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Eduardo Noble

The Rise and Fall of the German Combinatorial Analysis

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1 Introduction

At the end of the eighteenth century, the mathematical landscapes of German-speaking territories in Europe were dominated by the ideas of a particular group of mathematicians known as the "German combinatorial school". This book deals with those ideas. But before moving on to them, it is necessary to consider the existing historiography on the topic, around which a specific historical interpretation of the group and of its scientific work has taken decisive form. Getting acquainted with this traditional interpretation, which concerns the factors involved in the formation of the combinatorial school, its longevity in the context of German mathematics, the distinctive features of its scientific theory, and the relation of this latter to other scientific theories, opens the possibility of better understanding the motivations and goals of this research subject.

1.1 Historiography on the Combinatorial Analysis

Despite the limited number of studies devoted to the combinatorial school, it is not that easy finding your way around their historical narratives. A sort of intellectual struggle begins as soon as our curiosity on the subject leads us to the natural question of just what the combinatorial school is. Carl Immanuel Gerhardt was one of the first historians of mathematics to answer this question (Gerhardt 1877, 201–206). In 1877, he affirms that this school was founded by Carl Friedrich Hindenburg (1739–1808) and adds that Johann Friedrich Pfaff (1765–1825) enjoyed a prominent position in this academic organization. In 2003, Laurence Brockliss writes, in a volume centered on the history of science during the eighteenth century and published by the University of Cambridge, that Pfaff and Hindenburg were the founders of the combinatorial school (Brockliss 2003, 57). Although the vast majority of studies agrees with Gerhardt's opinion, Brockliss's remark affords an instance of the basic inaccuracies that still prevail in our knowledge of this school.

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This kind of inaccuracies does not only concern the question of who founded the combinatorial school, but also the question of who joined this group as an active member. Gerhardt claims that, besides Pfaff and the founder Hindenburg, Georg Simon Klügel (1739–1812), Hieronymus Christoph Wilhelm Eschenbach (1764–1797), and Heinrich August Rothe (1773–1842) were members of the combinatorial school. In his paper on combinatorics, which is about twenty pages long and which remains the most detailed and the most comprehensive study on our research subject to date, Eugen Netto adds another three names to the list established by Gerhardt: Heinrich August Töpfer (1758–1833), Christian Kramp (1760–1826), and Johann Karl Burckhardt (1773–1825) (Netto 1908, 201–221). However, while Gerhardt and Netto included these names in their respective lists of members of the combinatorial school on the basis of their partial analysis of some texts of these authors, other historians arbitrarily inflate the list by adding names without justifying their choices, often not even a brief description of any writing of the chosen author is provided. In Bell's account of the combinatorial analysis, one can read that Josef Hoëné Wronski (1776–1853) was linked to this group, but this claim is not supported by any kind of primary source analysis (Bell 1940, 290–291). As time passes, the list of the members of the combinatorial school increases and differs from one historian to another according to the needs of their respective narratives. For instance, Martin suggests that Heinrich Bürmann (?–1817) was an active collaborator in the construction of the combinatorial school's theory, but he does not discuss any of Bürmann's ideas, merely quoting the title of one of Bürmann's works (Martin 1996b, 83–84). Pradier analyzes the work of Johann Nicolaus Tetens (1736–1807) but he does not justify the assertion that Tetens was a member of this school (Pradier 2003, 146). Ferraro includes Bürmann, Tetens, and Moritz von Prasse (1769–1814) in the group (Ferraro 2007, 479–480), and instead of elaborating on his reasons for doing so he directs the reader to (Jahnke 1993) and (Panza 1992, vol. 2, 651–659) for further information, but Jahnke does not address the works of any of those authors in his paper and Panza deals with Prasse's thought exclusively. Admittedly, it is just not reasonable to expect historians to draw up a single and definitive inventory of the mathematicians belonging to the combinatorial school, for the discovery of new historical data is a constituent part of historical research. On the contrary, since the inclusion of a given mathematician in the combinatorial school must be based on methodological decisions as far as historical narratives are concerned, it is reasonable to ask what criteria they used to make those decisions. Given the recurrent lack of primary source analysis in many studies, as those referred to above, it must be recognized that, in most cases, those criteria become indiscernible and hard to assess.

Gerhardt suggests that the combinatorial school has headed the mathematical research carried out in Germany during the "last decades" (*den letzten Jahrzehnten*) (Gerhardt 1877, 201). Considering that his book was published in 1877 and that he thought that Hindenburg was the founder of the combinatorial school, his remark implies that the existence of this school extends over a period of seventy or eighty years. Georg Faber and Alfred Pringsheim seem to agree with Gerhardt's estimation when they say that the influence of the combinatorial school can be perceived in German textbooks until the middle of the nineteenth century (Pringsheim and Faber 1909, 3–4). In his French version of Faber's and Pringsheim's work, Jules Molk goes beyond a simple translation and transforms one statement of Faber and Pringsheim into an entire paragraph about the combinatorial school, in which Molk claims that the combinatorial school continues to be active between 1825 and 1850, and even later (Molk et al. 1911, 3). Similarly, Wilhelm Lorey states that Hindenburg founded the combinatorial school and that university professors of mathematics emerged from this school up to the mid-nineteenth century (Lorey 1916, 27, 29). However, views on this matter are far from unanimous. According to Ferraro, it is possible to put a precise date to the emergence and dissolution of the combinatorial school, which was formed in 1780 and dissolved in 1810 (Ferraro 2007, 479–480). Therefore, its existence corresponds to a shorter period of just thirty years. The foundation year of 1780 comes again and again in the writings of different historians. This date is used by Schubring too, though he is not as categorical as Ferraro, nuancing that this school was "in vigour in Germany since about 1780" (Schubring 2007, 110). He disagrees with respect to the second date. For him, the textbooks related to the combinatorial school's ideas that appeared between 1800 and 1825 prove that this school did not cease its activities before 1825 (Schubring 2005, 562). In fact, in his lecture presented at a Satellite Symposium of the first European Congress of Mathematics held in Paris from April 3 to April 6, 1992, Schubring dated the beginning of the combinatorial school around 1780 and its end around 1820 (Schubring 1996a, 370). Thus, for him, the existence of this school varies between forty and forty-five years. Jahnke, for his part, proposes to distinguish two periods in the history of the combinatorial school (Jahnke 1990, 171). The first took place from 1780 to 1808, where the starting year was probably chosen because of the publication of Hindenburg's book entitled *Infinitinomii dignitatum exponentis indeterminati historia leges ac formulae*, appeared in 1779, and the period ends when Hindenburg dies. The second took place from 1808 to 1840, and it is characterized by the combinatorial school's loss of hegemony in the German academic world, as well as by the lack of innovative mathematical research, which was replaced by the production of textbooks. Thus the temporal situation of the combinatorial school fluctuates within relatively wide limits, which range from thirty to eighty years, with an average of fifty years. There is what seems to be a tacit agreement on the year of the combinatorial school's emergence or foundation, located around 1780. However, a historical reconstruction based on the assumption that this school lasted thirty years necessarily bears little relation to another that attributes a lifetime of fifty or eighty years to the group. The difficulty is clear: given the deep differences between the historical and cultural contexts of a period of three decades and a period of almost a century, the composition, nature, and characteristics of the combinatorial school cannot be the same in both cases. In other words, each of these historical reconstructions deals with a different object of study, ambiguously referred to by the same term of "combinatorial school", and this ambiguity necessarily brings confusion to our comprehension of the subject.

