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Weight Function Methods in Fracture Mechanics

Theory and Applications

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Foreword

In 1985, as part of a NASA and CAE Symposium on Structural Mechanics held in Beijing, China, I had the pleasure of meeting Dr. X. R. Wu and his colleagues. Since then, I have known of the research work on the two- and three-dimensional (2D and 3D) weight function methods (WFMs) of Dr. Wu and coworkers through the NASA/CAE Cooperative Research Program on Fatigue and Fracture Mechanics on “small-crack behavior in aerospace materials.” A team at the NASA Langley Research Center and the Beijing Institute of Aeronautical Materials conducted the test and analysis program for about six years. At that time, Dr. I. S. Raju and I were using the finite element method (FEM) to develop stress intensity factor (SIF) solutions for many 2D and 3D crack configurations. Also, Drs. Wu and Carlsson, and coworkers, were developing WFMs for analysis of 2D and 3D crack problems. The small-crack test specimen selected, the single-edge-notch tension SEN(T) specimen, has become a standard for measuring small-crack behavior under a variety of cyclic loading conditions. In NASA RP-1309 (1994), the SIFs for a surface- and corner-crack at the semi-circular edge notch from FEM and WFM analyses agreed very well over a wide range in crack-configuration parameters. These SIF solutions were then used successfully to correlate small-crack-growth-rate data under cyclic loading and to predict the “fatigue” life of SEN(T) specimens made of two high-strength aluminum alloys and subjected to a simulated aircraft spectrum loading using a micro-structural defect size that nucleated the fatigue failures.

The SIF is the foundation of linear elastic fracture mechanics (LEFM) analyses for engineering materials and structures. WFM is an efficient, accurate, and easy-to-use method for computing SIFs due to arbitrary loadings. The concept is simple and the WF only depends upon the crack configuration and the boundary condition, and when the WF is known, the SIF solutions are easily calculated from the un-cracked stress distribution at the prospective crack site through a simple quadrature. In current aerospace structural damage-tolerance design software, such as NASGRO, AFGROW, and DARWIN, most of the SIF solutions, especially the more recent ones, are generated by using WFMs.

The first book on WFM by Wu and Carlsson “Weight Functions and Stress Intensity Factors” was published in 1991 by Pergamon Press, Oxford, UK. It has been widely used by many researchers and engineers all over the world in the past 30 years. Now from the new Wu-Xu book, I am glad to see the advances being made in WF theory, with a systematic and comprehensive treatment of the state of knowledge of WFMs. The book coverage has been greatly expanded to include (1) for 2D crack problems: standardized analytical WFs for a large number of 2D center- and edge-crack configurations, extension of the WFM from isotropic to orthotropic materials (as more composite materials are being used in structural applications), to mixed-mode crack configurations (opening mode, K_{I}, and shear mode, K_{II}), to multiple-site damage (MSD, a failure mode that is now considered in aircraft design), to the determination of residual-strength of panels containing MSD, and to simulate the effects of crack-tip plasticity by using the strip-yield model; (2) for 3D crack problems: various WFMs for embedded and surface/corner cracks subjected to uni- and bi-variant stresses; (3) a variety of new applications of the WFMs, such as crack analyses involving thermal and residual stresses, cohesive/Dugdale models, bridging stress and crack opening stress, WFMs for "substitute crack geometry," and an inverse WF approach for determining crack line stresses; (4) analyses and discussions of other 2D and 3D WFMs, although most of the work presented in the book has been conducted by the authors and coworkers.

The application to strip-yield modeling is well-suited for the WFM, since the model is a crack under various crack-surface loading. The book shows application to fracture using the crack-tip-opening displacement (CTOD) or angle (CTOA) fracture criteria with very encouraging comparisons of measured and calculated failure loads.

For many years, I have used the strip-yield model to simulate fatigue-crack growth using Elber's crack-closure theory in the FASTRAN code. The code is like AFGROW and NASGRO but has not used the WFM to calculate SIFs. The code uses closed-form SIF equations that were developed for many standard crack configurations that occur in aircraft structures. In the literature, other applications with the strip-yield model have used WFMs. FASTRAN currently has two models—a central through crack in a finite-width plate and two symmetric through cracks emanating from a circular hole. However, to predict the crack closure (opening) behavior of other crack configurations the code uses K-analogy. In this book, the WFM is used to predict the crack-closure (opening) behavior for other crack configurations. Again, the comparisons between the closed-form equations and the WFM were quite good. Therefore, the WFM can provide a very efficient tool for accurate determination of configuration-specific crack opening stresses in the Newman crack-closure model.

