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Hamad M. Yehia

Rigid Body Dynamics

A Lagrangian Approach

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Hamad M. Yehia (Deceased)
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura, Egypt

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This book is dedicated to my family:
Nagwa, Hani and Magid
and to the memory of
my teacher and friend Vladimir G. Dumin

Introduction

Rigid body dynamics is one of the oldest and most challenging subjects in classical mechanics. It was initiated by Leonhard Euler, who formulated the equations of motion for a general (asymmetric) torque-free body and obtained the first integrable, in quadratures, case known after his name. With the efforts of D'Alembert, Poinsot, Lagrange and Poisson, the equations of motion of a body about a fixed point under the action of forces were written in their present form known as the Euler–Poisson equations: a system of six first-order differential equations for which three integrals are known. Lagrange found the second integrable case, the case of a heavy axi-symmetric body, usually named as Lagrange's top. In both cases of Euler and Lagrange, an integral of motion followed from general principles of mechanics, constancy of the angular momentum in the first and due to the cyclic angle of rotation about the axis of symmetry in the second. An important moment was that in both cases, the equations of motion were solved to the end and the solution expressed through elliptic functions, invented by Jacobi, and certain integrals involving them.

The search for integrable cases continued, but, although the problem attracted the attention of several eminent mathematicians, the search did not lead to any other cases. A whole century later, Sofia Kowalevski found a new integrable case of the heavy rigid body. That was not in virtue of a physical conservation principle, but using a purely mathematical condition: all solutions of the equations of motion should have only poles as their singularities as functions of time in the complex t-plane. This property is satisfied by the solutions in the two known integrable cases of Euler and Lagrange, being expressible in terms of elliptic functions of time. Having isolated three cases of this type, two cases of Euler and Lagrange and a new third one, Kowalevski tried and found the complementary integral in the third case. That integral turned out to be the first instance ever of a polynomial integral of degree four in the dynamical variables in a dynamical problem. Kowalevski also reduced the problem to quadratures and expressed all the dynamical variables in terms of hyper-elliptic functions of time, which are far more complicated than elliptic functions, but share with them the property of having only poles as singular points in the complex plane.

From this point on, instead of searching for new integrable cases, researchers directed their efforts to find methods to prove non-integrability, non-existence of a fourth integral of motion, in addition to the elementary three general ones, which is algebraic (Liouville, Husson and Burgatti) or even single-valued (Poincaré). This trend reached its perfection after the KAM (Kolmogorov–Arnold–Mozer) theory for integrable Hamiltonian systems was established. In the last few decades, new concepts were applied to rigid body dynamics in the works of several authors, among which those of Kozlov and Ziglin play a distinguished role.

The last quarter of the nineteenth century was a golden era for rigid body dynamics. More complicated problems including certain generalizations of the classical problem were investigated. The integrable case of the motion of a body carrying a symmetric rotor (the gyrostat) was found by Joukovsky and Volterra. Brun found integrals of motion for the motion of a body acted upon by asymmetric Newtonian force of attraction. The equations of motion of a body in a liquid were constructed by Kirchhoff and integrable cases of that problem are associated with the names of Clebsch, Steklov and Lyapunov. The same period is also characterized by the appearance of several cases of exact particular solutions of the classical problem, which were obtained by Staude, Hess, Goryachev, Chaplygin, Kowalewsky, Bobylev and Steklov. Those cases constitute more than half of the exact solutions of the classical problem, known to us to date.

The next half-century or so has elapsed without significant advancements in problems of rigid body dynamics as concerns integrable cases. The second half of the twentieth century, on the other hand, witnessed a renaissance of the subject. Interest has grown in integrable problems in general and, in particular, in those of rigid body dynamics. Several important results were obtained, including new exact solutions of the classical problem and the problem of motion of a heavy gyrostat. New problems emerged and underwent intensive investigation. One of them was that of motion of a body acted upon by more general conservative potential and gyroscopic forces. The last four chapters of this book present mostly innovations brought into the subject in the past few decades, in which the author had some substantial contributions.

