Translational Systems Sciences 26

Kyoichi Kijima · Junichi Iijima · Ryo Sato · Hiroshi Deguchi · Bumpei Nakano *Editors*

Systems Research I

Essays in Honor of Yasuhiko Takahara on Systems Theory and Modeling

Springer

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Preface

It is a great pleasure to publish Systems Research I and II: Essays in Honor of Yasuhiko Takahara, to commemorate the 50th anniversary of Dr. Yasuhiko Takahara's research and education activities, who has been active at the world level in the field of systems research. We compile representative research of researchers and practitioners scented by Dr. Takahara from Japan and abroad who pay homage and gratitude to him into two volumes.

Dr. Takahara completed his research with Dr. M. Mesarovic at Case Western Reserve University. He brought up the results of General Systems Theory (GST), especially Mathematical General Systems Theory (MGST), from the United States to Japan to be appointed to Tokyo Institute of Technology in 1972. In Japan, it was a time the term "system engineering" began to attract interest in gradually establishing the system as a unique field and capturing the essence of what is recognized as a system.

We, editors-in-chief, who fortunately shared that era, still remember the shock we had when we first learned GST, especially MGST. MGST attempts to transparently understand the properties of systems such as interdependency and emergent property, by discussing the logic related to systems in a set-theoretic framework to formulate causal systems and hierarchical systems.

A wide range of books used as textbooks at the seminars in Takahara laboratory remind us of such good old days. They not only have worked as the soils supporting our research since then but also certainly reflect some part of the background of knowledge at that time. They include:*Universal Algebra* (George Graetzer); *Algebra* (Saunders MacLane and Garrett Birkoff); *Introduction to Topology and Modern Analysis* (George F. Simmons); *Topology* (James Dugundji); *Abstract and Concrete Categories: The Joy of Cats* (Jiri Adámek, Horst Herrlich, and George E. Strecker); *Model Theory* (C.C. Chang, H. Jerome Keisler), *Beginning Model Theory* (Jane Bridge); *Model-Based Systems Engineering* (A. Wayne Wymore); *Theories of Abstract Automata* (Michael A. Arbib); *The Specification of Complex Systems* (B. Cohen et al.); *The Structure of Scientific Theories* (Frederick Suppe (Editor)), *Goedel, Escher, Bach* (Douglas R. Hofstadter); *Forever Undecided* (Raymond Smullyan); *Introduction to Systems Philosophy* (Ervin Laszlo), *Living Systems*

(James Grier Miller); *Facets of Systems Science* (George J. Klir), *Systems Thinking Systems Practice* (Peter Checkland); *Heuristics* (Judea Pearl); *Multifaceted Modelling and Discrete Event Simulation* (Bernard P. Zeigler).

Over the 50 years since then, Dr. Takahara has developed the concept and theory of general systems centered on formal system research. By projecting mathematical general systems theory from meta-theory to the real world, he has powerfully promoted diverse but coherent systems research with a strong desire and intention to construct knowledge not only in theory (episteme) but also in engineering (techne) and practice (phronesis). They include systems modeling, information systems, decision support systems, and systems thinking. The results and findings there are now having a great impact on improving our socio-economic situations, which are becoming more and more complex, by providing the basis of ideas for the sophistication of business models and creation of new services by networks and ICT.

At the same time, Dr. Takahara has promoted educational activities with these studies and research as a nursery to give a great influence on many researchers and practitioners, not limited to students who received his direct scent. It triggers intellectual excitement and acts as a device to encourage their diversified but coherent systems study.

The Takahara School of GST is outstanding in its interdisciplinary and transdisciplinary approach. As you can see in the table of contents of the book, it ranges from highly abstract mathematics to practical applications of social science. It is very fortunate in systems research history that the Takahara school has created such a wide range of content in a deep dialogue using “system” as the common concept.

Some 20 authors gathered here have established their position by developing systems concepts in theory, models, methodologies, and applications in various ways, keeping in their mind the works by Dr. Takahara and co-authored with Mesarovic. Their activities include:

- Research to develop the strong intellectual desire for generality tackled by mathematical general systems theory into knowledge to connect different levels and approaches to the same object for solving actual problems. They are eager to catch the spirit that GST aimed at in the early days.
- Enterprise by which MGST and system engineering are practically fused by examining the basic concept of the system to develop knowledge applicable to hot topics these days.
- Development of what we call translational approach that connects theory and practice in a cyclic way, which follows the process of analyzing mechanisms, solving in an evidence-based way, and intervening in a problematic situation.
- Exploration of new disciplines such as Decision Systems Science and Service Systems Science by accommodating “hard” systems theory with “soft” systems thinking.
- Promotion of a formal approach in the field of Information Systems, for example, to identify isomorphisms between different recommendation systems and decision schemes known in social choice theory, and to the Enterprise Ontology in Design and Engineering Methodology for Organization.

Since the articles contributed by the twenty authors have a wide range of issues, we decided to structure them based on topics and approaches and publish them as a two-volume set. The first volume consists of 11 chapters divided into two parts dealing with the field of Systems Theory and Modeling.

Finally, we would like to express deep thanks to you Yutaka Hirachi of Springer Japan and Selvakumar Rajendran, and other staff of SPi Global. Without their patience and understanding, this volume could not be published.