An interesting feature of the "inverse" WFM is that the method can be used to determine the crack line stress distribution in an un-cracked body. The method uses the analytical WF and crack-mouth-opening displacement, CMOD, for the crack configuration as known inputs. An integral equation that relates the CMOD, WF and crack line stress is solved. The stress distribution at the prospective crack line location in the un-cracked body is determined segment by segment. The method is verified through comparisons to known stress distributions for a variety of 2D crack configurations and loadings. The inverse WFM was shown to be useful for the determination of residual stresses.

In summary, this book represents the persistent efforts of the authors and coworkers on the research of WFMs in fracture mechanics for well over 30 years. It has provided analytical WFMs and a wide variety of accurate and rigorously verified WFs for many crack configurations that would be useful for researchers, students, engineers, and designers to use in fatigue and fracture analyses on a variety of structural applications. It has been demonstrated through many examples that the analytical WFM can provide a versatile, efficient, and accurate analytical tool for complicated crack problems involving arbitrary crack-face loadings. I congratulate the authors for the pioneering work and significant achievements that are of high scientific and practical value to fracture mechanics and to aerospace and many other industries.

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June 2021

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Preface

Fracture mechanics has been an indispensable tool in many important technical areas for the design and safe operation of structures containing cracks or crack-like defects that are either introduced in materials processing, component fabrication or induced in service operation. The stress intensity factor (SIF), the characterizing parameter of the linear elastic crack tip stress–strain field, is the foundation of fracture mechanics analysis. The weight function method (WFM) is very powerful for the determination of SIFs and other important fracture parameters for cracked bodies. It is especially attractive for complex loadings and when a large number of SIFs are desired for wide range of crack sizes and multiple load conditions. Compared to various other solution methods, the WFM has several distinct advantages, being versatile, accurate, remarkably efficient, and easy-to-use.

According to the basic theory of WFM first introduced by Bueckner and further developed by Rice, SIFs for a crack with its faces loaded by an arbitrary stress distribution can be determined by a simple quadrature of the product of the weight function (WF), $m(a, x)$, for the given crack geometry and the "crack line stress" in the un-cracked body, $\sigma(x)$. Since $\sigma(x)$ can be readily obtained by using theory of elasticity or by numerical methods, accurate derivation of $m(a, x)$ for different crack geometries is crucial to the successful application of WFMs and therefore has been the central topic of WFM research. The main objectives of the present book are to present reliable methods for derivation of accurate WFs for different types of cracks, to provide accurate and rigorously verified WFs for various crack geometries, and to give a large amount SIF (and COD) data for many crack geometries and load conditions.

This book gives, in systematic manner, a detailed account of the research work on WFMs by the authors and coworkers over the past three decades. Theoretical background to the WF theory is described in detail, and various two- and three-dimensional (2D and 3D) WFMs are introduced and evaluated. A standardized procedure for the derivation of 2D analytical WFs for center and edge crack geometries with different boundary conditions based on crack opening displacements (CODs) is presented. By using the standardized procedure, a large number of analytical WFs for various crack geometries are derived, verified and validated, and utilized to determine SIFs and CMODs for various complex loadings. Accuracy of the results is rigorously evaluated. Many of these WFs and SIF solutions have been incorporated into relevant international standards and industrial codes of practice.

The book contains 15 chapters.

Chapters 1–3 give the theoretical background and overview, derivation, verification, and assessment of WFMs for 2D crack geometries. Chapter 1 presents a standardized analytical procedure based on CODs for deriving 2D crack WFs and analytical expressions for SIFs and CODs under basic load cases. Chapter 2 analyzes and discusses two other WFMs that are based on multiple reference states (MRS): the direct adjustment method and the universal WFM. Verification and accuracy evaluation of the three WFMs by using the Green's functions are made in Chap. 3. Furthermore, in-depth analyses of merits and shortcomings of different WFMs are made, and possible sources of error are analyzed.

Chapters 4–10 constitute the bulk portion of the book, providing detailed derivation of the WFs for several dozens of 2D center crack (Chap. 4) and edge crack geometries in simply and multiply connected regions (Chaps. 5 and 6), and a large number of SIF solutions for various loadings in graphical and/or tabular forms. Analytical WFMs are also presented for several important crack problems including cracks in orthotropic composite material (Chap. 7), collinear multiple site damage (MSD) WFs and their applications to residual strength analysis of panels with MSDs (Chap. 8), mode II crack WFs and SIFs for center/edge cracks (Chap. 9). Different types of 3D WFMs for embedded and part-through cracks are presented in Chap. 10, including the slice synthesis WFM and the point load weight function methods. SIF solutions for basic load case of crack face pressure of power-law type are tabulated to facilitate rapid determination of SIFs for more complex loadings. A large number of 3D SIFs for various load cases are generated, and their accuracy levels are verified against well-recognized analytical or numerical results and fitted equations.