The idea of writing this book emerged more than a decade ago. It was delayed so long due to the social and political upheaval that arose in Egypt at the time and had a direct impact on every aspect of life. The original motivation was two-fold, **first and foremost**, there was a need for a new survey on the subject of rigid body dynamics. The core of such a survey should be classification and a complete up-to-date account of all the known but scattered in the literature integrable cases and particular solutions of the diverse problems treated within this subject.

Only in integrable cases can one study the motion in the whole phase space and draw conclusions about the behaviour of the mechanical system over an infinite interval of time. It will be evident as we go through the book that integrable cases are a rarity in problems of rigid body dynamics, and in some problems, there are even proofs that no more integrable cases are there to be found in the future. This situation ensures the high importance of every integrable case and justifies that each one is recorded under the name of its discoverer. Also of great importance are particular exact solutions, obtained only under certain conditions on the initial state of motion of the body. Those constitute the only window through which one can visualize or study a little fragment of the solution in non-integrable problems over an infinite time interval.

Direct methods already used in the construction of new integrable problems since the late nineteenth century have been exhausted and are no longer capable of identifying integrable cases in more complicated problems. Completely new ways of thinking were required to establish new methods and to make substantial advancements in the subject. One of those ways was writing equations of motion in the form of the Lax pairs and using advanced tools of algebraic geometry. In this way, many integrable systems were constructed in the dynamics of particles and some in rigid body dynamics. Later, those methods helped to construct the formal exact solution of some problems. An example is the integrable case of motion of a body acted upon by two skew uniform fields, which generalizes Kowalevski's case of a single field due to Reyman and Semenov-Tian-Shansky. Two integrable versions of this problem were known when the problem admits a linear integral, but using the above method made it possible to obtain the case, with a quadratic integral, unifying those versions. The solution of the complexified equations was also pointed out in terms of Theta functions.

Another method of finding integrable problems in rigid body dynamics was an inverse one. The author of the present book developed a method for the construction of integrable generalized natural systems of two degrees of freedom, which admit integrals of motion in the form of polynomial in the velocities with coefficients depending on the position. This method led to the appearance of vast families of such systems, living on two-dimensional Riemannian manifolds. Designating special values for the parameters, it was possible to construct integrable cases on some known manifolds. In this way, a comparatively large collection of general and conditional integrable problems in rigid body dynamics was constructed over the past decades. Examples are the case of a gyrostat, which generalizes the classical case of Kowalevski, and a large set of conditional cases. Every one of the integrable cases poses new mathematical challenges: to investigate qualitative properties using integrals of motion, to achieve separation of variables and study topological properties of integral manifolds in the phase space. Another possible task is the construction of explicit time solution of the equations of motion, usually by inverting quadratures in case of separation of variables or by using Lax pair representation of a certain type.

Moreover, the application of certain transformations to known integrable cases with cyclic coordinates has led to the construction of much more general integrable cases. This produced new general integrable cases depending on several extra parameters and added physical effects to all the integrable cases known earlier.

Thus, the state of the subject has radically changed since the time of the well-known monographs of Leimanis and Magnus. In 2005, Borisov and Mamaev published their book "Rigid Body Dynamics" and an English translation appeared in 2017. This marvellous book lays emphasis on mathematical structures and generalizations of integrable cases to higher dimensions. Several important topics are not covered in it. Moreover, reading this book requires professional mathematical knowledge.