The theory elaborated by the combinatorial school goes under the name of "combinatorial analysis". In this case, there is absolute unanimity among scholars as to the nature of this theory, which focuses on studying the elementary operations of permutations and combinations in order to then apply them to solve problems belonging to the field of mathematical analysis. Three different terms are currently used to make reference to the combinatorial school's theory in the literature: "combinatorial analysis" (*kominatorische Analysis*), "theory of combinations" (*Kombinationslehre*), and "combinatorics" (*Kombinatorik*). Gerhardt and Netto employ all of them indistinctly (Gerhardt 1877; Netto 1908). When German is not the original language of the historical study, there is a predominant use of the first two terms. For instance, Eric Temple Bell adopts the term "combinatorial analysis", while Schubring and Heuser prefer "theory of combinations" or "combination theory" (Bell 1940, 290–291; Schubring 1996a, 370; Heuser 1996, 48–49). Schubring also uses "combinatorial analysis" when he writes in French (Schubring 1990, 98). Indeed, it seems clear that these terms are taken as synonyms in the relevant literature. Let's now examine in more detail what is involved in the combinatorial analysis according to the relevant literature.

According to a consensus reached by most of the scholars, the combinatorial school advanced the idea that the multinomial theorem is the ultimate principle of mathematical analysis. The most common explanation for this conviction of the combinatorial school is the increasing importance of the binomial theorem in mathematics during the eighteenth century (Gerhardt 1877; Netto 1908). In this sense, the conviction of the combinatorial school is regarded as an attempt to reach more general mathematical results by enlarging the domain of objects subject to the theorem from binomials to any polynomial. However, it is possible to find more adventurous explanations. For Séguin, there is no mathematical motivation behind the combinatorial school's choice of postulating the multinomial theorem as the foundation of analysis, but a philosophical and cultural contextualized motivation (Séguin 2005). Séguin claims that the cultural atmosphere of the time dominated by German philosophers that began to build systems of knowledge based on a single, ultimate principle influenced Hindenburg and lured him to believe that a similar principle should exist within the realm of mathematics. Séguin even dares to affirm without any kind of historical evidence that it was Fichte's philosophy that directly influenced Hindenburg.

However, there have been some dissenting voices, too, about which mathematical proposition played the role of the ultimate principle of analysis for the combinatorial school. At the end of the nineteenth century, Karl Fink seems to suggest that the members of the combinatorial school, except for Hindenburg, considered the binomial theorem as the most important mathematical proposition in analysis, but it is true that Fink's text remains somehow ambiguous on this question (Fink 1890, 115). On the contrary, Sebestik explicitly says that, for the combinatorial school, the "key element" of the entire field of analysis was Taylor's theorem (Sebestik 1992, 84–85). Although Sebestik does not elaborate on this assertion at all, his idea could be argued in two different ways. One could say that Taylor's theorem was considered as the most fundamental theorem in analysis by the combinatorial school because of its interest in problems of reversing series, in which Taylor's theorem plays a central role. Or one could simply say that this assertion was explicitly formulated by Bürmann. Anyway, what is really important at this point is to emphasize the fact that one can find incompatible historical interpretations about one of the crucial issues that deeply impinges on the understanding and explanation of what the German combinatorial analysis was.

An elementary theory of combinations, its application to mathematical analysis, and a fundamental principle represent the constitutive elements of the German combinatorial analysis developed by the combinatorial school, according to the existing historical reconstructions. This general picture, shared, as far as we know, by all researchers that work on the subject, sets the stage for a more general interpretation about the intellectual commitments made by the members of the combinatoiral school regarding the way in which they understood mathematical knowledge. The general conception of mathematics attributed to this school has been associated with some kind of "formalism", as already noted by Gerhardt and Netto (Gerhardt 1877, 205; Netto 1908, 201). But it is possible to distinguish between two different interpretative variants of what this formalism means. The first rests on Gerhardt's and Netto's works and affirms that the combinatorial school conceived mathematics as a general science of formulae, in which every problem can be reduce to the problem of transforming a given formula into another by formal manipulations. In this sense, mathematical analysis, which depends on the theory of series for the members of the combinatorial school, can be seen as a theory focused on the formal transformations of finite and infinite series by means of combinations, as pointed out by Ferreirós Domínguez 1993, 37–38, 1999, 11–13. On the other hand, the second interpretative variant does not originate from a historical narration, either that of Gerhardt and Netto or any other one. It rests instead on some texts written during the first half of the nineteenth century in which the topics discussed and their treatment are close to the combinatorial school's ideas, but in which the theory of combinations is presented as a sort of syntax. From this point of view, the combinatorial analysis can be regarded as a formal syntax that provides the rules for manipulating the symbols of mathematical theories. In this second interpretation, the formalism of the combinatorial school points to the formulation of a common symbolic grammar for mathematical analysis, while in the first interpretation there is no common grammar, but local manipulations of symbols according to algorithms formulated for a given specific case. The syntactic interpretation can be found in (Martin 1996b, 83–84, 1996a, 201; Schubring 1996b, 66). In particular, Schubring claims that, under the influence of the combinatorial school, Jacob Friedrich Fries (1773–1843) called "*Syntaktik*" the most fundamental branch of mathematics. Later, Pulte reproduces Schubring's interpretation about Fries and adds that Fries's conception also results from the influence of Kant's philosophy (Pulte 2006, 117).

In the literature, the combinatorial school's theory has been associated with two different scientific programs, one dating from the seventeenth century and the other from the eighteenth century. This latter program is now known under the name of "algebraic analysis", and the idea that there exists a theoretical relation between this program and the combinatorial school's work had not been formulated until recently. No historical reconstruction carried out in the nineteenth century or in the first half of the twentieth century proposes such an interpretation of the combinatorial school's work. It is worth saying that this school has always been associated with Joseph-Louis Lagrange (1736–1813), who is considered to be the first proponent of the algebraic analysis, but this association was traditionally based on the belief that certain aspects of the combinatorial school's manipulations of infinite series have their roots in Lagrange's thought, in particular those concerned with reversion of series, as can be seen for example in (Reiff 1889, 148–149). On the contrary, during the two last decades of the twentieth century, some historians have advanced the hypothesis that the most general objective of the German combinatorial analysis coincides with that of the algebraic analysis (see, for example, Friedelmeyer 1997, 17; Jahnke 1990, 169–232; Laugwitz 1996, 57–63; Schubring 2005, 562). In these historical narratives, the algebraic analysis is characterized as the Lagrangian project of founding differential and integral calculus on algebraic bases, thus excluding all problematic metaphysical assumptions inherent in questionable concepts, as the concept of "infinitesimal". In this order of ideas, under the influence of the Lagrangian project, the German combinatorial school pursued, according to these narratives, the goal of founding calculus on algebra, with the difference being that algebra was supposed to obey combinatorial laws.