Chapters 11–15 demonstrate various engineering applications of the 2D analytical WFMs. The topics discussed include SIFs for cracks in self-equilibrating residual and thermal stress fields (Chap. 11); calculation of CODs for arbitrary loadings (Chap. 12); analyses of bridging and cohesive models, and crack opening stresses required by fatigue crack growth life prediction models (Chap. 13); analysis of cracks in real-world complex crack geometries using WFM (Chap. 14); inverse WF approach for determination of un-cracked stress distributions with special applications to residual/thermal stresses (Chap. 15).

Throughout the book, the analysis has been carried out in term of normalized (non-dimensional) quantities; the normalization is necessary to avoid lengthy expressions. For each crack geometry, a characteristic length dimension, W, is chosen for normalization of the crack length a, the coordinate x, and the COD u. The corresponding normalized quantities thus become: $\alpha = a/W$, $\xi = x/W$, $u = U/W$. Similarly, for crack line stress $\sigma(x)$, a normalizing stress σ_0 is chosen, and the normalized crack line stress is written as $\sigma(x)/\sigma_0$ or $\sigma(\xi)/\sigma_0$. The non-dimensional SIF is denoted by f. The SIF is obtained by $K = f \cdot \sigma_0(\pi a)^{1/2} = f \cdot \sigma_0(\pi \alpha W)^{1/2}$.

The present book reflects the continuing efforts of further developing and improving the analytical WFMs since the publication of its predecessor, the book entitled "Weigh Functions and Stress Intensity Factors Solutions," Pergamon Press, Oxford, 1991, by Wu and Carlsson, that has been widely used by the international fracture mechanics community in the past 30 years. The basic ideas of 2D WFM are adopted by the present book, but the major part of the present book is new, including Chaps. 2, 3, 7–11, 13–15. Furthermore, Chaps. 1, 4–6 have been significantly expanded both in breadth and in depth. Also, it should be noted that a WF book in Chinese was published in 2019 by China Aviation Press. However, the present book is not a direct translation of the Chinese version; it is rewritten and significantly condensed.

The book is primarily intended to serve as a useful reference for researchers, designers, and engineers, and for university senior students and postgraduates, who are concerned with fatigue and fracture of engineering materials and structures. Principal areas for applications include, but not limited to aerospace, mechanical, civil, and material engineering. Readers are expected to have basic knowledge of fracture mechanics.

The book can be used for different purposes: (1) deriving analytical WFs for new crack geometries and boundary conditions by following the standardized WF derivation procedure for 2D through-thickness crack geometries and the slice synthesis WFM and the point load WFMs for 3D embedded and part-through crack geometries; (2) computation of SIFs and CODs for the crack geometries and crack face loadings of interest by using the relevant WFs contained in the present book; (3) use it as a handbook to find the required solutions of SIFs (and CODs) directly, or with the non-dimensional data tables for power-law stresses in this book, to calculate SIFs (and CODs) for crack line polynomial stresses by simple arithmetic; (4) solutions to various practical examples in the book, including those in Chaps. 11–15, can provide useful guidance to readers for determination of SIF and COD solutions to their specific crack problems.

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Acknowledgments

My research work on weight function methods (WFMs) in fracture mechanics began in the mid-1980s in the Royal Institute of Technology (KTH), Stockholm, Sweden, under the guidance of Prof. A. Janne Carlsson, who was my Ph.D. supervisor. We co-authored the first WF book entitled “Weight Functions and Stress Intensity Factor Solutions” that was published by Pergamon Press in 1991 and was recommended for publication by Profs. J. W. Hutchinson (Harvard University) and K. J. Miller (University of Sheffield) to whom I am very grateful.

First of all, my very sincere thanks go to Janne for his guidance, support, and cooperation. A year ago, he was very glad to agree being a coauthor of the present WF book, but it was too sad that he passed away last November at the age of 89. Also, I want to express my hearty thanks to Prof. M. G. Yan (BIAM former Technical Director) and Prof. Kezhi Huang (K. C. Hwang, Tsinghua University) for their long-term support and encouragement to my research work on WFM.

