Communications and Control Engineering

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Inverse Optimal Control and Inverse Noncooperative Dynamic Game Theory

A Minimum-Principle Approach

Springer

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To

My family and S.C. (T. M.)

My families in Peru and Germany (J. I. C.)

My wife and son Philipp (S. H.)

Jae and Oliver (T. P.)

Preface

This book aims to provide an introduction to selected topics within the theory of inverse problems in optimal control and noncooperative dynamic game theory. These topics have emerged relatively recently in data-driven problems that involve inferring the underlying optimality objectives of decision-makers (agents or systems) from quantitative observations of their behavior. For example, such problems have arisen in applications across systems and control, robotics, machine learning, biology, economics, and operations research including the development of robots that mimic the behavior of human experts; the quantitative study of biological control systems; the design of advanced driver assistance technologies; the efficient inference of agent intentions; and the estimation of competitive market and economic models in economics and operations research.

The origins of this book lie in our own research exploring inverse problems in optimal control and noncooperative dynamic game theory. We noticed a sparsity of literature treating such inverse problems in their data-driven forms. Most notably, almost no work on them had appeared in leading systems and control journals prior to 2018! Despite the broad practical significance and deep (intellectual) challenges of inverse optimal control and inverse noncooperative dynamic game theory, the powerful mathematical tools and fundamental theoretical insights offered by systems and control theory had, therefore, been missing from many popular treatments. The purpose of this book is thus to both expose systems and control researchers to inverse problems (providing a springboard to open problems) and to draw broader attention to useful systems and control techniques for solving them (specifically Pontryagin's minimum principle).

This book's intended audience are researchers and graduate students in systems and control, robotics, and computer science. It is intended to be mostly self-contained, but previous exposure to systems and control or (dynamic) optimization would be helpful. Given the significance of the minimum principle throughout this book, we provide a background chapter with a short introduction to its use in (forward) optimal control and noncooperative dynamic game theory. In particular, we collect the scattered results on the conditions for optimal and Nash equilibrium solutions, both in discrete and continuous time.

After presenting background fundamentals, the first half of this book seeks to illuminate key concepts underlying the rapidly growing literature on inverse optimal control for linear and nonlinear dynamical systems in discrete and continuous time with continuous state and control spaces. These concepts include the formulation of different inverse optimal control problems depending on the available data as well as the proposal of the techniques to solve them.

The second half of this book endeavors to generalize and extend inverse optimal control theory to inverse noncooperative dynamic game theory. Inverse problems in noncooperative dynamic game theory are concerned with computing the individual optimality objectives of competing decision-makers from data. Such inverse problems raise a host of new theoretical issues due to the information structures and (equilibrium) solution concepts unique to noncooperative dynamic games. Therefore, the book attempts to highlight both the similarities and differences between inverse optimal control and inverse noncooperative dynamic game theory.

Throughout the book, an emphasis is placed on fundamental questions and performance characterizations. For example, conditions analogous to identifiability and persistence of excitation are established under which inverse optimal control and inverse noncooperative dynamic game problems have either unique or functionally equivalent solutions.

It is hoped that this book will prove helpful and inspire future investigations of inverse optimal control and inverse noncooperative dynamic game theory.

Melbourne, Australia — Timothy L. Molloy
Heidelberg, Germany — Jairo Inga Charaja
Karlsruhe, Germany — Sören Hohmann
Brisbane, Australia — Tristan Perez

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The research on inverse problems at the Institute of Control Systems of the Karlsruhe Institute of Technology (KIT) started with the work on model-based design approaches for human–machine shared control systems by Dr. Michael Flad and Prof Hohmann in 2013. New results in biocybernetics revealed that human movement is well described by optimality principles, leading to dynamic games being natural candidates for modeling human–machine shared control, and raising the question of how the parameters of the human should be identified. Initially focused on human-driver behavior, the problem was later addressed further by the work of Dr. Inga Charaja, partially funded by the German Research Foundation's (DFG) research grant project "Inverse Noncooperative Dynamic Games in Automatic Control".

The undertakings of both research groups resulted in the presentation of two similar papers at the 2017 IFAC World Congress. This was the beginning of a fruitful collaboration between the groups over a number of years and publications, culminating in the completion of this monograph.