On the other hand, the second program is that of the "universal characteristic" or the "*ars combinatoria*" suggested by Gottfried Wilhelm Leibniz (1646–1716) during the second half of the seventeenth century. In this case, practically every single historian that has worked on the subject is convinced that the German combinatorial analysis is closely related to Leibniz's program (see, for example, Friedelmeyer 1997, 17; Gerhardt 1877, 203; Netto 1908, 201; Peckhaus 1997, 238–240; Schubring 1990, 98; Séguin 2005; Thiel 1995, 20). According to these historical reconstructions, the Leibnizian project consisted in developing a symbolic calculus on the basis of the theory of combinations and whose scope extends to all human knowledge. The influence exerted by Leibniz's thought on Hindenburg and his colleagues was so important that, according to these historians, the German combinatorial analysis can be seen essentially as an attempt to finally bring Leibniz's project into existence. Nevertheless, these scholars recognize that the plan of the German combinatorial analysis is less ambitious than that of the Leibnizian universal characteristic inasmuch as its scope is restricted to mathematics.

In fact, the German combinatorial analysis has also been related to a major cultural movement of the time by one of the scholars mentioned above. Séguin ventures the hypothesis that the early German Romanticism impacted the shape and composition of Hindenburg's mathematical work, principally through Fichte's philosophical thought (Séguin 2006, 64–72). Thus, the nature and characteristics of Hindenburg's mathematics were determined by the nascent romantic ideals, and, at the same time, his mathematics transmitted these ideals to other exponents of the German culture, notably to Friedrich von Hardenberg, Novalis (1772–1801). However, although Novalis's fascination with the German combinatorial analysis is well established (Dyck 1960; Hannah 1981; Jahnke 1991; Margantin 1999; Schefer 2005), Séguin's hypothesis does not seem to be supported by any kind of confirming evidence. It is well known that Novalis wrote some lines about Hindenburg's theory (Novalis 1993), but it cannot be inferred from that that Hindenburg was a Romantic. In fact, it is well known that Novalis was fascinated with new scientific discoveries in general, but this does not mean that such discoveries belong to the Romantic movement. Besides the fact that Séguin does not provide any textual documentation supporting his hypothesis about Hindenburg's Romanticism, it should be noted that this hypothesis depends on another questionable hypothesis, namely, that about Fichte's influence on Hindenburg's work which also rests on nothing more than vague assumptions. In sum, these hypotheses should be considered with caution.

Besides these general interpretations about the combinatorial school and its theory, the historians of the end of the nineteenth century and beginning of the twentieth expressed negative opinions of this period in the history of German mathematics. Gerhardt characterized it as "stagnant" (*stehend*) and Netto as "strange" (*merkwürdig*) (Gerhardt 1877, 201; Netto 1908, 201). Elaborating on Gerhardt's remark, Gino Loria described the mathematical works of the combinatorial school's members as "soporific" (*soporiferi lavori*) and placed on them the entire responsibility for the "lethargy" that had supposedly paralyzed German mathematics for half a century (Loria 1888, 335). It should be noted, however, that Loria provided a rather neutral description of this school in his capital work on the history of mathematics (Loria 1933, 349–351). In 1916, Lorey describes the theory of the combinatorial school as "an old combinatorics" (*alte Kombinatorik*) forgotten because of its "sterile subtleties" (*unfruchtbaren Spitzfindigkeiten*) that distorted the spirit of the "original and very interesting and important combinatorial problems" (*ursprünglich mathematisch gewiß sehr interessanten und wichtigen kombinatorischen Probleme*) of Leibniz, Moivre, Bernoulli, and Euler (Lorey 1916, 29–30). And he talks about Hindenburg as a "narrow-minded" (*Engherzigkeit*) man blinded by his obsession with combinatorics, and thus unable to see beyond his own mathematical horizon (Lorey 1916, 28, footnote 5). According to Bell, this historical episode was nothing but a "ridiculous interlude" in which the "patriotic disciples" of Leibniz (i.e., the members of the combinatorial school), lacking the talent for mathematical sciences, failed to understand the deep insights of the Leibnizian universal characteristic (Bell 1940, 290–291). Florian Cajori was convinced that the combinatorial school produced nothing worthy of esteem and History rewards it with deserved oblivion (Cajori 1919a, 231, 373, 1892, 1). He went as far as to mock their manipulations of divergent series as though the members of the combinatorial school had had the obligation to know beforehand Cauchy's results (Cajori 1890, 362–363). In a more moderated tone, Felix Klein refused to devote any part of his acclaimed lectures on the state of mathematics in the nineteenth century to the combinatorial school under the pretext that its ideas can rather be seen as a "ramification of old scientific trends" than the "beginning of a new scientific development" (Klein 1967, 113). Perhaps this recurrent stigmatization has played a dissuasive role as a deterring factor for scholars to pursue more detailed studies of this historical period. In any case, it was not until the end of the twentieth century that a fresh wave of historical studies appeared, in which the combinatorial school's ideas have been reevaluated by placing them in the mathematical contexts to which they belong. For instance, Hans Niels Jahnke analyzes the impact that this school had on the design and planning of education in the Germany of Humboldt's day (Jahnke 1990); Marco Panza analyzes it in the general framework of the development of mathematical analysis during the eighteenth century (Panza 1992, vol. 2, 651–659); Maarten Bullynck proposes a new approach centered on the components of Hindenburg's work that are related to the history of number theory (Bullynck 2006, 235–271); Gert Schubring shows that it is even possible to objectively address the topic of the strong opposition to the combinatorial school's ideas that emerged in the nineteenth century, without disqualifying them on the basis of personal opinions (Schubring 2007).

In sum, the existing reconstructions of the history of the combinatorial school do not always correspond to a coherent and consistent narrative. Some tensions and disagreements can be detected among historians on elementary matters concerning, for instance, the question of who was the true founder of the combinatorial school, or the more relevant question of what was the fundamental mathematical proposition on which the combinatorial analysis should be based. Nevertheless, there are common traits linking these historical interpretations. It seems clear that they explain the formation of the combinatorial school as the result of a conscious, voluntary decision made by several mathematicians with the aim of creating a scientific team. The research team is focused on developing the theory of combinations, which can also be called "combinatorics" or "combinatorial analysis", since all of these terms are synonyms. The theory of combinations developed by this school is characterized by its generality, its formalism, and its syntactic symbolism. Furthermore, there is no doubt that the general conception of this theory comes from the Leibnizian project of the universal characteristic, and its aims are determined by the aims of Leibniz's project, though Lagrange's algebraic analysis has exerted some influence on these goals. These common traits constitute what will be called here the "traditional interpretation" of our subject matter.