I was gratified at the positive response by which the international fracture mechanics community has appreciated the Wu–Carlsson WF–book. Many readers have used the analytical WFM and data in the book to solve various crack problems and have confirmed to me personally that the book was a useful help in performing their own crack analysis. A very successful international cooperation on fatigue and fracture mechanics of high-strength aluminum alloys between NASA and the Chinese Aeronautical Establishment in 1987–1994, technically led by J. C. Newman Jr (NASA–Langley Research Center, USA) and X. R. Wu (Beijing Institute of Aeronautical Materials, China), had prompted the research on three-dimensional WFM and finite element (FEM) analysis for surface and edge cracks at notches. I am most grateful to Prof. Newman for the successful collaboration and long term friendship and for kindly writing the Foreword for this book.

Many individuals have contributed to the research work that has led to the present WF book. Helpful discussions and communications with international colleagues have stimulated new ideas and led to deeper understanding of the key issues of WFMs. Specifically, I would like to thank Dr. T. Fett (Research Center Karlsruhe, Germany) for sending me many of his WF publications and IKM research reports, Prof. G. Glinka (University of Waterloo) for the in-depth and fruitful discussions a year ago on several important aspects of WFMs. The Wu–Carlsson analytical WFM has been successfully utilized by many researchers to tackle a large variety of practical crack problems. I am glad to express my thanks to those colleagues for their efforts of applying and developing the WFMs, in particular, to Prof. Petzow and Dr. Schneider (Max-Plank Institute, Stuttgart) on thermal shock crack problems in ceramics, Dr. R. John et al. (USAF Wright Laboratory) on developing WFs for clamped edge crack plate and bridging stress in metal matrix composites, Prof. P. Bowen (University of Birmingham) on crack analyses of metal matrix composites and bi-materials, Prof. R. O. Ritchie (University of California, Berkeley) on fatigue crack growth and comments on WFM, Prof. M. R. Hill (University of California, Davis) on crack problems related to residual stresses, Dr. D. Ball (L-M Co.) on damage-tolerance design tools of aircraft structures with WFMs, Dr. R. C. McClung (Southwest Research Institute, San Antonio), and Prof. H Millwater (University of Texas at San Antonio) on aerospace structure damage-tolerance design software NASGRO and DARWIN using WFMs.

The research work on WFMs presented in the present book has been a successful teamwork. Many of my students have contributed to various aspects of WFMs. Here, I would like specifically acknowledge the contributions made by Drs. W. Zhao (for his very good work on 3D WFM), X. G. Chen, J. Z. Liu, Y. J. Guo, B. Chen, D. H. Tong and X. C. Zhao, and Masters R. X. Xu and Z. Jing.

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Last but not least, X. R. Wu cordially wants to thank his wife, Xiu-Ling Sun, for her deep understanding and great support in so many years. W. Xu wants to thank his parents and wife for their love and support.

Beijing, China Xue-Ren Wu
May 2021

Contents

Abbreviations and Symbols

Abbreviations

2(3)D	Two(three)-dimensional
BCM	Boundary collocation method
BEM	Boundary element method
C(M)OD	Crack (mouth) opening displacement
CMSD	Crack mouth sliding displacement
COA	Crack opening area
CTOA(D)	Crack tip opening displacement (angle)
DAM	Direct adjustment method
FEM(A)	Finite element method (analysis)
GF	Green's function
MMC	Modified mapping collocation
MRS	Multiple reference states
MSD	Multiple site damage
PWF(M)	Point weight function (method)
PZS	Plastic zone size
SAWF(M)	Standardized analytical weight function (method)
SIF	Stress intensity factor
SSWFM	Slice synthesis weight function method
UWF(M)	Universal weight function (method) of Glinka–Shen
WCTSE	Weight function complex Taylor series expansion
WF(M)	Weight function (method)
WFD	Widespread fatigue damage
WRPWF(M)	Wide-range point weight function (method)

