech into writing, mathematics, science, computing and the Internet.

In Chap. 4 we study the impact of alphabetic writing on the ancients Greeks on their development of deductive logic and its application to geometry and abstract science. We describe the difference between deductive logic and abstract science and demonstrate that one cannot prove the truth of a proposition using the methods of science.

In Chap. 5 we review the evolutionary chain of the six languages of speech, writing, mathematics, science, computing and the Internet. We then show that the information overload created by science led to the emergence of computing and that the information overload created by computing led to the emergence of the Internet and the Web. We then describe the semantical and syntactical aspects of the fifth and sixth languages of computing and the Internet/Web. We conclude this chapter with a discussion of computer-based AI and show how great is the difference between human intelligence and artificial intelligence debunking the notion of the technological Singularity.

Chapter 1
The Origin of Mathematics in the Mind

Introduction

Before starting our study of the topology of the thought and communications that the human mind creates as part of the **Mathematics in Mind** series we asked ourselves the question: what is the origin of mathematics in mind? To answer this question, we must first ask, what is the origin of the mind and what is the origin of mathematics. We will start first with the question of what is mathematics in this chapter and then turn to the question of the origin of the mind in the next chapter. We will discover that one of the aspects of mathematics, namely set theory, played a significant role in the creation of verbal language and, subsequently, the human mind and propose that set theory and verbal language co-emerged. The human mind as we will show in the next chapter is a product of the brain and verbal language or put another way the verbal language extended the brain into the mind. But for now, let us explore what is the nature of mathematics.

What Is Mathematics?

Mathematics is many different things depending on who is defining it because it has so many different facets. Mathematics is concerned with quantity, number, enumeration, magnitude, order, arrangement, structure, form, shape, space, change, calculation, and measurement. There is in fact no simple definition of mathematics so rather than trying to define mathematics in a simple sentence we will merely describe the many forms that mathematics takes. The term mathematics etymologically speaking is derived from the ancient Greek term μάθημα (máthēma), which translates into English as knowledge, learning or study and hence describes a discipline. But long before the ancient Greeks began the formal study of mathematics, the Babylonian and the Egyptians also studied math as a discipline as evidenced by

R. K. Logan, I. Pruska-Oldenhof, *A Topology of Mind*, Mathematics in Mind,
https://doi.org/10.1007/978-3-030-96436-8_1

their respective texts Plimpton 322 (Freiberg 1981) and the Rhind Mathematical Papyrus (Neugebauer 1969) both circa 1900 BC. The appearance of mathematics, not as a discipline, but as a way of thinking or as a cognitive structure can be traced back to the very beginning of human existence and the simultaneous emergence of verbal language and the human mind as we will describe when we discuss the idea of the extended mind. Mathematical thinking is a universal characteristic of all human cultures as is evidenced by the fact that all languages have words for numbers and other ways of designating amounts and orderings. Mathematical thinking seems to have been a characteristic of humankind throughout our history. One piece of evidence for this is the archeological find of the 22,000-year-old Ishago Bone on which a numerical tally was etched (De Heinzelin de Braucourt 1962).

The level of mathematical activity varies from culture to culture. Hunting and gathering societies, for example, did not have much need for mathematical thinking and as a result the range of numerals in these societies is very simple consisting of the numbers one, two, and many (Boyer 1991, 3). Even though this is the case an experiment conducted by Brian Butterworth of University College London with Australian indigenous children revealed that they are able to distinguish numbers greater than two even though their number systems only designate 1, 2 and many. As a result of his study he remarked:

> Recently, an extreme form of linguistic determinism has been revived which claims that counting words are needed for children to develop concepts of numbers above three. That is, to possess the concept of 'five' you need a word for five. Evidence from children in numerate societies, but also from Amazonian adults whose language does not contain counting words, has been used to support this claim. However, our study of aboriginal children suggests that we have an innate system for recognizing and representing numerosities—the number of objects in a set—and that the lack of a number vocabulary should not prevent us from doing numerical tasks that do not require number words (https://www.ucl.ac.uk/news/news-articles/news-releases-archive/aboriginal).