The second motivation for writing the book was to showcase some novel methods developed by the author that have led to substantial results and completely new collection of integrable cases in rigid body dynamics. Although almost all these cases have definite physical interpretation, they have not been given due attention. In the mean time, as we essentially use the Lagrangian approach, against a currently prevailing trend that the Hamiltonian (or Poisson bracket) approach is the one that should be followed in all problems of mechanics whenever it is applicable. We had to explain why we intentionally use the Lagrangian formalism to present some features of mechanical systems that are vividly seen in that formalism, while disguised in the Hamiltonian formalism. Our standpoint is that the use of this or that approach is not an arguable issue. None of them can be said to be absolutely better than the other. Each approach must be applied, with no prejudice, to problems for which it is most suited. The problem of classifying and tabulating integrable cases of motion of a rigid body subject to potential and gyroscopic forces, which occupies most of the book, is an example. Those cases are time-irreversible, i.e. equations of motion are not invariant under the change of sign of the time variable. As will be established later in the book, every such case is completely determined by two (scalar and vector) functions V and μ. The scalar is the potential and the vector uniquely determines the moment exerted by gyroscopic forces. The pair (V, μ) uniquely characterizes the physics of the problem and its equations of motion and thus can be used as a basis for the classification and tabulation of integrable cases. On the other hand, in inverse methods used to construct integrable systems, gauge terms that arise as a part of the solution of partial and ordinary differential equations enter in the definition of momenta. The Hamiltonian function and the Hamiltonian equations of motion depend on those terms and obscure the necessary terms that determine potential and gyroscopic forces acting on the body together with the physics of the problem. Hamiltonian equations of motion of an irreversible mechanical system can be written in an infinite number of equivalent forms. In fact, to determine whether two Hamiltonians are equivalent, one has to do some steps that are equivalent to finding the equations of motion in the Lagrangian form. Concrete examples are given in the last chapters of the book, beginning with Chap. 10. Nevertheless, in Chaps. 10 and 12, after classification of integrable cases on the Lagrangian basis, to conform with the reference character of the book, we also give Hamiltonians and complementary integrals in terms of momenta for all integrable cases. These cases involve gyroscopic moments, which depend on the position in a complicated manner. For such systems, the Lagrangian approach is not just an awkward presentation, but it faithfully and uniquely presents the physics of the problem under consideration. On the other hand, a system of Hamiltonian equations of motion can represent a whole class of physically different mechanical systems on that level of complication. Nevertheless, the whole integrability theory is most easily and clearly in terms of the Hamiltonian approach.

In the plan of this book, it soon became clear that the original motivation to include all new changes in the field of rigid body dynamics to produce something similar to Routh's tractate of the late nineteenth century is too ambitious and rather impractical. The changes in the subject in the preceding half-century are far more extensive, to be included in one volume. A narrower line had to be set as an aim for the book. We have, thus, made the decision to make a survey of known integrable cases in various problems in the dynamics of a rigid body moving about a fixed point under given forces, including certain problems of motion of a body with no fixed point, but which reduce, after some transformation or reduction, to the first type of problems (with a fixed point). Examples of such problems are the problem of motion of a body in a liquid and that of motion of a satellite in a circular orbit about a spherical planet.

Thus, no place was left for some important problems. The problem of motion of a rigid body subject to a non-holonomic constraint, rolling on a plane or a surface, is an example. Other examples are the motion of a gyroscope in gimbals, problems involving motion of a system of connected bodies in a general state of motion and the motion of a body with a cavity, completely filled with a liquid in a state of vortex flow. The last problem is described by Poincaré–Joukovsky's equations of motion. We have also excluded a large set of existing solutions, which are not in finite exact form, like asymptotic solutions and series solutions or perturbations of exact solutions in power series of a small parameter. On the other hand, problems of rigid body dynamics in which integrability is not a principal issue are not considered. An example is the controlled rotational motion of the rigid body.

Even in the main course of the problem considered in the book, we had to make some definite selections of the material to be included. In the first place, we intended to make a complete up-to-date account of all known integrable cases in the subject. A considerable part of such content is scattered in the literature and would be presented in the form of a book for the first time. The information about integrable cases should contain conditions for their existence, full historical context of their discovery or development from former cases and sufficiently detailed forms of the first integrals in each case. This covers all general integrable cases, i.e. cases integrable in the whole phase space (for arbitrary initial conditions) as well as conditional integrable cases, i.e. integrable on a fixed level of the relevant linear integral of the motion (the areas integral). In most elementary cases, we tried to illuminate as much as possible the process of obtaining the explicit solution of the equations of motion. By this, we mean the expression of all the physical phase variables in terms of time. As will be seen in most of the integrable problems considered in this book, the separation of variables, inverting quadratures and constructing explicit solutions have turned into a separate art and in their majority still represent open mathematical challenges. Even in solved cases, a frequently met drawback is that some explicit solutions are expressed using complex functions of time, a situation that obstructs their use in numerical calculations or simulation. For this reason, greater importance is devoted to the construction of some particular solutions expressible in terms of elliptic, trigonometric or simpler functions of time. Apart from the integrable cases, complete, and somewhat detailed, account of all the twelve known exact particular solutions of the classical problem of motion of a heavy rigid body is given. But this could not be pursued in other higher problems of the hierarchy. That could simply double the size of the book and also the time to compile the existing information.