Symbols

a	Crack length
b_{in}	Coefficients of $\beta_i(\alpha)$
c	Semi-minor axis of an elliptical crack
D	Disk diameter
D_{n}	Fett–Munz weight function series coefficient
$E(E', E^*)$	Young's modulus (effective Young's modulus)
$f(f_{\mathrm{r}})$	Non-dimensional stress intensity factor (reference non-dimensional stress intensity factor)
f_n	Non-dimensional stress intensity factor for power stress applied on the crack surface
$G(\alpha, \xi)$	Green's function
H	Plate height
$K(K_{\mathrm{II}})$	Stress intensity factor
$k(t, s)$	Kernel function
$K(K_{\mathrm{r}})$	Stress intensity factor (reference stress intensity factor)
$m(a, x)$ $(m(\alpha, \xi/\alpha))$	Weight function, $m(\alpha, \xi\big/\alpha) = G(\alpha, \xi\big/\alpha)\big/\sqrt{\pi\alpha}$
M_i	Glinka–Shen weight function series coefficient
N_i	WCTSE weight function series coefficients
P	Point force
R	Disk/hole radius
$R_{\mathrm{o}}(R_{\mathrm{i}})$	Outer (Inner) radius of a circular ring/hollow cylinder/C-specimen/curved beam
t	Plate thickness
$u(a, x)$, $u(\alpha, \xi/\alpha)$	Crack opening displacement
$V_r(\alpha)$	Non-dimensional crack mouth opening displacement for reference load case
W	Characteristic length of cracked body
x, y, z	Coordinates
α	Normalized crack length $\alpha = a/W$
$\beta_{\mathrm{i}}(\alpha)$ $(\beta_{\mathrm{IIi}}(\alpha))$	Wu–Carlsson mode I (II) weight function series coefficient
λ_i	Non-dimensional stress intensity factor f_{r} series coefficient
γ_n	Non-dimensional crack mouth opening/sliding displacement V_{r} series coefficient
ξ	Normalized coordinate along crack line ($\xi = x/W$)
v	Poisson's ratio
σ_0	Normalizing (nominal) normal stress
τ_0	Normalizing (nominal) shear stress
$\sigma(x)$, $\sigma(\xi)$	Crack line stress in un-cracked body
$\tau_0(x)$, $\tau_0(\xi)$	Crack line shear stress in un-cracked body
Φ	The complete elliptic integrals of the second kind

Chapter 1
Standardized Analytical Weight Function Method Based on Crack Opening Displacements

Abstract The stress intensity factor (SIF) is the foundation of fracture mechanics analysis for engineering structures and materials. Development of solution methods for SIFs of cracked bodies has been one of the central topics in fracture mechanics. The weight function method (WFM) is a powerful, accurate, efficient and easy-to-use method for computing SIFs due to arbitrary loadings. In this chapter, the Wu-Carlsson standardized analytical WFM based on crack opening displacement, and the generalized WFM for mixed boundary conditions are presented. Detailed derivation procedures for analytical WFs are described for 2D center and edge crack geometries, respectively. Closed-form SIF-expressions for various basic crack face loadings are derived. Verification of 2D WF-accuracy using Green's function for point-by-point assessment is proposed. The influences of displacement boundary condition and the reference load case on analytical WFs are discussed. Analytical WFMs for crack analysis in mode II and in orthotropic composite material are briefly introduced.

Keyword Fracture mechanics analysis · Stress intensity factor · COD-based analytical weight function method · Center cracks · Edge cracks

1.1 Introduction

It is well known that most engineering materials and structures contain cracks or crack-like defects that are either introduced during material processing and fabrication or in service by damage due to e.g. overloading, fatigue or environmental effects. Within the anticipated operational life of structures, subcritical crack growth can take place and may eventually lead to catastrophic structural fracture/failure. It is therefore widely recognized that the presence of cracks and subsequent crack growth must be considered both in design and in service operations of engineering structures.

Fracture mechanics provides theoretical foundation and effective tools for analysis of crack problems in engineering materials and structures [1–9]. Being an important branch of solid mechanics, fracture mechanics focuses its attention on assessing, in quantitative manner, the behavior of cracks with singular stress/strain field in the

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crack tip region. Under the conditions of linear elastic fracture mechanics (LEFM), the singular crack tip stress/strain field is dominated by the stress intensity factor (SIF, or K). Under the condition of small scale yielding, i.e. yielding is confined in a region small relative to the crack length and other dimensions of the crack body, K still uniquely characterizes crack-tip stress/strain conditions. The parameter K describes the first order effects of stress magnitude and distribution as well as the geometry of structure/component and crack, and relates the crack geometry and applied load to the driving force for crack growth. This "one-parameter characterization" of crack tip condition is one of the most important concepts in fracture mechanics, which makes it possible to transfer test results from simple laboratory specimens to real world cracked structures via K. In many industries such as the aerospace industry, the reliability of airframes and engines are maintained by damage tolerance approaches that in turn depend on SIF-solutions. Therefore, accurate and efficient SIF-determination is regarded as "at the very heart of damage tolerance assessment" [9].