This result should come as no surprise given that there are many animals capable of numeracy including lions, monkeys, certain birds, and chimpanzees. Cultures that required keeping track of quantities developed more sophisticated number systems. It seems as a result of Brian Butterworth's finding that numeracy is a built-in feature of humans.

In terms of cultures that have limited number words we found the following example from the Gumulgal people of Australia ingenious. They developed a number system base two with just two number words used repetitively. The word urapon represents 1 and the word ukasar represents 2, but they are able to represent numbers greater than two as follows: ukasar-urapon = 3; ukasar-ukasar = 4; ukasar-ukasar-urapon = 5 and ukasar-ukasar-ukasar = 6 and so on and so forth. Odd numbers end in urapon and even numbers in ukasar (https://numberwarrior.wordpress.com/2010/07/30/is-one-two-many-a-myth, accessed February 22, 2017).

Number words are only one indicator of mathematical thinking. Mathematics is concerned with the comparisons of quantity, amounts and number of items, on the one hand, and order or structure, on the other hand. Number words, whether cardinal or ordinal are a precise way of making these comparisons, but there are many other words that can also be used to make such comparisons and are evidence of mathematics in mind. Here is a list of such words in English that have their counterpart in many other languages.

Table of Words Related to Quantitative Analyses or Comparisons

Many, much, none, amount;
Some, more, most, almost;
Less, lesser, least;
Little, littler, littlest,
Tiny, tinier, tiniest;
Small, smaller, smallest;
Big, bigger, biggest;
Large, larger, largest;
Great, greater, greatest;
Far, farther, farthest, further, away;
Close, closer, closest, near;
Beside, between, behind, in front of, after, under, over, above;
Before (position and time);
After (position and time);
Long, longer, longest (position and time);
Short, shorter, shortest (short can refer to height but also not having enough);
Heavy, heavier, heaviest (weight but also an emotion);
Light, lighter, lightest (weight, tone and emotion);
Equal, unequal, approximately;
Extreme, slight, intense;
Now, later, latest, past, present, future, before (time), after (time);
Soon, sooner, soonest;
Dawn, sunrise, morning, noon, afternoon, evening, sunset, night, midnight;
Start, begin, during, end, finish;
When, where, how many, how much, how often;
Seldom, often, never;
Open, closed;

We have also compiled a separate list of words that describe mathematical operations that are also used outside the formal study of the discipline of mathematics that have meaning in addition to their formal mathematical definition. These etymologies reveal that these mathematical operations have their roots in the cognitive structures of the mind and that formal mathematics has its roots in the linguistic structures of the mind that emerged with verbal language, a topic we will return to in the next chapter.

Table of Mathematical Terms and Their Etymology That Also Have Significance Outside of Mathematical Studies

Add; from ad + dare ["to give, put"] (www.dictionary.com/browse/**adding**); to join or unite (something to something else), from Latin addere "**add** to, join, attach, place upon" (www.dictionary.com/browse/**add**s).

Take away or subtract; 1530s, "withdraw, withhold, take away, deduct," a back-formation from **subtraction** (q.v.), or else from Latin subtractus, past participle of subtrahere "take away, draw off." Related: **Subtracted**; **subtracting**. Mathematical calculation sense is from 1550s (www.etymonline.com/index.php?term=**subtract**).

Multiply; From multus ["much, many"] + plicō ["fold, double up"] (math.stackexchange.com/questions/1150438/the-word-times-for-**multiplication**); Online **Etymology** Dictionary. mid-12c., multeplier, "to cause to become many," from Old French multiplier, mouteplier (12c.) "increase, get bigger; flourish; breed; extend, enrich," from Latin multiplicare "to increase," from multiplex (genitive multiplicis) "having many folds, many times as great in number" (www.etymonline.com/index.php?term=**multiply**).

Divide; Online **Etymology** Dictionary. early 14c., from Latin dividere "to force apart, cleave, distribute," from dis- "apart" (see dis-) + -videre "to separate," from PIE root *weidh- "to separate" (www.etymonline.com/index.php?term=**divide**).