1.2 An Alternative Historical Interpretation of the German Combinatorial Analysis

As our research advanced, we were forced gradually to abandon the traditional historical interpretation of the combinatorial school, and its work, summarized above. This traditional interpretation was in fact our starting point, and our goal consisted in consolidating and completing it in a single volume. However, we realized that the main ideas on which it rests do not stand up to close scrutiny in the light of the available primary sources. Because of this, we could not rely on these ideas and we were compelled to rethink our goals and rewrite the history of the combinatorial analysis from a new viewpoint. This new viewpoint comes from our analysis of primary sources, which allowed us to identify what one could call working assumptions, even if they are rather the result of our research than a general framework externally imposed upon our historical narrative. In what follows, we will present the assumptions on which our historical reconstruction has been built, assumptions discussed in detail throughout this book.

The subject matter of our historical reconstruction is the German combinatorial analysis at the end of the eighteenth century. The general aim of this study is to explain, from a historical point of view, what this mathematical theory was according to its creators. There is no doubt that the history of the combinatorial analysis walks hand in hand with that of the combinatorial school, but the historical reconstruction of the first does not presuppose the historical reconstruction of the second. The members of the German combinatorial school were related to specific academic environments, worked in particular academic institutions, each with their own defining features, they communicated their ideas by different means and interacted with various scientific communities, and each of them had his own intellectual interests and incentives, other than the combinatorial analysis. Each of these elements should be carefully analyzed in a study devoted to the German combinatorial school. In this book, since the combinatorial school is not the object of our historical reconstruction, these elements will be only discussed when necessary, that is, when their discussion is pertinent to provide further clarification around our main subject: the German combinatorial analysis. Thus, these pages tell the story of the emergence of a scientific theory, of its gradual conformation, and of its death.

In our research, we have identified two main historical facts concerning the history of the combinatorial analysis. For the purposes of this introductory section, we will express these historical facts in terms of working assumptions, in particular because their status of "historical facts" has not yet been established. They are incompatible with the traditional interpretation, but we hope that we will be able to show their historical adequacy. The first assumption concerns the historiographical category of "combinatorial school", and the second one that of "combinatorial analysis". They can be formulated as follows:

1. The historiographical category of "combinatorial school" makes reference to a group of mathematicians that worked together to achieve a specific common objective, like a scientific research team. This intellectual association was formed around 1794 and progressively disintegrated from 1800.
2. The historiographical category of "German combinatorial analysis", or simply "combinatorial analysis", makes reference to the mathematical system that the combinatorial school wanted to develop. This system was never completed, but a scientific research program was written by Hindenburg in which the main elements of the system are described. Its first formulation in 1781 appears before the formation of the group. In 1781, this research program is a personal project of Hindenburg and it is composed of four objectives: (a) Organizing mathematical knowledge as a scientific system, (b) Improving and developing combinatorial methods, (c) Creating new mathematical notations in order to express the results of the system, (d) Applying combinatorial methods to the solution of problems belonging to the theory of series. Hindenburg's personal project becomes the scientific research program of the combinatorial school.

According to the first assumption, the period of activity of the combinatorial school has been considerably reduced with respect to the temporal limits proposed by other historians.

This apparent discrepancy can be easily explained. The term “combinatorial school” has been inconsistently and ambiguously used in the historical literature. It has been used to designate, at the same time, both a scientific team and a mathematical current of thought. This confusion has led historians to multiply the temporal limits attributed to the existence of the combinatorial school and to propose excessive and incompatible historical datings, ranging from three decades to almost a century. However, the existence of a current of thought does not necessarily rely on the existence of a particular scientific team. In our view, it is possible to identify the existence of a “combinatorial current of thought” after 1800 in German mathematics, but at this time the combinatorial school no longer exists as a team. Thus, we propose to exclusively use the term “combinatorial school” to designate the group of mathematicians to which point (1) makes reference. To decide who was a member of this school, we propose the criterion that a scholar belonged to the combinatorial school if he contributed to the achievement of the scientific program described in point (2).

Assumption (2) allows us to pave the way for properly characterizing the German combinatorial analysis. With the benefit of hindsight, we know that the German combinatorial analysis should be considered as a project in progress or as an unfinished theory, despite the effort and energy of its creators to complete it and despite the partial goals achieved. This peculiarity of the German combinatorial analysis makes it harder to tackle the problem of its historical reconstruction, especially when it comes to unravel the results or the writings that agree with this program from those that were presented under the name of “combinatorial analysis” but that pursued different goals. Naturally, this question will be carefully discussed throughout the book. But, even before discussing this point in detail, if our historical interpretation is right, it is impossible to reduce the German combinatorial analysis to the theory of combinations. Certainly, the improvement of this theory is listed in point (b) of the combinatorial school’s program, but this is only one point of the program. Thus, the term “German combinatorial analysis” is not synonymous with “theory of combinations”. In this book, the terms “combinatorics” and “theory of combinations” will be considered synonymous, and, on the other hand, the expression “the combinatorial analysis” always refers to the German combinatorial analysis, unless otherwise indicated.

Other historical facts were discovered during this research, which did not conform to the traditional picture of the combinatorial school. Here again, their status of “historical facts” will be established later, when supporting historical data will be presented and discussed. For the purposes of this section, they are formulated here as subsidiary assumptions, and they complement our two main assumptions. Concerning assumption (1), we will defend the idea that the formation of the combinatorial school was not the result of a conscious, voluntary decision made by a group of scientists, and therefore there is no founder of this school. As remarked in assumption (2), Hindenburg’s personal research project became the research program of the combinatorial school, but this does not imply that Hindenburg founded a scientific team. Similarly, it will be seen that nobody took a conscious decision to disband the research team.

Concerning assumption (2), we have identify three subsidiary assumptions. The first one contradicts the only point of agreement reached by almost all historians in their historical reconstructions, and thus this subsidiary assumption is probably the most radical one in our historical narrative. We will argue, on the basis of textual evidence, that the combinatorial analysis does not have its roots in Leibniz's universal characteristic or *ars combinatoria*. In our research, we have observed, moreover, not only that the combinatorial analysis does not find its origins in Leibniz's project, but also that Hindenburg's project cannot be considered as an attempt to accomplish it since, in fact, it bears no relationship to Leibniz's ideas. The German combinatorial analysis was originally related to the Newtonian tradition of the theory of series, and was later explicitly founded by Hindenburg on the work of Abraham de Moivre (1667–1754). It is worth noting that a few studies have reconstructed this part of history without involving Leibniz; for instance, Panza stresses the importance of Moivre's work for the combinatorial school and leaves Leibniz aside (Panza 1992, vol. 2, 651–652).