The book is divided into two parts:

The first part, the elementary part, grew mainly as a course on rigid body dynamics delivered over years to undergraduate mathematics students of the Faculty of Science and it can be used for this purpose. This part includes Chaps. 1–7, covering the material necessary for a mathematics or physics student to get acquainted with the subject of rigid body dynamics, its main problems, techniques and historical development. A few sections of Chap. 8 can be selected to augment that content with some examples of a particular solution in rigid body dynamics.

We begin in Chap. 1 with a study of the characteristics of mass distributions: the centre of mass and the inertia matrix. Even in this classical material, some innovative element was introduced. A new theorem is given, determining natural bounds on the location of the centre of mass of a body with given moments of inertia. Chapter 2 is devoted to different ways of the description of finite rotations, infinitesimal rotations and the angular velocity vector. Introduced here are Euler's angles, the rotation matrix and quaternions for describing the orientation of the body.

Chapter 3 includes a brief study of the classical problem of motion of a rigid body about a fixed point under the action of its own weight. Different forms of the equations of motion and their integrals are derived in different reference frames fixed in the body and moving with it. Equations of motion are obtained in Lagrangian, Routhian and Hamiltonian forms.

In Chap. 4, the three general integrable cases of the classical problem known after the names of Euler, Lagrange and Kowalevski are presented in some detail. Explicit time solution of the equations of motion is given in terms of elliptic functions of time for Euler's case of a torque-free body. The solution of Lagrange's case is reduced to an elliptic quadrature, which may be used to express it in elliptic functions as well. However, we relied, following Poisson, on the use of integrals of motion to establish certain qualitative aspects of the motion, without referring to explicit time solution. Kowalevski's case is formally reduced to hyper-elliptic quadratures. Some degenerate cases are solved in elliptic or simpler functions in Appendix B. The conditional case of integrability bearing the names of Goryachev and Chaplygin is also presented with its separation of variables belonging to Chaplygin. Degenerations of hyper-elliptic quadratures are presented in some detail in Appendix C.

Chapter 5 is devoted to the study of the problem of motion of a heavy gyrostat. In its simplest form, the gyrostat is a rigid body in which a symmetric rotor is placed with its axis fixed in the carrier body by cylindrical smooth joint(s) and given a constant angular speed with respect to the main body. The gyrostatic effects are in wide use in several problems of science and technology. Equations of motion were formulated in the last decades of the nineteenth century. An integrable case is readily recognized, which is a trivial generalization of Lagrange's case, when the main body is axially symmetric and the rotor is aligned along its axis of symmetry. The second case generalizes Euler's case in the classical problem by adding a rotor in an arbitrary direction fixed in the body. This case was found by Joukovsky and shortly later by Volterra. The third general integrable case, Yehia's case, was found as a generalization of Kowalevski's case in the classical problem by adding a gyrostatic momentum. The conditional case of Goryachev and Chaplygin that was generalized to the gyrostat by Sretensky is also presented. The chapter concludes with some applications of the gyrostat dynamics to stabilize certain motions.

In Chap. 6, the problem of motion of a gyrostat about a fixed point while acted upon by the force of a Newtonian centre of attraction is presented. Especially interesting is the case when the attraction centre is far from the fixed point. This case is treated in some detail, for it has several applications in certain problems in astronomy and physics. Two integrable cases are known. They generalize Euler's and Lagrange's cases. In this chapter, we also present what is called Brun's problem, which is equivalent to a special version of the former problem. A quite interesting property is proved that Brun's potential is the only one that admits an integral of motion, quadratic in the velocities.