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Chapter 1
Introduction

1.1 Motivation

The notion that phenomena within the natural world, including human and animal behavior, arises from the optimization of interpretable criteria has inspired the study of *optimality* across almost all fields of human endeavor. Studies of optimality in nature date back to antiquity, with Heron of Alexandria discovering that rays of light reflected from mirrors take those paths with the shortest lengths and least travel times [33, pp. 167–168]. Optimality now underlies our understanding of the principle of least action and Fermat's principle of least time in physics, evolution and animal behavior in biology [42, 65, 66], human motor control in neuroscience [41, 64], and utility optimization in economics (among myriad other examples). Optimality has thus been described as "one of the oldest principles of theoretical science" [58] and "one of science's most pervasive and flexible metaprinciples" [59].

Despite the scientific quest to discover optimality principles and underlying optimality criteria from observational data, the study of mathematical optimization has principally focused on *forward problems* that involve finding the best or optimal values of decision variables under given optimality criteria. *Inverse problems* that instead involve finding criteria under which given values of decision variables are optimal have received less attention, particularly within the *optimal control* branch of mathematical optimization.

Optimal control is concerned with exerting optimal causal influence on a dynamical system evolving in (discrete or continuous) time, with the variables of influence called *controls* and the variables to be influenced called *states*. The forward problem of optimal control (or simply, *the optimal control problem*) specifically involves finding controls that lead to a given *cost functional* of the states and controls being minimized subject to the constraints imposed by a given dynamical system. Optimal control thus constitutes *dynamic* mathematical optimization with the decision variables being controls, and their optimality depending on time and the order in which they influence the dynamical system.

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Optimal control originated from the calculus of variations, and evolved significantly during the second half of the twentieth century with the celebrated work of Bellman on dynamic programming, Pontryagin on the minimum principle,[1] and Kalman on linear-quadratic (LQ) optimal control [10, 61]. Bellman's dynamic programming specifically led to the elegant result that the optimal controls for a dynamical system can be expressed as functions of its past states, with these functions being called optimal *feedback (control) laws*. In contrast, Pontryagin's minimum principle led to a set of conditions that trajectories or sequences of controls must satisfy in order to be optimal (i.e., a set of *necessary optimality conditions*). Finally, Kalman showed that optimal control problems involving linear dynamical systems and cost functionals that are quadratic in the state and control variables can be solved in an efficient manner via matrix equations. In recent years, optimal control has attracted much renewed attention due to its close relationship with reinforcement learning, which relaxes some of the (stronger) assumptions of optimal control such as having prior knowledge of the dynamical system (see, e.g., [9, 36, 37] for detailed discussions of the relationship between optimal control and reinforcement learning).

In this book, we investigate *inverse optimal control* problems (and their extensions in *noncooperative dynamic game theory*) that involve computing cost functionals under which given or measured state and control trajectories of dynamical systems are optimal. Interest in these inverse problems has grown significantly in recent years, sparked by their potential to model complex, dynamic decision-making tasks such as human navigation [5]; human arm movement [8, 62]; human pose adjustment and posture control [14, 56]; human eye movement [15]; the performance of human pilots, drivers, and operators [22, 26, 40, 43, 67, 68]; and other animal behaviors [18]. The solution of these inverse problems also raises the possibility of developing machines, robots, and autonomous agents that mimic the capabilities of human experts and highly evolved organisms [1, 2, 31, 46, 57].

1.2 Inverse Optimal Control

Rudolf Emil Kalman was the first to pose an inverse optimal control problem. In his famous 1964 paper, Kalman posed the question "*When is a Linear Control System Optimal?*", and considered the problem of finding all cost functionals under which a given feedback control law is optimal for a given dynamical system [32]. Importantly, he demonstrated that this inverse problem is frequently ill-posed, with a linear feedback control law often being optimal under more than one cost functional.

Kalman [32] originally posed and solved his inverse optimal control problem under several rather restrictive assumptions including that:

1. the dynamical system is linear and time-invariant;
2. the dynamical system has a single control variable;

[1] Pontryagin originally formulated the minimum principle as a maximum principle.

3. the given feedback control law is time-invariant and linear; and
4. the cost functionals considered are quadratic.

While subsequent works have focused on relaxing some of these assumptions (cf. [12, 29, 34, 47, 63]), most have remained concerned with the structural properties of optimal, mainly LQ, control problems given feedback control laws.

Within systems and control engineering, inverse optimal control has only recently expanded to encompass the data-driven (inverse) problem of computing cost functionals under which given or measured state and control trajectories are optimal. Indeed, Nori and Frezza in 2004 [53] appear to have been among the first in systems and control to examine this data-driven form of inverse optimal control. Similar *structural estimation* and *inverse reinforcement learning* problems had, however, earlier been examined in economics [24, 25] and computer science [52] (albeit mostly for systems evolving in discrete time with a finite number of states and/or controls).