We have also found that the combinatorial analysis was conceived as a foundational program of mathematical analysis. Indeed, it is possible to find explicit assertions in this regard in Hindenburg's works. It is more difficult to decide whether this foundation concerns the analysis of the finite or the analysis of the infinite, i.e. whether Hindenburg's project can be seen as an attempt to found differential calculus on less controversial bases. In our view, Hindenburg aimed to rebuild the theory of series on a firmer foundation, and thus he intended to reformulate the analysis of the finite by means of combinatorial concepts and methods, which should then be used to reconstruct the theory of series. In this sense, his program aimed to provide mathematical analysis with an alternative that may be less controversial that differential calculus. Hindenburg thought that the multinomial theorem was the link between the combinatorial reconstruction of the analysis of the finite and the theory of series, and as a consequence this theorem should be considered as the mathematical fundamental proposition on which the mathematical system mentioned in point ($2a$) rests.

Our third subsidiary assumption regarding point (2) states that the combinatorial analysis vanished around 1800. Needless to say, this does not mean that the influence of the combinatorial school's work on German mathematics stopped all of the sudden in 1800. On the contrary, its influence can be perceived far beyond the first half of the nineteenth century, and this is what can be called a combinatorial current of thought in German mathematics during the nineteenth century. Since the accomplishment of the project objectives that characterized the German combinatorial analysis cannot be identified as a goal to be pursued by the mathematicians belonging to this combinatorial current of thought, it can be affirmed that the work of those mathematicians did not contribute to the construction of the German combinatorial analysis, which, hence, disappeared at the turn of the eighteenth and nineteenth centuries.

The German combinatorial analysis is an example of the different programs appeared during the second half of the eighteenth century in order to better explain the relatively new field of mathematics called "analysis". It was an attempt to understand the concepts and methods of this new, mysterious field in terms of the mathematical methods and concepts available at the time. Mocking this program, or any other of the period, on the basis of personal preferences just shows incapacity to discern how science evolves and how historical processes are constituted. In this book, we have tried to understand and explain a historical period of the German scientific thought, as well as the mechanisms involved in the creation of a mathematical theory.

1.3 Content of the Book

This book is composed of five chapters, without counting the introduction and the epilogue. Chapter 2 intends to give a picture of the conceptual genealogy of the German combinatorial analysis, which is of paramount importance to understand its genesis and further development. This conceptual genealogy is not so much a sort of conceptual background lying behind the combinatorial school's work as it is a kind of conceptual branching structure impacting on the evolution of the combinatorial analysis at specific times of its history. Although all the events included in Chap. 2 are chronologically prior to the combinatorial school's works, they are incorporated into the formation of the combinatorial analysis's identity in different time periods, and some of them keep coming back modifying the course of history and the theoretical orientation of the program. These events constitute, certainly, the past of the combinatorial analysis, but they cannot be taken as a unified background which remains homogeneous and unproblematic no matter what happens in the future. The different branches of this conceptual genealogy disrupt the present of the combinatorial analysis at definite moments, causing sensible inflections in its configuration. To better capture these inflections, it would have been convenient if these disruptive events had been incorporated in our narrative in accordance with the moments in which they were taken into account by the members of the combinatorial school. However, from a methodological point of view, it is a better choice to bringing them all together in a single chapter in order to avoid frequent and long digressions in our discourse. Thus, these events form the subject matter of the second chapter.

More precisely, Chap. 2 first presents a historical reconstruction of the binomial theorem from its first formulation by Isaac Newton (1642–1727) up to 1781, that is to say, up until the first draft of Hindenburg's project appeared. Special attention will be paid to the nature of the methods used to demonstrate this theorem, emphasizing its central place in the general conception of mathematical analysis at the time. In particular, it will be seen that the enormous amount of proofs that were carried out during this period reflects the vivid interest of mathematicians from all Europe in understanding what would be the best method to correctly justify the binomial theorem, and at the same time it reflects the great difficulty of finding a correct answer. This also means that this subject had remained in force for a century and it was still a topical theme in the 1770s, as Hindenburg turned his attention to it. Later in the chapter, a similar historical reconstruction of the multinomial theorem is presented for the same period of time. Particular emphasis is placed on the way the problem of raising a multinomial to a given power was stated and on the different methods used to prove it. It is of particular interest for us to understand that the research on the multinomial theorem was considered in general as a secondary topic, whose importance and relevance for mathematics depended on the binomial theorem and it was seen, at best, as a way of fully analyzing the consequences of the binomial theorem. Nevertheless, the research on the multinomial theorem gave rise to a new topic that was profusely discussed during the eighteenth century, namely, that of reversing series. Although no historical reconstruction of the theorem on reversion of series has been included in this chapter, we will examine some elements of its history that are related to the combinatorial analysis.

Chapter 3 is organized around the idea, completely neglected and unexplored to date, that the origin of the German combinatorial analysis is found in Hindenburg's early mathematical works and in his mathematical training. Concerning this latter point, it will be argued that Borz, one of his close teachers, had intended to show that differential methods could be successfully replaced by algebraic ones, at least in certain domains of mathematical inquiry. Hindenburg collaborated with his teacher on this work around 1769, and it is possible that this collaboration planted in him the belief that methods based on the analysis of the finite could be developed as an alternative to differential calculus. This could explain why Hindenburg never used differential methods in his writings and why he strove his entire life to create new mathematical methods similar to algebraic ones. In our view, this is a better explanation of his "algebraic analysis" than the traditional hypothesis that Lagrange's algebraic analysis influenced him at an unknown time by unknown means, since our explanation rests, at least, on existing textual documents.

Concerning Hindenburg's early mathematical work, it will be argued that Hindenburg's interest in mathematical tables during the 1770s convinced him that, on the one hand, mathematical tables were an essential tool to drive progress in mathematics, and, on the other hand, they also convinced him that there is a close relationship between arithmetic and combinatorics. These two beliefs inspired him to approach, for the first time, the study of a particular problem of the theory of series from a new perspective, namely, that of finding a non-recursive formula for calculating the general coefficient of the power series expansion of a multinomial raised to a rational power. Later, he generalized his innovative method to deal with a relatively long list of problems belonging to the theory of series. It will be seen that, through this generalization, the multinomial theorem was overshadowed by the combinatorial method itself. Then, it will be analyzed how Hindenburg wrote the first draft of his research program on the basis of his new combinatorial method, and how this draft can be considered as the first exposition of what is known as the German combinatorial analysis.

Chapter 4 deals with the growing period of the combinatorial analysis from around 1792 to 1798. A main aim of this chapter is to carefully describe the process that led to the formation of the German combinatorial school. It will be discussed the idea that three main factors are involved in this process. First, Hindenburg forged a new generation of students that adopted his mathematical techniques and his general conception of mathematics. Second, he had at his disposal a scientific journal, founded by himself, that served as an optimal vehicle to transmit his combinatorial views on mathematics. Third, Hindenburg's scientific program was plagiarized in 1792, which triggered an intellectual fightback against the plagiarist, coordinated by Hindenburg's former students. The German combinatorial school raised from this intellectual conflict, in which it is possible to identify the characteristic organization of a scientific team. In particular, a combinatorial non-recursive formula for reversing series, established and demonstrated by Hindenburg's former students, was stolen by the plagiarist. Later, some members of the combinatorial school proved that this formula was equivalent to a formula established by Lagrange on the basis of differential calculus. It will be argued that this equivalence could be interpreted by the combinatorial school as a corroborative evidence that differential calculus could be replaced by combinatorial methods.