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Part I
General Systems Theory

Chapter 1
Mesarovic-Takahara Time Systems Under the Effect of Feedback Mechanism

Jeffrey Yi-Lin Forrest, Zhen Li, Yohannes Haile, and Liang Xu

Abstract This chapter focuses on the study of Mesarovic-Takahara (MT) time systems and demonstrates how various properties of such systems are feedback invariant. In particular, after a brief introduction of the concepts of MT time systems and feedback systems, this work shows how these concepts play an important role in establishing the relationship between manufacturing and further industrial transformations. With such practical importance of feedback systems established, this presentation turns its focus to the theoretical study of various feedback invariant properties of MT systems. As an application, the concept of feedback systems is employed to explore when and how government economic policies can become effective in terms of assisting with and stimulating economic growth.

Keywords Attractor · Chaos · Consumer surplus · Economic system · Government policy · Market demand · Stationary system · Time-invariable realization

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1.1 Introduction

At this special moment of honoring one of the great thinkers of our time, the first coauthor of this chapter likes to express his sincere and heartfelt appreciation to Professor Yasuhiko Takahara. It is Professor Takahara's publication of various important monographs and papers that majorly shaped this coauthor's entire professional career since the mid-1980s. As a tribute to Professor Takahara, this chapter continues the scholarly works started by him in the field of systems research.

To achieve this research objective, this chapter investigates various classes of Mesarovic-Takahara (MT) time systems, such as those that are strongly stationary or precausal or causal, and various important properties, such as time-invariable realization, chaos, and attractors. The focus is on the effect of the feedback mechanism on these particular systems and properties. Other than continuing the tradition of Professor Takahara's works in the language of set theory, this chapter employs recent developments in the research of economics to illustrate how some of the derived set-theoretic results can be beneficially applied to enrich our knowledge on

- How races between market exchange and manufacturing can be purposefully employed for a nation to kick-start and maintain a self-sustaining momentum of growth.
- Under what conditions government economic policies can be effective in terms of economic development.

The rest of this chapter is organized as follows: Sect. 1.2 introduces the basic concepts needed for the work and demonstrates how feedback systems function in real life. Section 1.3 features the main properties of MT time systems, such as strong stationality, (pre-)causality, time-invariant realization, chaos, and attractors. Following this theoretical study, Section 1.4 demonstrates how some of the basic concepts and established conclusions of MT time systems can be practically employed to study when government economic policies become effective. Section 1.5 concludes this presentation.

1.2 Basic Concepts

This section presents the basic terms needed for the rest of this research. After introducing the fundamental terminologies in the first subsection, such as the concepts of input-output systems and MT time systems, the second subsection illustrates the practical importance of feedback systems by examining how such a systemic structure appears naturally within a growing economy.

1.2.1 The Underlying Concepts

Let X and Y be two sets. Then the following binary relation S

$$\varnothing \neq S \subset X \times Y$$

is known as an input-output system (Mesarovic & Takahara, 1975) with X being the input space and Y the output space of S, respectively. It is evident that most economic entities in the business world are input-output systems, because various inputs are needed for these entities to survive and particular outcomes are offered to the marketplace (Forrest, 2010; Porter, 1985).

Let the positive half of the real number line, written as $T = [0, +\infty)$, be the time axis, and A and B be two linear spaces over the same field $\mathcal{A}$. Define sets A^T and B^T as follows:

$$A^T = \{f : f \text{ is a mapping } T \to A\} \quad \text{and} \quad B^T = \{f : f \text{ is a mapping } T \to B\}.$$

As conventionally known, these sets A^T and B^T become linear spaces over $\mathcal{A}$ if the following operations are introduced. For any elements f, $g \in A^T$ (respectively, $\in B^T$), and any scalar$\alpha \in \mathcal{A}$,

$$(f + g)(t) = f(t) + g(t) \quad \text{and} \quad (\alpha f)(t) = \alpha \cdot f(t), \quad \text{for each } t \in T.$$

Each input-output system S defined on A^T and B^T, or symbolically, $S \subset A^T \times B^T$, is referred to as an Mesarovic-Takahara (MT) time system (Mesarovic & Takahara, 1989).

Let $\mathcal{A}$ be a field, A and B linear spaces over $\mathcal{A}$, and S an input-output system satisfying that.

1. $\varnothing \neq S \subset A \times B$
2. $s \in S$ and $s' \in S$ imply $s + s' \in S$
3. $s \in S$ and $\alpha \in \mathcal{A}$ imply $\alpha \bullet s \in S$

where $+$ and $\cdot$ are addition and scalar multiplication in $A \times B$, respectively, defined as follows: For any $(x_1, y_1), (x_2, y_2) \in A \times B$ and any $\alpha \in \mathcal{A}$,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \quad \text{and} \quad \alpha (x_1, y_1) = \left(\alpha x_1, \alpha y_1\right),$$

then S is then known as a linear system (Forrest, 2010).