In its data-driven form, inverse optimal control has begun to attract the attention of control theorists equipped with the powerful tools of (nonlinear) optimal control theory. Specifically, its data-driven form has been observed to naturally lend itself to solution and analysis via Pontryagin's minimum principle due to the principle's focus on optimal trajectories rather than optimal feedback control laws. In this context, Chaps. 3 and 4 present a control-theoretic introduction to (data-driven) inverse optimal control in both discrete and continuous time using Pontryagin's minimum principle.

1.3 Inverse Noncooperative Dynamic Game Theory

Game theory provides a mathematical theory of interaction between multiple rational decision-makers, called *players*; it is *dynamic* if the players interact by each exerting causal influence on a common dynamical system (in either discrete or continuous time); and it is noncooperative if the players pursue their own individual objectives, which may conflict with those of the other players. Noncooperative dynamic game theory is thus a natural extension of optimal control to settings in which the controls of a single dynamical system are divided between multiple different players, each with their own cost functional. However, unlike optimal control, the (forward) problem of finding optimal player strategies given the dynamical system and the player cost functionals is ambiguous since the notion of optimality itself ceases to be a well-defined concept.

A variety of optimality (or *solution*) concepts for *(forward) noncooperative dynamic games* have been developed by varying factors including the order in which the players make decisions, and what information the players have or believe about the other players and state of the dynamical system.[2] In this book, we shall focus on *Nash equilibrium* solutions that arise when all players act simultaneously and

[2] A detailed discussion of solution concepts for noncooperative dynamic games is beyond the scope of this book, but is given in [7, Chap. 1].

seek to minimize their individual cost functionals under the (correct) belief that all other players act likewise. A precise definition of Nash equilibria is deferred until the next chapter, but intuitively a player following their Nash equilibrium strategy has no incentive to unilaterally adopt a different strategy.

Nash equilibrium solutions to (forward) noncooperative dynamic games can be analyzed and obtained using the modern tools of optimal control including Bellman's dynamic programming, Pontryagin's minimum principle, and Kalman's matrix equations in the case of a linear dynamical system and quadratic player cost functionals [7]. Historically, however, noncooperative dynamic game theory evolved alongside optimal control (rather than after it), with Isaacs first introducing two-player noncooperative dynamic games in the 1950s and 1960s [28], and Starr and Ho [60] introducing N-player noncooperative dynamic games in 1969.[3]

Noncooperative dynamic game theory has since developed a rich literature and numerous applications in mathematics, economics, engineering, and biology including vehicle collision avoidance [7, 45, 49], modeling markets [17, 35], control of power systems [13], decentralized control of electric vehicles [39], vehicle formation control [23, 38], advanced driver assistance systems [19, 20, 30, 50], and modeling collision avoidance in birds [44]. In addition, recent experiments show the descriptive power of noncooperative dynamic games in modeling human–machine interaction or shared control systems [20, 27, 30, 48]. These results can be seen as a natural extension of the conjecture that human motion is governed by an optimality principle asserting the minimization of individual costs (see, e.g., [16, 54, 64]). Consequently, interactions between humans and machines (as players) modify the costs incurred by individuals, and hence the actions they respond with.

While noncooperative dynamic game theory evolved in parallel to optimal control, surprisingly little attention has been paid to its inverse problem of computing player cost functionals such that given state and player control trajectories (or feedback control laws) constitute a Nash equilibrium. Indeed, *inverse noncooperative dynamic game theory* appears to have only emerged within the last four decades, with most developments found in the economics literature. Notable early treatments include Fujii and Khargonekar [21] in 1988, and Carraro [11] in 1989, who both considered linear dynamical systems, quadratic player cost functionals, and given (or estimated) linear player feedback control laws (in the same spirit as Kalman's early work on inverse optimal control). Subsequent treatments in economics have focused on (data-driven) inverse noncooperative dynamic game problems (called *inverse noncooperative dynamic games*) involving given state and player control trajectories, with the vast majority considering relatively simple dynamical systems in discrete time with a finite number of states and/or controls (cf. [3, 6, 55], the survey paper of [4] and references therein). More recently, the related problem of multiagent

[3] Isaacs was the first to extend the concept of a Nash equilibrium, proposed by John Nash [51] for (static) game theory, to describe the (forward) solution of two-player noncooperative dynamic games. Starr and Ho [60] generalized Issacs' work to N-player noncooperative dynamic games, and were the first to explicitly note that it was no longer obvious what should be deemed a solution.

inverse reinforcement learning has received some attention in computer science, but again mostly in discrete time.