Chapter 7 contains a brief account of the problem of motion of a body having no fixed point. Equations of motion will be useful in certain topics later to be exposed in this book. A quite interesting example, the motion of a top on a smooth plane (called Poisson's top), is considered at least for its educational importance.

The second part of the book contains mostly new research material that was not compiled before in book form.

The original course included a few examples of particular solutions of the equations of motion. In the final plan of the book, I found it necessary to make separate Chap. 8 collecting the basic information and results about all the twelve known exact particular solutions of the equations of motion in the classical problem. This information, which accumulated in the period from the 1890s to 1970s, to the date the last case was found, has never been presented in a source in the English language. Just a few cases are pointed out in Leimanis' book [256], not all of them are correct. The same situation applies to Magnus' monograph [270]. Borisov and Mamaev mention only half of these cases, with somewhat detailed analysis of the earlier results of Staude, Hess, Bobylev, Steklov and Grioli. In our presentation, some new features were added. In most cases of a particular solution, we show the curve drawn during the motion by the apex of the vertical unit vector γ on the unit sphere fixed in the body. This graph gives a full idea on how the body moves relative to the vertical through the fixed point, i.e. up to a rotation about it.

Exact particular solutions of the gyrostat problem are considered only in one brief section of Chap. 8. Those solutions were intensively studied almost exclusively by the school of Mechanics in Donetsk. All those cases are listed with the essential information on each case. Those are nine cases generalizing their counterparts of the classical problem and four new cases with no classical analogs. For each case, we provided necessary information and references that would help the interested reader to track every case in original works.

In Chap. 9, we consider the analogy between the motion of a rigid body about a fixed point and the problem of motion of a particle on a smooth ellipsoid. This analogy, noted first by Minkowski, furnishes several easy ways for the reduction of the order of equations of motion, using the known integrals of motion. The climax in this direction is the maximal reduction to a single differential equation of the second order named as the "orbital equation". This equation settles once for all the question of maximal reduction of order raised in rigid body dynamics the since 1890s and discussed by a long list of authors.

Chapter 10 is devoted to the problem of motion of a body by inertia in an ideal incompressible fluid extending to infinity in all directions. This problem originally belongs to the field of hydrodynamics, which is described by boundary-value problems on partial differential equations. Nevertheless, the efforts of several authors led to the result that the pressure of the fluid on the body can be completely avoided, leaving us with a mechanical system of six degrees of freedom. Equations of motion of that system were given by Kirchhoff and Clebsch for a simply connected body and by Lamb for a multi-connected (perforated) body.

Also in this chapter, a new form of the equations of motion and a transformation have changed the way of presentation so deeply, gave a new insight into the problem and revealed its inherent relation to other physical problems that were treated before as completely separate from each other. An analogy has been established between this problem and a special form of the problem of motion about a fixed point of a heavy and magnetized body, which carries immovable in it electric charges under the action of an axi-symmetric combination of gravitational, electric, magnetic and Lorentz forces. This opened the way to study systematically, for the first time, the motion of a heavy, magnetized and electrically charged body or gyrostat. The full list of known integrable cases, seven general and two conditional, valid for the two equivalent problems, is given in a unified form. For each case, we give relevant historical information and essential contributions to its study. For completeness, we also provide the Hamiltonian and the complementary integral for every integrable case in the tables, beginning from Chap. 10, where gyroscopic forces will play a more prominent role and Hamiltonians found by inverse methods are usually obscured by gauge terms.

The analogy just described above of the problem of motion of a body in a liquid and the alternative problem has placed the last problem on the top of a hierarchy of the problems considered in all previous chapters and paved the way to create a higher and richer level of that hierarchy that was never treated before. It may have been considered as hopelessly complicated to yield significant results.

In Chap. 11, the use of the Lagrangian approach and certain peculiarities of the equations of motion has pushed the whole subject beyond its common limits. Equations of motion are given in their historical context. They formally generalize the new alternative form of equations of motion of a body in a liquid and go much further from the physical point of view. Transformations are given, which generate new integrable cases of the most complicated nature from the ones in the lower hierarchies, by adding more parameters into their structures.