Let $S \subset A \times B$ be a linear system and $S_f : B \to A$ a linear function or known as a functional system. Then the feedback system S' of S by S_f is defined (Forrest, 2010) as follows: for any $(x, y) \in A \times B$

$$(x, y) \in S' \leftrightarrow (\exists z \in A)\left((x + z, y) \in S \quad \text{and} \quad (y, z) \in S_f\right). \tag{1.1}$$

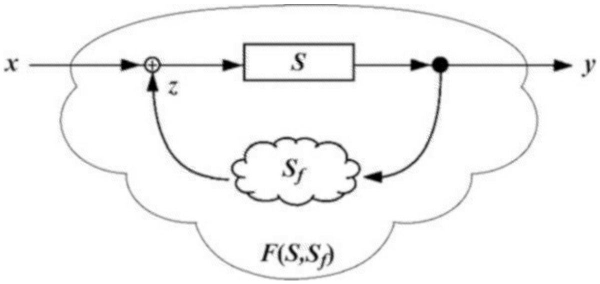


Fig. 1.1 The structure of a feedback system

System S is known as an original system; and S_f as a feedback component system. Let

$$\mathcal{S} = \{S \subset A \times B : S \text{ is a linear system}\},$$

$$\mathcal{S}_\mathrm{f} = \{S_\mathrm{f} : B \to A : S_\mathrm{f} \text{ is a linear functional system}\},$$

and

$$\mathcal{S}' = \left\{S' \subset A \times B : S' \text{ is a subset}\right\}.$$

Then a feedback transformation $F : \mathcal{S} \times \mathcal{S}_\mathrm{f} \to \mathcal{S}'$ is defined by Eq. (1.1), and known as the feedback transformation over the linear spaces A and B (Forrest, 2010; Lin & Ma, 1990; Saito, 1986). Figure 1.1 shows the geometric meaning of the concept of feedback systems, where the output y is jointly affected by the input x and the feedback loop $F(S, S_\mathrm{f})$.

Scholars from different disciplines have employed the concept of feedback to develop important insights and conclusions (Bayliss, 1966; Deng, 1985; Forrest et al., 2018a, b; Henig, 1983; Milsum, 1966; Negoita, 1992; Saito, 1986, 1987; Saito & Mesarovic, 1985; Takahara & Asahi, 1985; Wonham, 1979; Wu, 1981; Zadeh, 1965). The definition of general feedback systems is first introduced by Saito and Mesarovic (1985). For relevant theoretical studies, see Lin and Ma (1990).

1.2.2 How the Feedback Mechanism Functions in an Economy: An Example

To demonstrate the practical importance of the concept of feedback systems, this subsection shows how the feedback mechanism, as introduced above, organically associates market exchange and manufacturing for a nation that desires to kick-start and maintain its self-sustaining momentum of growth. By self-sustained momentum of growth, it means a self-sustained societal development that penetrates all economic, social, and political aspects of the society. As a result of such societal development, advanced technologies appear and alter how people live and how government operates. Meanwhile, a population explosion follows the societal development, while the methods of production and service are revolutionized (Heaton, 2017; Wen, 2016).

Beyond confirming what *consumers really want in life and satisfying market demands, market exchanges dictate the level of competitive supplies and necessitates needed innovations through demonstrating consumer demands* (Forrest et al., 2017). *The level of competitive supplies and the quality of innovations are determined by sufficient market depth (or purchasing power). They in turn directly further* the development of technologies due to intensified competitions. On the other hand, f*or a nation to develop its needed market depth, it mobilizes a portion of the available labor force, freed from producing life necessities, into the manufacturing of luxurious goods.* As a result of the relocation of labor, citizens have much more incomes than before, which raises their purchasing powers. That helps deepen the product market. That is, market competition encourages the development of new technologies and helps industries advance to higher levels. At the same time, these developments simultaneously expand various markets, be they financial, product, or labor, with greater varieties of products. For more discussions along this line, see Forrest (2010).

This discussion naturally reveals how a feedback mechanism plays its role in a nation's economic development. In particular, let us visualize the input x as a particular government policy and y the corresponding economic output produced by adopting the policy. It is intuitive to see that although the output y is a consequence of the policy x, it represents the combined effect of the policy and the market reaction, denoted by $S_f(S(x))$, to the policy. In reality, of course, the situation can take one of two possible scenarios—the economy is a relatively closed system from the outside world or an open system. For the former case, this discussion does not involve any externalities and related costs, while for the latter case, the market reaction $S_f(S(x))$ includes all those from domestic and foreign markets and the environment.

Speaking differently, the previous paragraph indicates that the race between market exchange and manufacturing production is indeed a feedback system, within which market demands excite manufacturers to produce more and better products and entrepreneurs to introduce new and novel offers. Such a feedback mechanism forces firms to satisfy the increasing need of production by hiring additional employees with rising salaries. The race simultaneously reinforces the market depth and the demand of the market. Such mechanism underlying the working of the described feedback system is figuratively shown in Fig. 1.2.

One good example that illustrates the abstract discussion above is the phenomena of industrial revolutions that occurred one after another in the recent history from around the world (Forrest et al. 2018a, b; Rostow, 1960; Wen, 2016). In each of these occasions, it was the intensifying race between market exchanges and manufacturing productions that brought forward with the desired self-sustaining

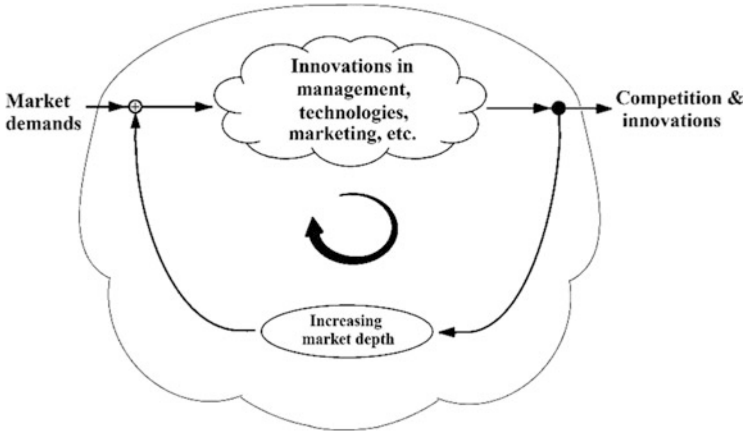


Fig. 1.2 The feedback race between market exchange and manufacturing

momentum of economic growth. However, contrary to this example of positive confirmation, the third industrial revolution in the United States did not really help the US economy enjoy rising salaries even though the economy nearly reached its full employment in February 2020. The reason behind the appearance of this counter-intuitive situation is that within the ongoing economic globalization, the US economy represents merely a local one, while what is discussed above expresses what holds true within the entire economic system that entertains an intensifying race between market exchanges and manufacturing productions. More specifically, instead of the US economy, it is those foreign economies involved in relevant cross-national productions that enjoy the fruitful benefits of the third industrial revolution that originated in the USA.