Despite having numerous potential applications in control beyond those covered by inverse optimal control including in multiagent systems and collaborative control, inverse noncooperative dynamic game theory has only recently been explored in its data-driven formulation using control-theoretic tools. Pontryagin's minimum principle is thus yet to be fully explored as a tool for analyzing and solving inverse noncooperative dynamic games. In this context, Chaps. 5 and 6 generalize and extend the inverse optimal control treatments of Chaps. 3 and 4 to inverse noncooperative dynamic game theory in both discrete and continuous time using Pontryagin's minimum principle (henceforth referred to simply as *the minimum principle*). Chapter 6 will specifically consider noncooperative dynamic game theory in continuous time with dynamical systems defined by differential equations. Following convention, hereon in this book we shall refer to (inverse) noncooperative dynamic games in continuous time as *(inverse) noncooperative differential games*, and (inverse) noncooperative dynamic games in discrete time as simply *(inverse) noncooperative dynamic games*.

1.4 Outline of this Book

This book is divided into seven chapters. This first chapter has served as an introduction to inverse problems in optimal control and noncooperative dynamic game theory, motivating their investigation using the minimum principle.

Chapter 2 gives the necessary mathematical background on static optimization, (forward) optimal control, and dynamic games. In particular, we present optimality conditions derived from minimum principles, which lay the foundation of the presented inverse optimal control and inverse dynamic game methods of this book.

Chapters 3 and 4 address inverse optimal control problems in discrete and continuous time, respectively. The first part of each chapter formulates specific inverse problems that may arise depending on the given state and control data. Direct approaches for solving inverse optimal control problems, called bilevel methods and based on bilevel optimization, are then discussed. Motivated by the limitations of these direct methods, we use the minimum principle to develop alternative methods along with theoretical results that characterize the existence and uniqueness of inverse optimal control solutions they may yield. We complete each chapter by examining the relationship between (data-driven) inverse optimal control and the feedback-law-based problem posed by Kalman as inverse LQ optimal control.

Chapters 5 and 6 extend the inverse optimal control methods and analysis of Chaps. 3 and 4 to inverse noncooperative dynamic games and inverse noncooperative differential games. Analogous to Chaps. 3 and 4, in Chaps. 5 and 6 we pose specific inverse problems before discussing direct methods for solving them. We then use the minimum principle in the form of (necessary) conditions for Nash equilibria to formulate efficient alternative solution methods with associated theoretical results characterizing the existence and uniqueness of the solutions they may yield. In addition, we complete each chapter by examining the specific solution of inverse LQ dynamic or differential games when player feedback laws are given rather than state and control trajectories.

Finally, Chap. 7 presents various simulation examples and an experimental case study of human driver behavior identification toward advanced driver assistance technology. The simulation examples and experimental case study serve to illustrate and compare the methods presented in the other chapters.

Each chapter in the book finishes with a section called "Notes and Further Reading", where we give additional information to help the reader find related work or extensions of the ideas presented, with the aim of illuminating current and potential future research directions and trends.

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Chapter 2
Background and Forward Problems

In this chapter, we briefly revisit concepts in (static) optimization, (forward) optimal control, and (forward) noncooperative dynamic game theory that will prove useful in later chapters on inverse optimal control and inverse noncooperative dynamic (and differential) game theory. Detailed treatments of these topics are provided in numerous books (e.g., [1, 3, 6, 12]), so we shall refer to these and other primary sources for rigorous mathematical proofs.

2.1 Static Optimization

Static optimization is an important precursor to optimal control and noncooperative dynamic (and differential) game theory.

2.1.1 General Formulation

Consider a real-valued *cost (or objective) function* $V : \mathscr{U} \mapsto \mathbb{R}$ defined on a *control-constraint set* $\mathscr{U}$ that is either a subset of $\mathbb{R}^m$ or the entirety of $\mathbb{R}^m$. The static optimization problem

$$\begin{aligned} \min_u \quad & V(u) \\ \text{s.t.} \quad & u \in \mathscr{U} \end{aligned} \tag{2.1}$$

involves determining an optimal *control* (or *decision*) variable $u^* \in \mathscr{U} \subset \mathbb{R}^m$ that leads to the cost function V attaining its minimum value over $\mathscr{U}$ in the sense that