Another main objective of Chap. 4 is to describe the process of consolidation of the combinatorial school's scientific research program. It will be seen that the consolidation of the German combinatorial analysis depends on two factors. On the one hand, Hindenburg improved his combinatorial methods by modifying the mathematical tables on which they were based. On the other hand, this improvement led Hindenburg to reevaluate Moivre's work in the light of his new mathematical tables. As a result, Hindenburg claimed that Moivre had already implicitly used those combinatorial methods when proving the multinomial theorem. It will be shown that this reevaluation of Moivre's thought was the key to consolidate the German combinatorial analysis inasmuch as it led to the formulation of the theoretical bases on which the combinatorial school's mathematical system of science should rest. There are two theoretical bases. First, the multinomial theorem becomes the fundamental proposition in the field of the theory of series, and therefore in mathematical analysis. Second, the entire system should be based on a new theory of combinations that can be seen, from today's perspective, as a sort of abstract algebra. It was Hindenburg who sketched the ideas behind this new theory of combinations, but no member of the combinatorial school, including Hindenburg, ever undertook to further develop this theory.

In Chap. 5, it will be discussed the idea that the German combinatorial analysis disappeared as a result of four main factors. First, no member of the combinatorial school was ever able to show that the multinomial theorem was effectively the most fundamental proposition in mathematical analysis, particularly because no one was ever able to prove this theorem independently of the binomial theorem and using exclusively the new theory of combinations of the school. This fact cast doubt on whether the multinomial theorem could be regarded as a legitimate foundation of a system. Second, the binomial theorem gained importance in the texts related to the combinatorial analysis that appeared at the beginning of the nineteenth century, and thus these texts did not convey anymore the image of the combinatorial analysis as a well-structured system of science. Third, it is possible to identify the emergence of rival theories at the end of the eighteenth and the beginning of the nineteenth century, which were systematically confused with the German combinatorial analysis, but their goals and general conception of mathematics were significantly different from those of the combinatorial school's project. Fourth, some mathematicians disagreed with the idea of the combinatorial nature of mathematical analysis, but they praised Hindenburg's mathematical results and tried to eliminate their combinatorial elements.

In Chap. 6, the historical consequences of Chap. 5 are discussed in detail. It will be seen that the dissolution of the combinatorial school in the first decade of the nineteenth century led to the production of isolated research, instead of the collaborative research that characterized the work of this school. Particular attention will be paid to the reconstruction of what has been called above the combinatorial current of thought of the nineteenth century inspired by the ideas of the combinatorial school. It will be argued that this current is divided into four main branches. Each branch is characterized by a particular position about the role that the theory of combinations plays in mathematical analysis, but this does not mean that the authors of a given branch hold the same conception of mathematics. A brief final section of Chap. 6 deals with the German combinatorial analysis in non-German-speaking countries.

Given the nature of the subject matter, the discussion of mathematical symbols occupies an important place throughout the book. It would be necessary to distinguish between use and mention of a symbol by means of quotation marks. However, in an effort to lighten the text, quotation marks will be omitted when a mathematical symbol is mentioned, the context being enough to avoid any possible confusion.

As a final remark, we note that all the translations of the primary sources quoted in this book are ours, and we have decided to complement them by including the corresponding original texts in footnotes.

A History of the Binomial and Multinomial Theorems

2

2.1 The Binomial Theorem

Some historians hold that the history of the binomial theorem goes back as far as the *Elements* of Euclid (fl.300 BC) by interpreting proposition 4 of Book II as a particular case of squaring $(a + b)$, where a and b are segments (Coolidge 1949, 147; Heath 1908, 379). In the same vein, there are historical interpretations of the work of Heron of Alexandria according to which Heron used this theorem in the context of extracting cube roots, and something similar applies to the work of Theon of Alexandria (ca. 335–ca. 405) (Bourbaki 1984, 95 note; Deslauriers and Dubuc 1996; Heron of Alexandria 1903, 176–179; Rome 1936, 473; Nikolantonakis and Yao-Yong 2011, 172–175).

This kind of interpretations is not restricted to the Greco-Latin tradition. According to some scholars, in the Chinese tradition a square root algorithm similar to Heron's can be found in the *Nine Chapters on the Mathematical Art*, and it was later reformulated by Liu Hui, in the third century AD, in order to calculate both square and cube roots (Deslauriers and Dubuc 1996, 178–180; Martzloff 1988, 215–216). In a related topic, it has been also argued that Jia Xian invented the arithmetical triangle in the eleventh century to solve certain root extraction problems (Chemla 1994, 210–231; Lay-Yong 1980, 416–423). In India, some applications of an early version of the binomial theorem have been attributed to Āryabhaṭa (476–550 AD) (Ayyangar 1926, 172–173; Datta and Singh 1962, 175; Prakash 1968, 167–168; Raju 2007, 128–130). It has been argued that, in Arabic mathematics, al-Karajī (953–1029) invented a method for calculating any positive integer power of a binomial expression, which eventually led him to the discovery of the arithmetical triangle, and other Arabic mathematicians, such as al-Khayyām (1048–1131), al-Zanjānī (fl. 1257), al-Tūsī (1201–1274), and Ibn al-Bannā (1256–1321), consolidated a vigorous field of study around the work of al-Karajī (Djebbar and Rashed 1981; Djebbar 2005, 86 ff.; Rashed 1972, 1984; Jushkevich 1976, 80; Yadegari 1980).

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In the European tradition, no systematic study about the arithmetical triangle was carried out until the second half of the sixteenth century. Although Jordanus de Nemore (fl. 1225) discussed the procedure for generating the triangular pattern in his *De arithmetica*, the earliest printed reproduction of the pattern itself appeared only in 1527, decorating the frontispiece of Petrus Apian's arithmetical treatise (Apian 1527; Hughes 1989). And here too, as in the other mathematical traditions, the main application of the arithmetical triangle concerned the solution of root extraction problems. That is the case for Michael Stifel (1487–1567) and Johann Scheubel (1494–1570) in Germany, and for Jacques Peletier (1517–1583) in France (Peletier 1549; Scheubel 1545; Stifel 1544, 44 verso). But in Italy Niccolò Tartaglia (1500–1557) and Gerolamo Cardano (1501–1576) examined the properties of the arithmetical triangle in the context of the theory of combinations (Cardano 1570, 131, 185; Tartaglia 1556, vol. 1: 69 verso). There is no doubt that the long history of advances in this field culminated in the conception of the *Traité du triangle arithmétique*, printed by Pascal in 1654 but published until 1665, in which Pascal exposed the relation between triangular numbers, combinatorial numbers, and binomial coefficients (Boyer 1950; Edwards 1987; Pascal 1665). Indeed, his treatise is reputed to contain the first proof of the binomial theorem for positive integral exponents.