1.3 Properties of MT Time Systems

This section presents results on how the feedback transformation affects various properties of linear time systems, such as strong stationality, (pre-)causality, time-invariable realization, chaos, and attractors.

1.3.1 Linear Time Systems That Are Strongly Stationary

In this section, all symbols and their relevant assumed conditions are maintained as given previously. Then, the following result implies that the feedback transformation

$F : \mathcal{S} \times \mathcal{S}_{\mathrm{f}} \rightarrow \mathcal{S}'$, given in Eq. (1.1), is well defined on the class of all MT time systems.

Theorem 1.1 *If $S \subset A \times B$ is a linear system and $S_{\mathrm{f}} : B \rightarrow A$ a linear functional system, then the feedback system $F(S, S_{\mathrm{f}}) \subset A \times B$ is also a linear system.*

Proof For arbitrarily chosen $(x_1, y_1), (x_2, y_2) \in F(S, S_{\mathrm{f}})$ and $\alpha \in \mathcal{A}$, Eq. (1.1) implies that there are $z_1, z_2 \in A$ satisfying

$$l\,(x_1 + z_1, y_1) \in S \quad \text{and} \quad (y_1, z_1) \in S_{\mathrm{f}} \quad \text{and} \quad (x_2 + z_2, y_2) \in S \quad \text{and} \quad (y_2, z_2) \in S_{\mathrm{f}}. \tag{1.2}$$

Because both S and S_{f} are linear systems, Eq. (1.2) implies

$$\left(\alpha x_1 + \alpha z_1, \alpha y_1\right) \in S \quad \text{and} \quad \left(\alpha y_1, \alpha z_1\right) \in S_{\mathrm{f}},$$

from which we have $\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1) \in F(S, S_{\mathrm{f}})$.

Additionally, because of the linearity of S and S_{f}, Eq. (1.2) implies

$$((x_1 + x_2) + (z_1 + z_2), (y_1 + y_2)) \in S \quad \text{and} \quad ((y_1 + y_2), (z_1 + z_2)) \in S_{\mathrm{f}}.$$

Hence, we have $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in F(S, S_{\mathrm{f}})$.

By combining what are established above, we conclude that the feedback system $F(S, S_{\mathrm{f}})$ is a linear system over $A \times B$. QED

For two arbitrary time moments $t, t' \in T$, satisfying $t < t'$, we denote the following intervals abbreviately

$$T^t = \left[0, t\right), \quad T_t = [t, +\infty), \quad T_{tt'} = \left[t, t'\right), \quad \overline{T}^t = [0, t], \quad \overline{T}_{tt'} = \left[t, t'\right].$$

Accordingly, the restrictions of any $x \in A^T$ with respect to these intervals of time are respectively denoted by

$$x^t = x\left|T^t\right., \quad x_t = x\,|T_t, \quad x_{tt'} = x\,|T_{tt'}, \quad \overline{x}^t = x\left|\overline{T}^t\right., \quad \overline{x}_{tt'} = x\left|\overline{T}_{tt'}\right..$$

More generally, for sets and vectors similar notations will be utilized. For instance, for $(x, y) \in A^T \times B^T$, we denote

$$(x, y)^t = \left(x^t, y^t\right), \quad (x, y)_t = (x_t, y_t), \quad \text{etc.}$$

A subset $S \subset X \times Y \subset A^T \times B^T$ is called a linear time system (Lin, 1989; Lin & Ma, 1987; Ma & Lin, 1992) if S is a linear subspace of $X \times Y$. Let the domain of S be written as $D(S)$, known as the input space of S. It is defined as follows:

$$D(S) = \{x \in X : \exists y \in Y \ni (x, y) \in S\}.$$

Without loss of generality, assume that $D(S) = X$ and $R(S) = Y$, where $R(S)$ is the range of S, known as the output space of S, defined by

$$R(S) = \{y \in Y : \exists x \in X \ni (x, y) \in S\}.$$

For any $x, x^{'} \in X$ and any $t \in T$, assume that $x^t \circ x_t' \in D(S)$, where $x^t \circ x_t'$ is the concatenation of x^t and x_t', which is defined as follows: for $s \in T$,

$$x^t \circ x_t'(s) = \begin{cases} x(s), & \text{if } s < t \\ x'(s), & \text{if } s \geq t \end{cases}.$$

For a given real number τ, let σ^τ be the shift operator defined on X as follows: for any $x \in X$, $\sigma^\tau(x) \in X$ satisfying that

$$\sigma^\tau(x)(\xi) = x(\xi - \tau), \quad \forall \xi \in T_\tau.$$

Figure 1.3a–c show the geometric meaning of the concept of the shift operator σ^τ respectively for the cases when $\tau = 0$, $\tau > 0$, and $\tau < 0$. In Fig. 1.3b, c the dotted $A \times T$ planes indicate the locations of the $A \times T$ plane before the shift operation σ^τ is applied. In other words, if $\tau > 0$, the shift operator σ^τ moves the graph of time function x τ units to the right; if $\tau < 0$, the shift operator σ^τ emphasizes on the portion of the graph of x on the right of the vertical line $t = \tau$

When a linear time system $S \subset X \times Y$ satisfies

$$\forall t \in T \left(\sigma^{-t}\left(S|T_t\right) = S\right),$$

then S is said to be strongly stationary (Lin, 1989; Lin & Ma, 1987; Ma & Lin, 1992).