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$V(u^*) \leq V(u)$ for all $u \in \mathscr{U}$. The value of a control variable u^* that minimizes V (i.e., a minimizing argument of V) is written as satisfying

$$u^* \in \arg\min_{u \in \mathscr{U}} V(u).$$

An important technical concern is that (2.1) may be *infeasible* in the sense that no u that minimizes the cost function V belongs to $\mathscr{U}$. For example, (2.1) is infeasible if the set $\mathscr{U}$ arises from contradicting constraints and is thus empty (denoted by $\mathscr{U} = \emptyset \triangleq \{\}$); it is also infeasible if V decreases without bound on $\mathscr{U}$ (such as in the case $V(u) = u$ with $\mathscr{U} = \mathbb{R}$ where $V(u) \to -\infty$ as $u \to -\infty$). The later example, in particular, highlights that for (2.1) to be feasible, it is necessary (though not always sufficient) for V to be bounded from below on $\mathscr{U}$ by some value $\kappa \in \mathbb{R}$ in the sense that $V(u) \geq \kappa$ for all $u \in \mathscr{U}$. The greatest value of κ that bounds V from below on $\mathscr{U}$ is called the *infimum* of V (on $\mathscr{U}$), and is written as

$$\begin{aligned} \inf_{u} \quad & V(u) \\ \text{s.t.} \quad & u \in \mathscr{U}. \end{aligned} \tag{2.2}$$

More precisely, the infimum of V is the greatest lower bound on the values of $V(u)$ with $u \in \mathscr{U}$ in the sense that $\inf_{u \in \mathscr{U}} V(u) \leq V(\bar{u})$ for all $\bar{u} \in \mathscr{U}$. The infimum can exist when the minimum does not since it need not correspond to a value of V attained on $\mathscr{U}$ (i.e., it may be that $\inf_{u \in \mathscr{U}} V(u) \neq V(\bar{u})$ for all $\bar{u} \in \mathscr{U}$). In this book, we shall often avoid explicitly assuming the existence of minima by instead considering infima, noting however that they correspond when (2.1) is feasible.

2.1.2 Necessary Optimality Conditions

Let us define $\nabla_u V(\bar{u}) \in \mathbb{R}^m$ as the gradient of the cost function V at $\bar{u} \in \mathbb{R}^m$. That is, the gradient of the cost function V at $\bar{u} \in \mathbb{R}^m$ is the vector

$$\nabla_u V(\bar{u}) = \begin{bmatrix} \left.\frac{\partial V(u)}{\partial u_{(1)}}\right|_{u=\bar{u}} \\ \left.\frac{\partial V(u)}{\partial u_{(2)}}\right|_{u=\bar{u}} \\ \vdots \\ \left.\frac{\partial V(u)}{u_{(m)}}\right|_{u=\bar{u}} \end{bmatrix}$$

where the components are the partial derivatives of V with respect to the components of the vector $u = \begin{bmatrix} u_{(1)} & u_{(2)} & \cdots & u_{(m)} \end{bmatrix}' \in \mathscr{U}$ evaluated at $\bar{u} \in \mathbb{R}^m$. Here and throughout the book, we use $'$ to denote the vector (or matrix) transpose.

If the cost function V is *continuously differentiable* on $\mathbb{R}^m$ (i.e., the gradient $\nabla_u V(\bar{u})$ exists and is a continuous function of $\bar{u}$) and $\mathscr{U}$ is a closed and convex subset of $\mathbb{R}^m$, then optimal solutions u^* to (2.1) lie either on the boundary of the set $\mathscr{U}$ (with a gradient directed outwards) or in the interior of the set $\mathscr{U}$ (with a zero gradient). Thus, if some $u \in \mathscr{U}$ is an optimal solution to (2.1) (i.e., if $u = u^*$), then

$$\nabla_u V(u)'(\bar{u} - u) \geq 0 \tag{2.3}$$

for all $\bar{u} \in \mathscr{U}$, which simplifies to $\nabla_u V(u) = 0$ if u is in the interior (i.e., not on the boundary) of $\mathscr{U}$. Here, we use 0 to denote either the scalar number zero, or a vector (or matrix) of appropriate dimensions with all zero elements.