Development of various numerical and analytical methods for determining SIFs has been one of the central topics in fracture mechanics. Many analytical and numerical methods for K-solutions have been developed [8–12]. However, after a long history of over 60 years, from the current structural integrity assessment procedures/standards [13–18], industrial design software [19, 20] and development trends [21], it is clear that besides several well-known SIF-handbooks [22–25] that are mostly for simple load cases, numerical methods such as the finite element method (FEM) and the boundary element method (BEM), and weight function methods (WFMs) now remain the two types of most viable methods for K-determination, especially when complex load conditions are involved. In the aerospace structural damage tolerance design software, e.g. the well-known NASGRO [19] and DARWIN [20], most of the K-solutions, especially the more recently added ones, are in fact generated by using various WFMs.

Numerical methods provide powerful technique for analyzing crack problems. However, each numerical calculation can only produce one K-value for a given crack length and load case. Furthermore, because of the crack tip singularity, modeling and computation for crack problems are much more complicated than crack-free problems. Special crack-tip elements, fine meshes and experiences are required for proper treatment of the crack-tip stress/strain singularity. For small cracks, it becomes quite difficult for numerical methods to achieve accurate solution due to meshing difficulties. Because SIFs are functions of crack length, repeated modeling and computation work must be carried out for many crack length steps from the initial crack size (a_0) to critical crack size (a_c). For fatigue crack growth analysis, this can be very laborious and time-consuming. There is no doubt that commercial FEM codes offer the capability of analyzing complicated crack problems. However, as pointed out recently by McClung [9]: numerical methods "can be attractive option for solving very specific problems (such as a critical field cracking issue), but the resource requirements (including the computational time itself) still render this approach impractical as a general design tool for complex structures with many fracture-critical locations." In short, numerical methods for SIFs are capable, but "can be expensive and impractical for real-world design applications". This may well explain why almost all the

SIF-solutions in the probabilistic damage tolerance design software DARWIN [20] for aero-engines are determined by using WFMs, and why WFMs are regarded as a "critical component of a damage tolerance fracture plan".

The powerfulness of WFMs stems from the fact that the WFs are only property of the crack geometry and the traction-displacement boundary composition, and is independent of the crack line stress (the stress along the prospective crack location in the crack-free body). For a given crack geometry, once determined based on one or more reference load cases to which SIFs are known, the WF can be used unlimitedly to calculate SIFs and other crack parameters for arbitrary load cases, by simple integration of the product of WF and crack line stress. Repeated modeling and computation for each load case and for many crack length steps that are required by numerical methods are eliminated. Quite often, closed form solution can be obtained (for two dimensional (2D) crack problems). In comparison to numerical methods, distinct advantages of WFMs include: versatility, efficiency, easy-to-use and reliability. And therefore, WFMs present "the best balance between accuracy and required computing power".

The WFM for determination of crack SIFs was initially envisioned by Bueckner [26] and further by Rice [27], and was subsequently developed by many researchers until present. According to Bueckner and his superposition principle [28], when using WFMs to determine SIFs, only the WF for the considered crack geometry and the stress distribution at the prospective crack line in the crack-free body are required. The determination of crack line stress presents no difficulties since it can be obtained either exactly by using the classical theory of elasticity or, in most cases, very accurately by numerical methods such as FEM. Therefore, accurate determination of the WFs for various practical crack geometries is the key to successful application of WFMs. However, it is virtually impossible to derive exact WFs for finite crack geometries. Consequently, various numerical and analytical approaches for the determination of WFs have been developed in the past decades. Numerical approaches generally require large amount of work, and can only give WFs at some discrete points along the crack line. Such discrete WFs are inconvenient to use. Analytical approaches are able to provide closed form WF-expressions that are convenient to use and computationally much more efficient, but the accuracy levels of analytical WFs derived with different approaches may differ significantly. Therefore, analytical WFs need careful verification for their accuracy prior to applications. The analytical approaches require one or more reference load cases, and can be further divided into two types, one requires a single reference load case, and the other requires multiple reference state (MRS). The two analytical approaches are presented in Chaps. 1 and 2, respectively, and are rigorously evaluated and compared in Chap. 3.

This chapter presents the basic theory of the generalized 2D WFM under mixed boundary conditions, and a standardized analytical WFM based on crack opening displacement (COD) associated with one reference load case. Detailed derivation procedures for analytical WFs are given for center cracks and edge cracks, respectively. Closed-form SIF-expressions for the two crack types under several basic crack face loadings are derived. Verifications of accuracy for the derived WF are made by using the Green's function (GF). The effects of displacement boundary condition and the reference load case are studied. Furthermore, analytical WFMs for crack analysis in mode II and in orthotropic composite material are briefly introduced.