In Chap. 12, unprecedented and quite complicated integrable cases involving large numbers of parameters were constructed in an exotic, but effortless, way. In fact, we have used certain tricky properties of the Lagrangian formalism to add extra parameters of physical significance to the structures of the known general integrable cases of a body in a liquid. The number of those additional parameters depends on the structure of the potential part of the Lagrangian of the integrable problem. The new cases are the only known examples in our days of integrable cases of motion of a rigid body acted upon by potential and gyroscopic forces of the most complicated structure. The variety of those cases may shed some light on some of the most intractable problems of mechanics concerning the motion of natural and artificial bodies in extreme conditions when the fields applied to the body have comparable effects and none of them can be treated as negligible.

In this chapter, we also introduce a new type of generalization of each general case of integrability of motion of a body in a liquid a conditional integrable case involving an arbitrary function. This type of generalization is valid on a fixed level of the cyclic (areas) integral.

It was argued by some authors that those two types of generalization are trivial and lose their value if the problem under consideration is written in Hamiltonian formalism. We use the Hamiltonian formalism to show that this view of the subject is not factual.

In Chap. 13, we give a full list of the known up-to-date conditional integrable cases of the problem of motion of a rigid body. The tables for those cases are given here in their last and most general form, with no regard to their physical interpretation. Cases in those tables are ordered according to the degree of the complementary integral as a function of the components of the angular velocity of the body. Some of them acquire a physical meaning only for certain values of the parameters present in them. Other ones do not seem presently to have physical meaning at all, mostly because their potentials involve singular terms of certain types, not usually attainable by natural fields. The 22 cases known at present of this class were obtained mainly in the works of the author and some with his coworkers. Most of those cases have resulted as special cases of certain generalized natural multi-parameter integrable systems that were constructed by the author over the last few decades. Some of those cases have even stimulated research to find new ways for the separation of variables and other mathematical topics.

Chapter 14 is devoted to a systematic presentation of the present status of the problem of motion of a rigid body about a fixed point under the action of an asymmetric combination of potential and gyroscopic forces (crossed fields). Equations of motion are derived in the Euler–Poisson variables. Known integrable cases are collected and classified. First presented are integrable cases of a body acted upon by two and three skew uniform fields, then cases with a potential that is quadratic in the direction cosines. Apart from some special cases, in both types of problems, the mechanical system has strictly three degrees of freedom, i.e. does not admit a cyclic integral. In the last two sections of this chapter, we present two classes of problems admitting a cyclic coordinate. The problem of motion of a (physically) axisymmetric body under the action of asymmetric forces admits the Eulerian proper rotation angle as a cyclic coordinate. The symmetry leading to a cyclic integral in the second problem is not about a fixed axis, neither in space nor in the body. It may be interpreted as axial symmetry in the quaternion space. The cyclic coordinate is the sum (or difference) of the two Eulerian angles of precession and proper rotation. For the last two classes, the method described in Chap. 11 and applied in Chap. 12 gives some exotic generalizations of the well-known cases.

Exercises constitute an essential component of the book. A large number of them provide some supplemental information to the main text or introduce in a brief way additional topics of special interest that could not be presented in more detail.

The elementary part of the present book should be easily readable by anyone who has completed courses on calculus, differential equations and analytical dynamics. Some parts require some knowledge of elliptic integrals and Jacobi's elliptic functions. The advanced part provides mostly new material but it is written in the most elementary way. It will be tractable for readers of different mathematical backgrounds: students, Mathematicians, Physicists and Engineers. I hope that it will stimulate research in the field. Many of the new cases may be investigated for explicit solution either by separation of variables or by the use of Lax pairs. Topological classification and qualitative properties of motion can be studied for every case, for which separation of variables is achieved. Many cases are waiting for appropriate concrete physical interpretation.

This book is intended to be a reference book for integrable cases and exact solutions. I have taken possible care of checking the large number of formulas involved, mostly by using computer packages of symbolic computation.