Theorem 1.2 *Assume that $S \subset X \times Y$ is a linear time system and $S_{\mathrm{f}} : Y \to X$ a strongly stationary linear functional time system. Then a sufficient and necessary condition for the feedback system $F(S, S_{\mathrm{f}})$ to be strongly stationary is that system S is strongly stationary.*

Proof ($\Rightarrow$). For the proof of this necessity, the feedback system $F(S, S_{\mathrm{f}})$ is assumed to be strongly stationary. Then the following holds true:

$$S = \{(x + S_{\mathrm{f}}(y), y) : (x, y) \in F(S, S_{\mathrm{f}})\}.$$

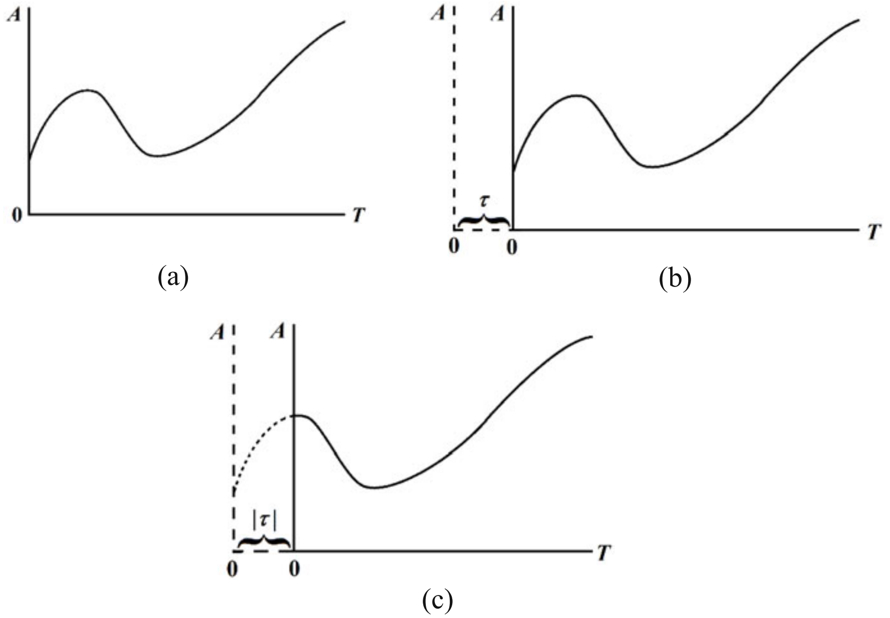


Fig. 1.3 How the shift operator σ^{τ} affects the graph of x. (**a**) The original x in plane $T \times A$. (**b**) After the shift σ^{τ} is applied if $\tau > 0$. (**c**) After the shift σ^{τ} is applied if $\tau < 0$

To complete the proof of this part, we only need to show that

$$\forall (x, y) \in S \forall t \in T \left(\sigma^{-t} \left((x, y) \left| T_t \right.\right) \in S\right).$$

To demonstrate this end, let $(x, y) \in S$ and $t \in T$. Then, we have

$$\begin{aligned} \sigma^{-t} \left((x, y) \left| T_t \right.\right) &= \sigma^{-t} \left((x - S_{\mathrm{f}}(y) + S_{\mathrm{f}}(y), y) \left| T_t \right.\right) \\ &= \left(\sigma^{-t} \left(\left(x - S_{\mathrm{f}}(y)\right) \right| T_t) + \sigma^{-t} \left(S_{\mathrm{f}}(y) \right| T_t) , \sigma^{-t} \left(y \left| T_t \right.\right)\right). \end{aligned}$$

Because the systems $F(S, S_{\mathrm{f}})$ and S_{f} are assumed to be strongly stationary, we have

$$\left(\sigma^{-t} \left(\left(x - S_{\mathrm{f}}(y)\right) \right| T_t) , \sigma^{-t} \left(y \left| T_t \right.\right)\right) \in F \left(S, S_{\mathrm{f}}\right),$$

and

$$\left(\sigma^{-t} \left(y \left| T_t \right.\right), \sigma^{-t} \left(S_{\mathrm{f}}(y) \right| T_t)\right) \in S_{\mathrm{f}}. \tag{1.3}$$

Let $x^{'} = \sigma^{-t}((x - S_{\mathrm{f}}(y))|T_t)$ and $y^{'} = \sigma^{-t}(y|T_t)$. Then, Eq. (1.3) indicates that $S_{\mathrm{f}}(y^{'}) = \sigma^{-t}(S_{\mathrm{f}}(y)|T_t)$. That is, we have

$$\sigma^{-t}\left((x, y)|\, T_t\right) = \left(x' + S_{\mathrm{f}}\left(y'\right), y'\right), \tag{1.4}$$

where $(x^{'}, y^{'}) \in F(S, S_{\mathrm{f}})$. So, Eq. (1.4) implies that $\sigma^{-t}((x, y)|T_t) \in S$.