1.2 The Generalized Weight Function for Mixed Load-Displacement Boundary Conditions

1.2.1 Weight Functions of Bueckner and Rice

The concept of weight function (or alternatively influence function, or Green's function) was first introduced by Bueckner based on analytical function representation of elastic fields for isotropic materials [26]. He showed that the stress intensity factor, K, due to an arbitrary set of loads can be obtained by integrating the product of crack line stress $\sigma(x)$ induced by these loads and the weight function $m(a, x)$ of the considered crack geometry. The SIF, $K(a)$, as a function of crack length a, is calculated using the WFM by Eq. (1.1):

$$K(a) = \sqrt{W} \int_0^{\alpha} \sigma(\xi) \cdot m(\alpha, \xi/\alpha) \mathrm{d}\xi \tag{1.1}$$

where α and ξ are non-dimensional crack length and coordinate, respectively, defined by $\alpha = a/W$, $\xi = x/W$ (a and x are the crack length and coordinate respectively, and W is a characteristic dimension of the considered crack geometry); $\sigma(\xi)$ is crack line stress at the fictitious crack line in the crack-free body, resulting from all applied loads and displacement and body force as well as internal stress (see Sect. 1.2.3). The analysis of $\sigma(\xi)$ is made for un-cracked body, and is a conventional problem of elasticity. Therefore, accurate determination of WF, $m(\alpha, \xi/\alpha)$, is the key issue in WFMs. Bueckner presented the basic theory of WFM for the determination of K, Eq. (1.1). However, the problem of methods for deriving WFs $m(\alpha, \xi/\alpha)$ for cracked bodies remained to be solved at that time. Although an approximate expression of $m(\alpha, \xi/\alpha)$ for an edge crack in finite width plate was later derived by Bueckner using the integral equation method, the applicable range is limited to $\alpha = 0 \sim 0.5$ (see Sect. 5.2.1 and Fig. 5.9 for details). The integral equation method seemed impractical as a general WF-derivation approach.

The same equation as Eq. (1.1) was derived independently by Rice [27] in a study of linear elastic crack tip fields employing Irwin's relation between energy release rate and the SIF K [3]. He also showed for the first time that WF can be determined by taking partial derivative of a known elastic solution of crack opening displacement (COD) for a reference load case, $U_{\mathrm{r}}(a, x)$, with respect to crack length a [27], as given by Eq. (1.2).

$$m(\alpha, \xi/\alpha) = \frac{E'\sqrt{W}}{K_r(a)} \cdot \frac{\partial U_r(a, x)}{\partial a}, \; K_r(a) = f_r(\alpha)\big(\sigma_0\sqrt{\pi a}\big) \tag{1.2}$$

where E' is effective Young's modules, with $E' = E$ for plane stress, and $E' = E/(1-\nu^2)$ for plane strain, ν is Poisson's ratio, $K_r(a)$ is a known SIF for the reference load case, and $f_r(\alpha)$ is non-dimensional SIF. The approach proposed by Rice, Eq. (1.2), paved a very effective way for the derivation of various WFs, and has been utilized by many researchers.

The combination of WFM [26, 27] with the superposition principle [28] provides a very powerful method for determining SIFs and other important crack parameters in LEFM. Since the mid-1970s, further studies on the theory and applications of WFM have been made by many researchers. References [29–43] are some earlier representative work on analytical WFMs of 2D crack problems, and Refs. [44–48] are those on numerical WFMs. References [49–58] are on WFMs for 3D crack problems. Some more recent publications on numerical WFMs are referred to [59–64]. In recent years, analytical WFMs for multiple-site-damage (MSD) collinear cracks in aircraft structures have been developed [65–68]. These new developments show that research on WFMs is still quite active, and there is even a recent resurgence of interest in the development and more engineering applications of the WFMs [9]. References [69–79] reflect efforts made by the present writers in recent years. For a comprehensive review on this topic, see [79]. The present book presents a systematic account of the WF theory; provides numerous analytical WFs with SIF-solutions for 2D and 3D crack problems[1]; and demonstrates various practical applications.