Throughout the book, I used the usual notations for mathematical terms and operations. Vectors and matrices are denoted by bold symbols, and scalar and vector products by dot and cross, respectively. We have found it much easier and more consistent to use for multiplication of a vector $\mathbf{v}$ by a matrix $\mathbf{M}$ the usual matrix form $\mathbf{vM}$, instead of the mostly used operator form $\mathbf{Mv}$. The first produces vectors in the usual row form and brings some advantage in avoiding the need in many sources to switch between row and column forms of vectors.

During work on this book, I enjoyed generous help from many friends, to whom I express my sincere gratitude. A conversation with David Gao at a conference in Poland was encouraging and inspiring. Michael and Irina Kharlamov and Pavel Ryabov provided me with some old papers in Russian. Gennady Gorr secured for me issues of MTT and longtime chats discussing many details of the subject. Our long-years friendship was in no way affected by our differences on some of the content of books coauthored by him. Alexey Borisov made some publications of RCD available to me, in addition to his books coauthored by Ivan Mamaev, including their marvellous book on rigid body dynamics (2005) and its recent English translation (2017). On the other hand, Ahmed Ghaleb has read extensive parts of the manuscript and made many suggestions to improve the English text. Adel Elmandouh and Ashraf Hussein helped me by checking some mathematical calculations and resolving some issues concerning LaTeX editing of the manuscript. Hani Yehia and Ashraf Hussein helped me to improve the quality of some graphics.

Mansoura, Egypt Hamad Yehia
March 2021 hyehia@mans.edu.eg

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Part I
The Elementary Part

Chapter 1
Distribution of Mass

In the study of the dynamics of a rigid body, we deal in a natural way with certain quantities which are determined by the distribution of mass in that body. In this chapter, we introduce those quantities and study their properties and the relations between them.

1.1 The Moment of Mass—The Centre of Mass

A rigid body is defined as a finite distribution of mass in which the relative positions of all its mass elements do not change with time, regardless of the position of the body in space and the external forces exerted on it. We do not assume any conditions on the structure or shape of the body, which may be composed of invariable rigidly connected parts that can comprise in any way discrete point masses and continuous line, surface or volume distributions of mass. The term "element of mass" we use below should be interpreted in each case accordingly.

1.1.1 Moments of a Mass Distribution

The moment of mass of a given rigid body is a vector defined by the integral

$$\boldsymbol{\sigma} = \int \mathbf{r} dm, \tag{1.1}$$

where dm is an infinitesimal mass element, $\mathbf{r}$ is the position vector of that element and the integral is taken over all mass elements of the body. In a given system of axes $Oxyz$, the components are

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$$(\int xdm, \int ydm, \int zdm), \tag{1.2}$$

Those are the first moments of the mass distribution with respect to the given coordinate system.

Generally, moments of arbitrary degree n for a mass distribution are also defined

$$\boldsymbol{\sigma}(n_1, n_2, n_3) = \int x^{n_1} y^{n_2} z^{n_3} dm, \tag{1.3}$$

where n_1, n_2, n_3 are non-negative integers and $n = n_1 + n_2 + n_3$. In dynamics of rigid bodies, zeroth-, first- and second-degree moments ($n = 0, 1, 2$) appear naturally in the equations of motion. Higher moments are also met when the potential of a rigid body is calculated in certain models of gravitational potential in the field of attraction of other bodies. The simplest case is that of approximating the potential of the body in the Newtonian field of a far centre of attraction. We shall return to this point later with more detail.

1.1.2 Centre of Mass

Let M be the total mass of the body

$$M = \int dm. \tag{1.4}$$

Obviously, M is positive and finite. The vector

$$\mathbf{r}_0 = \boldsymbol{\sigma}/M = (\frac{1}{M}\int xdm, \frac{1}{M}\int ydm, \frac{1}{M}\int zdm) \tag{1.5}$$

defines a unique point in the body, called the centre of mass. This point has the fundamental property that the resultant of forces exerted on the body by an arbitrary uniform gravity field always passes through it. However, this property can be lost for any non-uniform gravitational field.