($\Leftarrow$). This part of the sufficiency proof follows from the fact that for a linear functional system $S : X \to Y$, $F(S, S_f)$ is injective, for each arbitrarily $S_f \in \mathcal{S}_{\mathrm{f}}$, if, and only if S is injective (Theorem 10.3 of Forrest, 2010) and the necessity part of this proof. QED

1.3.2 Linear Time Systems That Are Precausal or Causal

If a linear time system $S \subset X \times Y$ satisfies

$$(\forall t \in T)\,(\forall x \in X)\left(\overline{x}^t = \overline{0}^t \to R(S)\left|\overline{T}^t\right. = S(0)\left|\overline{T}^t\right.\right),$$

then S is referred to as a precausal system (Lin, 1989; Lin & Ma, 1987; Ma & Lin, 1992), where for $x \in X$, $S(x) = \{y \in Y : (x. y) \in S\}$.

Theorem 1.3 *A sufficient and necessary condition for a linear time system* $S \subset X \times Y$ *to be precausal is that* $(\forall t \in T)\,(\forall x, y \in X)\left(\overline{x}^t = \overline{y}^t \to S(x)\left|\overline{T}^t\right. = S(y)\left|\left|\overline{T}^t\right.\right.\right)$.

Proof ($\Rightarrow$). For this necessity part of the proof, S is assumed to be precausal. Hence, for any $x, y \in X$, we have

$$\overline{(x - y)}^t = \overline{0}^t \to S\,(x - y)\left|\overline{T}^t\right. = S(0)\left|\left|\overline{T}^t\right.\right. .$$

This expression means that

$$\overline{x}^t = \overline{y}^t \to S(x)\left|\overline{T}^t\right. = S(y)\left|\left|\overline{T}^t\right.\right. ,$$

because the condition $\overline{(x - y)}^t = \overline{0}^t$ is equivalent to that of $\overline{x}^t - \overline{y}^t = \overline{0}^t$. So, we have

$$\overline{x}^t = \overline{y}^t \text{ and } S\,(x - y)\left|\overline{T}^t\right. = S(0)\left|\left|\overline{T}^t\right.\right. \to S(x)\left|\overline{T}^t\right. - S(y)\left|\left|\overline{T}^t\right.\right. = S(0)\left|\left|\overline{T}^t\right.\right. .$$

Therefore, we have $S(x)\left|\overline{T}^t = S(y)\right|\overline{T}^t + S(0)\left|\overline{T}^t = S(y)\right|\overline{T}^t$.
($\Leftarrow$) For this part of the sufficiency proof, assumed is

$$(\forall t \in T)\,(\forall x, y \in X)\left(\overline{x}^t = \overline{y}^t \to S(x)\left|\overline{T}^t = S(y)\right|\left|\overline{T}^t\right.\right).$$

This assumption implies

$$(\forall t \in T)\,(\forall x \in X)\left(\overline{x}^t = \overline{0}^t \to S(x)\left|\overline{T}^t = S(0)\right|\overline{T}^t\right).\text{QED}$$

Theorem 1.4 *Assume that both* $S : X \to Y$ *and* $S_\mathrm{f} : Y \to X$*:* Y *are time systems that are linear functional causal. Then a sufficient and necessary condition for the feedback system* $F(S, S_\mathrm{f})$ *to be causal is that the time system* $S \circ S_\mathrm{f} \circ F(S, S_\mathrm{f}) : D(F(S, S_\mathrm{f})) \to Y$ *is causal.*

Proof The necessity condition follows from the fact that for any given causal functional time systems, their composition is also causal.

($\Leftarrow$). For the proof of the sufficiency condition, we assume that $S \circ S_\mathrm{f} \circ F(S, S_\mathrm{f})$ is a causal system. Hence, we have

$$\forall t \in T \forall x \in D\left(F\left(S, S_\mathrm{f}\right)\right)\left(\overline{x}^t = \overline{0}^t \to S \circ S_\mathrm{f} \circ F\left(S, S_\mathrm{f}\right)(x)\left|\overline{T}^t = \overline{0}^t\right.\right).$$

If we let $y = F(S, S_\mathrm{f})(x)$, we need to show that $\overline{y}^t = \overline{0}^t$. To this end, Eq. (1.1) implies

$$S\left(x + S_\mathrm{f}(y)\right) = y.$$

Therefore, we have $S(x) + S \circ S_\mathrm{f}(y) = y$, where

$$S(x)\left|\overline{T}^t = \overline{0}^t\right. \quad \text{and} \quad S \circ S_\mathrm{f}(y)\left|\overline{T}^t = S \circ S_\mathrm{f} \circ F\left(S, S_\mathrm{f}\right)(x)\right|\overline{T}^t = \overline{0}^t.$$

Thus, we know

$$\overline{y}^t = \left[S(x) + S \circ S_\mathrm{f}(y)\right]\left|\overline{T}^t = S(x)\right|\overline{T}^t + S \circ S_\mathrm{f}(y)\left|\overline{T}^t = \overline{0}^t + \overline{0}^t = \overline{0}^t\right..$$

This end means that $F(S, S_\mathrm{f})$ is a causal system. QED

1.3.3 Time-Invariable Realization

Assume that $S : X \to Y$ is a linear functional time system. If the system S satisfies

$$\forall t \in T \forall x \in X \left(\lambda^t S\left(0^t \circ \sigma^t(x)\right) = S(x), \tag{1.5}$$

then it is referred to as time invariably realizable, where the relation λ^t is defined by $\lambda^t(\bullet) = \sigma^t(\bullet|T_t)$.