Although the algebraic interpretation of ancient mathematics on which depends the ancient history of the binomial theorem can be deemed anachronistic and has been questioned by several scholars, this is not the place to try to decide that question. Suffice it to say that it is a plausible way to interpret them, but that is not the perspective assumed in this chapter. For the purposes of this section, the history of the binomial theorem begins in the seventeenth century when Newton invented it and it was named after him in the eighteenth century. Moreover, the principal aim of this section is not to give an account of Newton's theorem for its own sake, but rather for its historical significance in determining the way in which the binomial theorem was conceived during the eighteenth century. Particular attention will be paid to the reception of Newton's theorem in eighteenth century Germany.

2.1.1 Newton's Theorem

The binomial theorem was discovered by Isaac Newton (1642–1727) as a result of his early mathematical research on the method of quadrature developed by John Wallis (1616–1703) and on the method of tangents of René Descartes (1596–1650). His early mathematical research has been analyzed in excellent monographs and critical editions of his work, which can be consulted with respect to the genesis of Newton's theorem (Panza 1995, 2005; Whiteside 1961, 1967). The German combinatorial school had no access to Newton's early research, and, for the members of this school, the history of the binomial theorem begins with the *Epistola prior* in which Newton states his theorem. This then will be the starting point for the story we are interested in here.

On 12 May 1676, Gottfried Wilhelm Leibniz (1646–1716) addressed a letter to Henry Oldenburg (1618–1677) in which he asks for information regarding the power series expansions of the sine and arcsine functions (Leibniz 1676a). Leibniz found out about these expansions from Georg Mohr (1640–1697) and he was curious about how they have been constructed. The expansions had been calculated by Newton. For instance, the power series expansion of arcsine can be found in one of Newton's papers collected by Whiteside (Newton 1665a, 110). Oldenburg informed Newton about Leibniz's interest and strongly advised him to make public his method, otherwise he might lose priority for his idea. Shortly afterwards, on 13 June 1676, Newton addressed to Oldenburg the so-called *Epistola prior*, which was transmitted to Leibniz on 26 June 1676 and which contains the first statement of the binomial theorem for rational exponents. The theorem is enunciated as follows (Newton 1676b, 49, or Turnbull 1960, 21–54):

> It is much easier to extract a root by this *theorem*:
>
> $$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.,$$
>
> where $P + PQ$ is the quantity whose root, or even any power, or the root of a [given] power, is to be found. P is the first term of this quantity, and Q stands for the other terms divided by the first. And $\frac{m}{n}$ is the numerical index of the power $P + PQ$, whether it is an integer power, or whether it is, so to speak, a fractional one, or whether it is a positive or negative one.[1]

The alphabetical notation consisting in capital letters is a Newtonian innovation, and means: $A = P^{\frac{m}{n}}$, $B = \frac{m}{n}AQ$, $C = \frac{m-n}{2n}BQ$, and so on. This notation works as an abbreviation for the more complex expression of binomial coefficients: $1, \frac{m}{n}, \frac{m}{n} \times \frac{m-n}{2n}, \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}$, and so on.

To show what kind of problems can be solved with the help of his theorem, Newton provides a lot of examples the aim of which is, more precisely, to illustrate the theoretical and practical importance of his discovery. We will consider here only one example of each kind of problem.

Not surprisingly, the first application regards the problem of extracting roots. In order to calculate the fifth root of the trinomial $c^5 + c^4x - x^5$, Newton proposes two solutions depending on whether the terms of the trinomial are grouped from right to left so as to

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[1] Sed Extractiones Radicum multum abbreviantur per hoc *Theorema*.

$$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

Ubi $P + PQ$ significat Quantitatem cujus Radix, vel etiam Dimensio quaevis, vel Radix Dimensionis, investiganda est. P, Primum Terminum quantitatis ejus; Q, reliquos terminos divisos per primum. Et $\frac{m}{n}$, numeralem Indicem dimensionis ipsius $P + PQ$: sive dimensio illa Integra sit, sive (ut ita loquar) Fracta; sive Affirmativa, sive Negativa.

form a binomial, or they are grouped from left to right (Newton 1676b, 50; or Turnbull 1960, 22). In the first case, the terms $c^4x - x^5$ are taken together, and one obtains:

$$\sqrt{5 : c^5 + c^4x - x^5 :} = \overline{c^5 + c^4x - x^5}|^{\frac{1}{5}} = c + \frac{c^4x - x^5}{5c^4} + \frac{-2c^8xx + 4c^4x^6 - 2x^{10}}{25c^9} + \&c.,$$

for $m = 1, n = 5, P = c^5$, and $Q = \frac{c^4x - x^5}{c^5}$. In the second case one gets:

$$\sqrt{5 : c^5 + c^4x - x^5 :} = \overline{c^5 + c^4x - x^5}|^{\frac{1}{5}} = -x + \frac{c^4x - c^5}{5x^4} + \frac{2c^8xx + 4c^9x + c^{10}}{25x^9} + \&c.,$$

for $m = 1, n = 5, P = -x^5$, and $Q = \frac{c^4x + c^5}{-x^5}$. Newton shows in this way that the binomial theorem furnishes the basis upon which it is possible to generalize the series expansions from a binomial to any polynomial. As will be seen in this chapter, this idea was decisive in the eighteenth century.

The second application is related to the method of long division. Let be $m = -1$, $n = 1$, $P = d$ and $Q = \frac{e}{d}$, then:

$$\frac{1}{d + e} = \overline{d + e}|^{-1} = \overline{d + e}|^{-\frac{1}{1}} = \frac{1}{d} - \frac{e}{dd} + \frac{ee}{d^3} - \frac{e^3}{d^4} + \&c.$$

Newton attached great importance to this example because Nicolaus Mercator (1620?–1687) used, in 1668, the method of long division to expand in series the function $\frac{1}{1+x}$. Then, Mercator obtained the expansion of $\ln(1 + x)$ by means of that of $\frac{1}{1+x}$ (Mercator 1668, 28–34; Naux 1971, vol. 2: 57–67). Mercator's procedure was one of the first to be published concerning the manipulation of infinite series. In 1665, Newton worked on the quadrature of the hyperbola $y = \frac{1}{1+x}$, but he did not explicitly relate his research on this subject to the expansion of $\ln(1 + x)$ (Newton 1665b, 113; Whiteside 1961, 255–259). In 1669, Isaac Barrow (1630–1677) transmitted to him a copy of Mercator's *Logarithmotechnia*, and, apparently, this book was a source of incentive for Newton to go deeper into the study of the theory of series. It seems that this example of application suggests that Newton considered the binomial theorem to be the fundamental basis of Mercator's procedure.