1.2 Second Moments and Inertia Matrix of a Mass Distribution

In the course of our study of rigid body dynamics, we deal with two related matrices (in fact, tensors):

1.2.1 Second Moments Matrix of Mass Distribution

Define the symmetric matrix $\bar{\mathbf{I}} = (\bar{I}_{i,j})_{i,j=1}^{3}$

$$\bar{I}_{i,j} = \int r_i r_j dm, \qquad i, j = 1...3, \tag{1.6}$$

where r_i stands for the i-th component of the position vector $\mathbf{r} = (x, y, z)$ of the mass element dm of the body, i.e.

$$\bar{\mathbf{I}} = \begin{pmatrix} \int x^2 dm & \int xydm & \int xzdm \\ \int xydm & \int y^2 dm & \int yzdm \\ \int xzdm & \int yzdm & \int z^2 dm \end{pmatrix}. \tag{1.7}$$

Diagonal elements $\int x^2 dm$, $\int y^2 dm$ and $\int z^2 dm$ are called moments of inertia of the body with respect to the planes yz, zx and xy, respectively.

1.2.2 Inertia Matrix of Mass Distribution

In most dynamical considerations, we more frequently meet the inertia matrix defined as

$$\mathbf{I} = tr(\bar{\mathbf{I}})\delta - \bar{\mathbf{I}}, \tag{1.8}$$

where δ is the unit matrix. This makes

$$\mathbf{I} = (I_{i,j})_{i,j=1}^{3} = \begin{pmatrix} \int (y^2 + z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (z^2 + x^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int (x^2 + y^2)dm \end{pmatrix}. \tag{1.9}$$

The diagonal elements of the inertia matrix I_{11}, I_{22} and I_{33} are called moments of inertia of the mass distribution with respect to the axes x, y, z, respectively, while the off-diagonal ones are termed the products of inertia with respect to the three coordinate planes. As the mass element is always positive, the moments of inertia are non-negative. Moreover, a moment of inertia of a body about an axis vanishes only if the body mass is distributed (continuously or discretely) on that axis.

Note that

$$tr(\mathbf{I}) = 2tr(\bar{\mathbf{I}}), \tag{1.10}$$

so that the inverse of the relation (1.8) can be written as

$$\bar{\mathbf{I}} = \frac{1}{2} tr(\mathbf{I})\delta - \mathbf{I}. \tag{1.11}$$

1.3 Properties of the Inertia Matrix

1.3.1 The Triangle Inequalities

Although the moments of inertia are always positive, not every three positive quantities can represent moments of inertia of some body about three perpendicular axes. In fact, moments of inertia of a body satisfy the inequalities

$$I_{11} + I_{22} - I_{33} \geqslant 0, I_{22} + I_{33} - I_{11} \geqslant 0, I_{11} + I_{33} - I_{22} \geqslant 0. \tag{1.12}$$

The equality holds only for plane mass distributions. Those inequalities suggest that the three moments of inertia about three perpendicular axes can be represented by lengths of three sides of a triangle. We shall return to this point with more detail later in this chapter.

To prove those inequalities, we notice from (1.7) that the diagonal elements of $\bar{\mathbf{I}}$ are non-negative, for being quadratic moments of mass with respect to the coordinate planes. The three inequalities follow from the relation (1.11). Equality holds only for bodies whose mass is distributed in one of the coordinate planes.

1.3.2 Theorem of Parallel Axes

Let $\mathbf{I}$ be the inertia matrix of a given body of total mass M with respect to some Cartesian frame $Oxyz$ with origin O at the centre of mass of the body. We shall calculate the inertia matrix $\mathbf{I}'$ with respect to another Cartesian frame $O'x'y'z'$ parallel to the first, so that O' has relative to O the position vector $\mathbf{r}_1 = (x_1, y_2, z_1)$. The first of those elements is

$$\begin{aligned} I'_{11} &= \int (y'^2 + z'^2)dm \\ &= \int [(y - y_1)^2 + (z - z_1)^2]dm \\ &= \int (y^2 + z^2)dm + (y_1^2 + z_1^2)\int dm - 2y_1 \int ydm - 2z_1 \int zdm. \end{aligned}$$

Since $\int ydm = \int zdm = 0$, we get

$$I'_{11} = I_{11} + M(y_1^2 + z_1^2). \tag{1.13}$$