Theorem 1.5 *Assume that $S : X \to Y$ is a linear functional time system. Then a sufficient and necessary condition for S to be time invariably realizable is that*

$$\forall t \in T \forall x \in X \left(S\left(0^t \circ \sigma^t(x)\right)|T_t = \sigma^t S(x)\right). \tag{1.6}$$

Proof ($\Rightarrow$) For this part of necessity argument, the assumption that S is time invariably realizable implies that the system S satisfies Eq. (1.5). Next, we show that Eq. (1.6) can be derivable from Eq. (1.5). To achieve this end, we have

$$\forall t \in T \forall x \in X, \quad \lambda^t S\left(0^t \circ \sigma^t(x)\right) = \lambda^t(y) = \sigma^{-t}\left(y\left|\overline{T}^t\right.\right) = S(x), \tag{1.7}$$

where $=S(0^t \circ \sigma^t(x)$. Therefore, we obtain

$$\sigma^t \circ \sigma^{-t}\left(y\left|\overline{T}^t\right.\right) = y|T_t = S\left(0^t \circ \sigma^t(x)\right)|T_t = \sigma^t S(x).$$

($\Leftarrow$) For this part of sufficiency argument, assume for any $t \in T$ and any $x \in X$ system S satisfies the condition in Eq. (1.6). Then, we have

$$\sigma^{-t} S\left(0^t \circ \sigma^t(x)\right)\left|\overline{T}_t\right. = \sigma^{-t}\sigma^t \mathrm{S}(x) = S(x). \tag{1.8}$$

Comparing Eqs. (1.5) and (1.8) leads to the conclusion that the given linear functional time system $S : X \to Y$ is time invariably realizable. QED

Theorem 1.6 Assume that *linear functional time systems $S : X \to Y$ and $S_f : Y \to X$ are time invariably realizable. Then a sufficient and necessity condition for the feedback system $F(S, S_f)$* to be *time invariably realizable is that*

$$\forall t \in T x \in D\left(F\left(S, S_f\right)\right) \ni S \circ S_f \circ F\left(S, S_f\right)\left(0^t \circ \sigma^t(x)\right)|T_t = \sigma^t S \circ S_f \circ F\left(S, S_f\right)(x), \tag{1.9}$$

Speaking differently, the fact that the feedback system $F(S, S_f)$ is time invariably realizable is equivalent to that $S \circ S_f \circ F(S, S_f)$ is time invariably realizable.

Proof ($\Leftarrow$) For the part of sufficiency argument, the assumption that $S \circ S_{\mathrm{f}} \circ F(S, S_{\mathrm{f}})$ is time invariably realizable implies that for any $t \in T$ and any $x \in D(F(S, S_{\mathrm{f}}))$, Eq. (1.9) is true. Hence, Theorem 1.5 implies that Eq. (1.6) is true. So, we have

$$S\left(0^t \circ \sigma^t(x)\right)|T_t + S \circ S_{\mathrm{f}}\left(F\left(S, S_{\mathrm{f}}\right)\left(0^t \circ \sigma^t(x)\right)|T_t\right) \\ = \sigma^t S(x) + \sigma^t S \circ S_{\mathrm{f}} \circ F\left(S, S_{\mathrm{f}}\right)(x),$$

which means

$$S\left[0^t \circ \sigma^t(x) + S_{\mathrm{f}}\left(F\left(S, S_{\mathrm{f}}\right)\left(0^t \circ \sigma^t(x)\right)\right)\right]\Big| T_t = \sigma^t S\left(x + S_{\mathrm{f}} \circ F\left(S, S_{\mathrm{f}}\right)(x)\right).$$

This end is equivalent to

$$F\left(S, S_{\mathrm{f}}\right)\left(0^t \circ \sigma^t(x)\right)|T_t = \sigma^t F\left(S, S_{\mathrm{f}}\right)(x).$$

Hence, Theorem 1.5 implies that $F(S, S_{\mathrm{f}})$ is time invariably realizable.

($\Rightarrow$) For this part of the necessity argument, $F(S, S_{\mathrm{f}})$ is assumed to be time invariably realizable. So, Theorem 1.5 implies that

$$\forall t \in T \forall x \in D\left(F\left(S, S_{\mathrm{f}}\right)\right) \;\; F\left(S, S_{\mathrm{f}}\right)\left(0^t \circ \sigma^t(x)\right)|T_t = \sigma^t F\left(S, S_{\mathrm{f}}\right)(x). \tag{1.10}$$

By letting $F(S, S_{\mathrm{f}})(0^t \circ \sigma^t(x)) = y$ and $F(S, S_{\mathrm{f}})(x) = y^{'}$, Eq. (1.10) implies

$$S\left(0^t \circ \sigma^t(x) + S_{\mathrm{f}}(y)\right)|T_t = \sigma^t S\left(x + S_{\mathrm{f}}\left(y'\right)\right),$$

and

$$S\left(0^t \circ \sigma^t(x)\right)|T_t + S \circ S_{\mathrm{f}}(y)\,|T_t = \sigma^t S(x) + \sigma^t S \circ S_{\mathrm{f}}\left(y'\right). \tag{1.11}$$