1.2.2 Generalized Weight Function Method for Crack Problem with Mixed Boundary Conditions

The boundary condition of cracked body considered by Bueckner [26] and Rice [27] formulation of WFs was for surface tractions prescribed type. Generalizations of the WF theory to mixed boundary condition involving both prescribed surface tractions and displacements were later made by several workers [30, 32, 80, 81]. A generalized WFM for crack problem with mixed boundary conditions was proposed by Wu and Carlsson [32, 36], based on the Betti's reciprocal theorem. SIF-calculations using WFMs for crack problems of mode I, II and III are given in [32, 36]:

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[1] *Note* The terms "2(3)D crack problems" and "2(3)D weight function (WF)" adopted in this book follows the theory of elasticity, and they refer to the dimension of the cracked body instead of the crack itself. Thus, 2D means through-thickness cracks in a two-dimensional (*x*–*y*) plane; 3D means embedded and part-through cracks in a three-dimensional body (*x*–*y*-*z*, *z* being thickness direction). These terms have been traditionally used in the literature. On the other hand, the terms "1(2)D crack problem" in more recent literature on WFMs refer to the dimension of the crack itself instead of the cracked body. The two types of terms are both correct and will not be confused.

$$K_{\mathrm{I}}^{(2)}(a) = \frac{4\mu}{(\kappa+1)K_{\mathrm{I}}^{(1)}(a)} \left[\int_{C_T} T_i^{(2)} \frac{\partial U_i^{(1)}}{\partial a} \mathrm{d}s - \int_{C_U} U_i^{(2)} \frac{\partial \sigma_{ij}^{(1)}}{\partial a} n_j \mathrm{d}s + \int_B b_i^{(2)} \frac{\partial U_i^{(1)}}{\partial a} \mathrm{d}B \right] \tag{1.3}$$

$$K_{\mathrm{II}}^{(2)}(a) = \frac{4\mu}{(\kappa+1)K_{\mathrm{II}}^{(1)}(a)} \left[\int_{C_T} T_i^{(2)} \frac{\partial U_i^{(1)}}{\partial a} \mathrm{d}s - \int_{C_U} U_i^{(2)} \frac{\partial \sigma_{ij}^{(1)}}{\partial a} n_j \mathrm{d}s + \int_B b_i^{(2)} \frac{\partial U_i^{(1)}}{\partial a} \mathrm{d}B \right] \tag{1.4}$$

$$K_{\mathrm{III}}^{(2)}(a) = \frac{\mu}{K_{\mathrm{III}}^{(1)}(a)} \left[\int_{C_T} T_3^{(2)} \frac{\partial U_3^{(1)}}{\partial a} \mathrm{d}s - \int_{C_U} U_3^{(2)} \frac{\partial \sigma_{3j}^{(1)}}{\partial a} n_j \mathrm{d}s + \int_B b_3^{(2)} \frac{\partial U_3^{(1)}}{\partial a} \mathrm{d}B \right] \tag{1.5}$$

where the superscripts $^{(1), (2)}$ represent the load cases (1) and (2); C_{T} and C_{U} refer to the boundaries with tractions (T_i) and displacement (U_i) prescribed, respectively; B is the surface of crack body, and b_i is body force; $\kappa = 3$–$4v$ (plane strain), $\kappa = (3 - v)/(1 + v)$ (plane stress); v is Poisson's ratio. Note there is a difference of $(\kappa + 1)/4$ between mode III and mode I/II. Details are referred to [32, 36]. It is noted that the generalized WFs in Eqs. (1.3)–(1.5) are difficult to use. However, by using the superposition principle for crack problems with mixed boundary conditions, these expressions can be converted into simple forms that are much easier to use, see next sections.

1.2.3 Superposition Principle

Although SIFs for any other load cases can be determined by using the generalized WFM, practical applications of Eqs. (1.3)–(1.5) may encounter difficulties. This is because the use of the equations requires both the knowledge of the SIF $K^{(1)}(a)$ and other quantities: e.g. $\partial U_i^{(1)}/\partial a$ on C_{T} and $\partial \sigma_{ij}^{(1)}/\partial a$ on C_{U}, which are not easily determined. These limitations on the direct application of Eqs. (1.3)–(1.5) can be overcome by using the Bueckner superposition principle in LEFM [28]. The idea here is to transform Eqs. (1.3)–(1.5), that require complete elastic solutions including the displacement $U_i^{(1)}$ on C_{T}, and the traction $\sigma_{ij}^{(1)}$ on C_{U}, to some equivalent expressions that will require much less knowledge of the reference crack problem.

Consider a crack-free body which is subjected to prescribed tractions T_i over the boundary C_{T}, and prescribed displacement U_i over the boundary C_{U}; it may also contain a self-equilibrating internal stress σ_{int}, Fig. 1.1a. On the line MN, there will be a stress distribution $\sigma(x)$ resulting from the loads on the boundary and internal stress systems. If a crack is now introduced along MN, at the same time a traction $-\sigma(x)$ is applied on the crack faces, Fig. 1.1b, the crack will remain perfectly closed,