Equal; late 14c., "identical in amount, extent, or portion;" early 15c., "even or smooth of surface," from Latin aequalis "uniform, identical, **equal**," from aequus "level, even, flat; as tall as, on a level with; friendly, kind, just, fair, equitable, impartial; proportionate; calm, tranquil," which is of unknown origin (www.etymonline.com/index.php?term=**equal**).

These etymologies reveal that these mathematical operations have their roots in the cognitive structures of the mind and that formal mathematics has its roots in the linguistic structures of the mind that emerged with verbal language, a topic we will return to in the next chapter.

Although mathematics is a discipline that is studied for its own sake mathematical thinking is an integral part of human cognitive structures and many forms of human activity. Enumeration or counting and the subsequent practice of arithmetic including adding, subtracting, multiplying and division are an essential part of commerce. Another form of mathematical thinking entails the measurement of amounts by volume or by weight; distances between points and land areas, all of which are essential for the organization of agriculture and the commerce associated with it, which resulted in standard measures of volumes, weights and distances and in the case of land areas to geometry.

Organization and Classification as a Form of Mathematical Thinking

Mathematics is primarily about quantitative analysis but it is also about qualitative analysis and organization as well. The organizing of objects and/or concepts in groups, sets, classes, classifications, categories, taxonomies, or schemas is an intrinsically mathematical operation. In a certain sense qualitative analysis comes before quantitative analysis because in order to enumerate or count things one has to first determine if the objects being counted or enumerated belong to the same class or group. Before one can say whether there are five dogs one must first determine that the objects being counted are in fact dogs. So, classification must precede enumeration. We therefore conclude that the ability to group objects in categories or classes is basically a mathematical skill. We will make use of this notion when we come to understand how the emergence of verbal language by the first Homo sapiens is connected to the mathematics in mind of connecting a group or set of percepts or experiences to a single concept represented by a word. Here we are using the terms group and set not as they are defined in group theory and set theory respectively but rather as a collection of percepts that are similar. We are suggesting that a primitive form of set theory and verbal language co-emerged.

Mathematics: A Practical Tool and a Theoretical Discipline

The etymology of the word mathematics from the Greek word for study or knowledge reveals that mathematics is considered as both a discipline of study for its own sake for scholars as well as a subject to be learned in school for practical reasons as mathematics is an integral part of many of the practical aspects of human life including the day to day organization of daily life as well as the following human activities that are unique to our species, namely, agriculture, trade, commerce, accounting, statistics, probability, insurance, business, management, finance, construction, engineering, science, medicine, computing, government, politics, and music. This list is by no means complete but it does indicate the diversity of activities in which mathematics plays a role.

The Different Divisions and Sub-disciplines of Mathematics

The field of mathematics includes many different divisions and subdivisions, which we will enumerate in roughly the order in which they emerged historically. The most basic form of mathematics is simply counting or enumeration. Next comes the simple arithmetic of adding, subtracting, multiplying and dividing whole numbers, fractions and decimals (actually the use of decimals are only 500 years old whereas fractions date back almost 4000 years to Egypt and Babylon). There then follows geometry in both two and three dimensions followed closely by syllogistic logic. The next development is that of the concept of zero, negative numbers and algebra developed in India approximately 2000 years ago and transmitted to Europe in the fourteenth century by Arab mathematicians; followed by the notion of square roots, quadratic equations and raising numbers to exponential values. The Renaissance also saw establishment of double entry bookkeeping. Next came analytic geometry which gave rise to calculus in the seventeenth century as well as the introduction of logarithms and probability theory. The eighteenth century ushered in the use of imaginary numbers, graph theory, topology, combinatorics, statistics and number theory. The nineteenth century saw the development of functions of complex variables, non-Euclidean geometry, vector spaces, noncommutative algebra, Boolean algebra, group theory, set theory, symmetry studies, and mathematical logic. The twentieth century saw the development of differential geometry, game theory, category theory, ergodic theory, knot theory, catastrophe theory, fractals, emergent dynamics, complexity theory, Lie theory, Turing machines, information theory, and the incompleteness theorem.