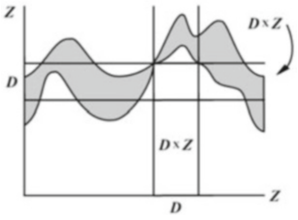
Both the time-invariable realizability of S and Theorem 1.5 indicate that Eq. (1.11) can be restated as follows:

$$S \circ S_{\mathrm{f}}(y)\,|T_t = \sigma^t S\left(S_{\mathrm{f}}\left(y'\right)\right). \tag{1.12}$$

Because Eq. (1.12) is the same as Eq. (1.9), we complete the proof. QED

1.3.4 Chaos and Attractor Under the Effect of Feedback Mechanism

Let S be an input-output system, satisfying $\varnothing \neq S \subset X \times Y$. Define $Z = X \cup Y$. Then S is a binary relation on Z and a subset $D \subset Z$, satisfying $D^2 \cap S = \varnothing$, is said to



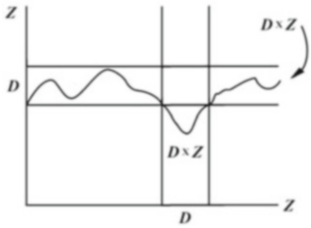


Fig. 1.4 The geometry of an attractor. (**a**) D is an attractor of S, and $S(x)$ contains at least one element for each x in $Z - D$. (**b**) D is an attractor of S, and $S : Z \to Z$ is a function

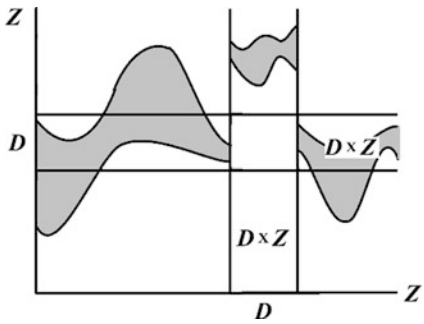


Fig. 1.5 D is a strange attractor of the system S.

be a chaos of S (Forrest, 2010; Zhu & Wu, 1987). Intuitively, D is seen as a chaos because system S has no control over the elements in D.

A subset $D \subset Z$ is known as an attractor of S (Forrest, 2010; Zhu & Wu, 1987) if for each $x \in Z - D$, $S(x) \cap D \neq \varnothing$. Figure 1.4a, b show the geometry of the concept of attractors. When S is not a function, Fig. 1.4a shows that the graph of S outside the vertical bar $D \times Z$ overlaps the horizontal bar $Z \times D$. When S is a function, Fig. 1.4b shows that the graph of S outside the vertical bar $D \times Z$ must be contained in the horizontal region $Z \times D$.

A subset $D \subset Z$ is said to be a strange attractor of system S (Forrest, 2010; Zhu & Wu, 1987), if D is both a chaos and an attractor of S. Figure 1.5 shows the case when a subset D of Z is a strange attractor of an input-output system S, where the square $D \times D$ is the only portion of the band $Z \times D$ over which the graph of S does not touch.

Theorem 1.7 *As long as $S \subset X \times X$ is a linear system, there must be a linear feedback component system $S_f : X \to X$ so that the following are equivalent:*

- *A subset $D \subset X$ is a chaos, or an attractor, or a strange attractor of system S; and.*
- *The subset D is a chaos, or an attractor, or a strange attractor of $F(S, S_f)$, correspondingly.*

Proof Evidently, no matter how a linear system $S \subset X \times X$ is defined, the desired learn feedback component system $S_f : X \to X$ can be defined as follows:

$$S_f(x) = 0_x, \quad \forall x \in X.$$

To finish the proof, we only need to check that $S = F(S, S_f)$. QED

Theorem 1.8 *Let $S \subset X \times X$ be a linear system, $S_f : X \to X$ a functional linear system and $D \subset X$, satisfying*

$$(X - D) \pm S_f(D) \subset X - D. \tag{1.13}$$

Then D is a chaos of S if and only if D is a chaos of the feedback system $F(S, S_f)$.

Proof $(\Rightarrow)$ For this part of the necessity argument, suppose that $D \subset X$ is a chaos of S while not a chaos of the feedback system $F(S, S_f)$. Then for $(x, y) \in D^2 \cap F(S, S_f)$, the following holds true:

$$(x + S_f(y), y) \in S.$$

So, we have

$$x' = x + S_f(y) \notin D \quad \text{and} \quad x = x' - S_f(y).$$

This end contradicts Eq. (1.13), which implies that D must be a chaos of $F(S, S_f)$.

$(\Leftarrow)$ For this part of the sufficiency argument, the assumption that $D \subset X$ is a chaos of $F(S, S_f)$ but not a chaos of S implies that if $(x, y) \in D^2 \cap S$, $(x - S_f(y), y) \in F(S, S_f)$ holds true. That implies that $x' = x - S_f(y) \notin D$. This last equation is equivalent to $x = x' + S_f(y)$, which is contradictory to Eq. (1.13). Therefore, we can conclude that D must be a chaos of S. QED

Theorem 1.9 *Let $S \subset X \times X$ be a linear system and $S_f : X \to X$ a functional linear system, satisfying the following two conditions:*

$$X - D \subset \{x - S_f(y) : (x, y) \in S \cap ((X - D) \times D)\}, \tag{1.14}$$

and

$$X - D \subset \{x + S_f(y) : (x, y) \in F(S, S_f) \cap ((X - D) \times D)\}. \tag{1.15}$$

Then, a sufficient and necessary condition for $D \subset X$ to be an attractor of S is that D is an attractor of the feedback system $F(S, S_f)$.