It is important to note that $u \in \mathscr{U}$ must satisfy (2.3) in order to constitute an optimal solution to (2.1). However, if $u \in \mathscr{U}$ satisfies (2.3) we cannot, in general, conclude that it is an optimal solution to (2.1) since (2.3) is satisfied by all $u \in \mathscr{U}$ that are local (potentially non-global) minima, maxima, or inflection points of V. Thus, we say that (2.3) is a *necessary*, though not always *sufficient*, condition for $u \in \mathscr{U}$ to constitute an optimal solution to (2.1). An important special case in which (2.3) is both a necessary and sufficient condition for $u \in \mathscr{U}$ to be an optimal solution to (2.1) is when both the cost function V and constraint set $\mathscr{U}$ are convex.

2.1.3 Quadratic Programs

A *quadratic program* is a static optimization problem ((2.1) or (2.2)) in which the cost function V is given by a quadratic form in the sense that

$$V(u) = \frac{1}{2}u'\Omega u + b'u \tag{2.4}$$

for $u \in \mathscr{U}$ where $\Omega \in \mathbb{R}^{m \times m}$ is a given real *symmetric matrix* (i.e., $\Omega = \Omega'$), and $b \in \mathbb{R}^m$ is a given real column vector. In this book, we shall primarily concern ourselves with the solution of unconstrained quadratic programs of the form

$$\begin{aligned} \inf_u \quad & \frac{1}{2}u'\Omega u + b'u \\ \text{s.t.} \quad & u \in \mathbb{R}^m. \end{aligned} \tag{2.5}$$

The gradient of V when V is the quadratic form (2.4) is

$$\nabla_u V(u) = \Omega u + b.$$

The necessary optimality condition (2.3) for u to be a solution to the unconstrained quadratic program (2.5) is thus

$$\Omega u + b = 0.$$

If Ω is positive definite (denoted by $\Omega \succ 0$ and meaning that $u'\Omega u \geq 0$ for all $u \in \mathbb{R}^m$ with equality if and only if $u = 0$), then V is (strictly) convex and this condition becomes both necessary and sufficient for u to be an optimal solution to (2.5). Equivalently, if Ω is positive definite then u solves (2.5) if and only if

$$u = -\Omega^{-1} b \tag{2.6}$$

since Ω has an inverse Ω^{-1} when it is positive definite (i.e., Ω is *invertible* or *nonsingular*). If, however, Ω is positive semidefinite (denoted by $\Omega \succeq 0$ and meaning that $u'\Omega u \geq 0$ for all $u \in \mathbb{R}^m$), we require the following *Moore–Penrose pseudoinverse* and *singular value decomposition* (SVD) concepts to present necessary optimality conditions for (2.5).

Definition 2.1 (*Moore–Penrose Pseudoinverse*) A matrix $A^+ \in \mathbb{R}^{n \times m}$ is the *Moore–Penrose pseudoinverse* (or pseudoinverse) of a matrix $A \in \mathbb{R}^{m \times n}$ if it satisfies the four conditions:

$$AA^+A = A \tag{2.7a}$$
$$A^+AA^+ = A^+ \tag{2.7b}$$
$$(AA^+)' = AA^+ \tag{2.7c}$$
$$(A^+A)' = A^+A. \tag{2.7d}$$

Definition 2.2 (*Singular Value Decomposition of Positive Semidefinite Matrix*) For a positive semidefinite matrix Ω, the pair (U, Σ) is called a *singular value decomposition* (SVD) of Ω if $\Omega = U \Sigma U'$ where $\Sigma \in \mathbb{R}^{m \times m}$ is a diagonal matrix with nonnegative entries and $U \in \mathbb{R}^{m \times m}$.

Detailed discussions of these definitions are given in [2, Chap. 1] and [7, Chap. 14]. Importantly, they lead to the following proposition characterizing the solutions to unconstrained quadratic programs of the form of (2.5) when Ω is positive semidefinite.

Proposition 2.1 (Solutions to Unconstrained Quadratic Programs) *Consider the unconstrained quadratic program* (2.5) *where Ω is positive semidefinite with Moore–Penrose pseudoinverse Ω^+ and with a SVD (U, Σ) such that $\Omega = U \Sigma U'$. If $(I - \Omega\Omega^+) = 0$, then all $u \in \mathbb{R}^m$ satisfying*

$$u = -\Omega^+ b + U' \begin{bmatrix} 0 \\ z \end{bmatrix} \tag{2.8}$$

for any (arbitrary) $z \in \mathbb{R}^{m-r}$ are optimal solutions to (2.5) *where I denotes the identity matrix of appropriate dimensions, and $r \triangleq \operatorname{rank}(\Omega)$ is the matrix rank of Ω.*

Proof See [7, Proposition 15.2]. □