The third example concerns the so-called Newton–Raphson method that Newton applied to approximate the roots of algebraic equations. The method was exposed for the first time in the *De analysi*, and it was simplified by Joseph Raphson (1648?–1715?) in 1690 (Cajori 1911; Kollerstrom 1992; Newton 1711, 8–14; Raphson 1690; Ypma 1995). In the *Epistola prior*, the example consists in finding an approximation for the root of the equation $y^3 - 2y - 5 = 0$. One can propose $y = 2$ as an approximate solution for it, and take $y = 2 + p$ as being the exact solution. By substituting $y = 2 + p$ into the original equation and by applying the binomial theorem, one obtains:

$$\begin{aligned} y^3 &= 8 + 12p + 6pp + p^3, \\ -2y &= -4 - 2p, \\ -5 &= -5, \end{aligned}$$

that is: $-1 + 10p + 6pp + p^3 = 0$. In order to find an approximate solution for this new equation, one can pick $10p - 1 = 0$, that is $p = 0.1$, as the approximation, so that the exact solution will be $p = 0.1 + q$. By substituting $p = 0.1 + q$ into the second equation and by applying the binomial theorem, one obtains another equation: $0.061 + 11.23q + 6.3qq + q^3 = 0$. The exact value of $q = -0.0054 + r$ follows from $11.63q + 0.061 = 0$. Likewise, one can calculate the approximate value of $r = -0.00004852$. Then, an approximate solution for the original equation is given by $y = 2 + 0.1 - 0.0054 - 0.00004852 = 2.09455148$.

A last kind of problem treated in the *Epistola prior* involves the power series expansion of some functions, such as the trigonometric functions arcsine, sine, cosine, and the exponential function. This latter is, however, expanded in a power series by means of a method called, in the eighteenth and nineteenth centuries, "the reversion of series", which was also known as "the inversion of series".

Newton's new method intrigued Leibniz, who asked Newton to comment more thoroughly on how it could be used to obtain Newton's results (Leibniz 1676b, 63). The response to Leibniz's letter is contained in the *Epistola posterior*, addressed to Oldenburg on 24 October 1676, but which was not transmitted to Leibniz until June 1677 (Newton 1676a, 67–70; or Turnbull 1960, 110–129). In the *Epistola posterior*, the first statement of the theorem on reversion of series was formulated in the following two propositions, where $a, b, c, \ldots$ are known coefficients (Newton 1676a, 85):

Proposition 2.1 *Let be $z = ay + byy + cy^3 + dy^4 + ey^5 + \&c.$, then:*

$$y = \frac{z}{a} - \frac{b}{a^3}z^2 + \frac{2bb - ac}{a^5}z^3 + \frac{5abc - 5b^3 - aad}{a^7}z^4 + \frac{3aacc - 21abbc + 6aabd + 14b^4 - a^3e}{a^7}z^5 + \&c.$$

Proposition 2.2 *Let be $z = ay + by^3 + cy^5 + dy^7 + ey^9 + \&c.$, then:*

$$y = \frac{z}{a} - \frac{b}{a^4}z^3 + \frac{3bb - ac}{a^7}z^5 + \frac{8abc - aad - 12b^3}{a^{10}}z^7 + \frac{55b^4 - 55abbc + 10aabd + 5aacc - a^3e}{a^{13}}z^9 + \&c.$$

Newton did not give any proof at all for these propositions, but showed how to calculate theirs coefficients in two particular cases. Concerning Proposition 2.1, he assumed the following series:

$$z = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \&c., \tag{2.1}$$

which corresponds to the power series expansion of $\ln\left(\frac{1}{1-x}\right)$, with the aim of expressing x as function of z. As seen above, one of Newton's examples suggests that the binomial theorem can be considered as the basis of the series expansions of powers of polynomials by virtue of the technique of grouping terms. Thus, in particular, the binomial theorem is the theoretical justification of the following powers of z, even if they were certainly calculated by simple multiplication in practice:

$$\begin{aligned}
z^2 &= x^2 + x^3 + \frac{11}{12}x^4 + \frac{5}{6}x^5 + \&c.,\\
z^3 &= x^3 + \frac{3}{2}x^4 + \frac{7}{4}x^5 + \&c.,\\
z^4 &= x^4 + 2x^5 + \&c.,\\
&\vdots
\end{aligned}$$

Then he eliminated the terms involving the higher powers of x from the right-hand side of Eq. 2.1. To eliminate x^2, one subtract $\frac{1}{2}z^2$ from both sides of Eq. 2.1:

$$z - \frac{1}{2}z^2 = x - \frac{1}{6}x^3 - \frac{5}{24}x^4 + \frac{13}{60}x^5 + \&c.$$

By adding $\frac{1}{6}z^3$ to both sides of this latter equation, one gets:

$$z - \frac{1}{2}z^2 + \frac{1}{6}z^3 = x + \frac{1}{24}x^4 + \frac{3}{40}x^5 + \&c.$$

By subtracting $\frac{1}{24}z^4$ from both sides of this latter equation, one gets:

$$z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 = x + \&c.,$$

and so on. In this way, Newton derived the power series expansion of $1-e^{-z}$. In the second case, corresponding to Proposition 2.2, Newton obtained the power series expansion of the arctangent function using a procedure similar to that just described (Newton 1676a, 84–85).

Even though Newton provides no rigorous justification of any of these propositions, his examples insinuate that the justification of the theorem on the reversion of series rests on the binomial theorem. But on the other hand, such examples do not suffice to establish the law of formation of coefficients in Propositions 2.1 and 2.2. The systematic search for this law was a main task to which considerable efforts of many mathematicians were devoted, as we shall see, during the eighteenth century.

In short, the binomial theorem has proved to be, in the epistolary exchange between Newton and Leibniz, a versatile mathematical tool that can be put to many uses. And from a theoretical point of view, it appears to be the foundation of:

1. The method of root extraction.
2. Mercator's method.
3. The Newton-Raphson method of root finding.
4. The method of raising a polynomial to an arbitrary rational power.
5. The procedure for finding the power series expansion of a given function.
6. The theorem on reversion of series.

Probably only in this perspective can one understand that the binomial theorem for rational exponents became the theorem of Newton in the eighteenth century; that is to say, it was Newton's theorem in the sense that it was conceptualized as the basis of Newton's method of infinite series.

2.1.2 Newton's Theorem and the Method of Fluxions

The first printed version of Newton's binomial theorem appeared in Wallis's *Treatise of algebra* in 1685, which was translated into Latin 8 years later to ensure widespread dissemination (Wallis 1685, 318–320, 330–333; 1693, 357–359, 368–371). However, the systematic reproduction of Newton's theorem in most of eighteenth century manuals on mathematical analysis and algebra cannot be explained solely by reference to the publication of Wallis's treatise. In some measure, the prominent place that the binomial theorem occupied in these mathematical disciplines was due to its relation to Newton's method of fluxions. And this is true even of mathematical practices developed outside of the English mathematical community, like those developed by German mathematicians in the eighteenth century. The purpose of this section is to analyze the relation between the binomial theorem and the method of fluxions in the first half of the eighteenth century.

At about the same time that Newton discovered the binomial theorem, he also discovered the rudiments of his new method of fluxions. In October 1666, Newton organized his discoveries about the method of fluxions into some principles in a tract that was never published in its day, but some of his results were included in the *De analysi* (Newton 1666; Panza 2005, 433–514; Whiteside 1966). Some 5 years later, in 1671, Newton prepared the *Tractatus de methodis serierum et fluxionum* for publication,