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Claudia Miller
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Emily E. Witt *Editors*

Women in Commutative Algebra

Proceedings of the 2019 WICA Workshop

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Association for Women in Mathematics Series

Volume 29

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ISSN 2364-5733 ISSN 2364-5741 (electronic)
Association for Women in Mathematics Series
ISBN 978-3-030-91985-6 ISBN 978-3-030-91986-3 (eBook)
https://doi.org/10.1007/978-3-030-91986-3

Mathematics Subject Classification: 13-XX, 14-XX, 18-XX

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This volume is the result of research activities that took place during the first workshop for Women in Commutative Algebra at Banff International Research Station for Mathematical Innovation and Discovery. The workshop brought together researchers from a diverse list of institutions and fostered research collaborations among women at different career stages and from different research backgrounds.

This volume has several purposes. First and foremost, it is a celebration of the high-level research activities that took place at the workshop. Further it wants to testify to the successful model behind the high research achievements, a model put in place by several fields in mathematics and, with Emily E. Witt, brought to commutative algebra by the hard work of Karen Smith, Sandra Spiroff, Irena Swanson, to whom we are extremely grateful. We have intended this volume to support and expand the goal of the workshop in re-enforcing the network of collaborations among women in commutative algebra by displaying in one place those connections, and bringing in new ones. The volume indeed contains articles or research advances that were made by the research groups, as well as some contributions related to the area of commutative algebra and, survey articles.

Commutative algebra is the study of the properties of rings that historically rose in algebraic and arithmetic geometry. With the development of several techniques and a rich theory, commutative algebra has become a thriving research area that feeds to and from several fields of mathematics such has topology and combinatorics, beyond the classical algebraic and arithmetic geometry. It would not be fair not to mention that significant advances have been made recently in the field with the the breakthrough of new techniques in positive and mixed characteristic methods and homological algebra, and the consequent solution of long-standing conjectures.

The volume reflects the ripple effect of such breakthroughs that have inspired a great deal of activities in commutative algebra. Our volume delivers readings that span from case studies to survey articles and cover a wide range of topics in commutative algebra. The study of characteristic p rings is present in this volume through the classification of Frobenius forms in certain dimension, the Frobenius singularities of certain varieties, and through a comprehensive survey of the Hilbert–Kunz function; further, the reader can find results of a homological flavor in the articles that deliver resolutions of powers of the homogeneous maximal ideal of graded Koszul algebras, the construction of the truncated free resolution for the residue field, or the notion of Tor-independent modules; finally, the volume contains several articles that expand on the connection between homological and combinatorial invariants.

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Acknowledgments

We would like to thank Emily E. Witt's co-organizers, Karen Smith, Sandra Spiroff, and Irena Swanson, of the Women in Commutative Algebra workshop at the Banff International Research Station for gathering such a great group of 3 researchers

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On Gerko's Strongly Tor-independent Modules

Hannah Altmann and Keri Sather-Wagstaff

Keywords Differential graded algebras · Semidualizing modules · Syzygies · Tor-independent modules

1 Introduction

We are interested in how existence of certain sequences of modules over a local ring $(R, \mathfrak{m}_R)$ imposes restrictions on R. Specifically, we investigate what Gerko [6] calls strongly Tor-independent R-modules: A sequence $N_1, \ldots, N_n$ of R-modules is *strongly Tor-independent* provided $\operatorname{Tor}^R_{\geqslant 1}(N_{j_1} \otimes_R \cdots \otimes_R N_{j_t}, N_{j_{t+1}}) = 0$ for all distinct $j_1, \ldots, j_{t+1}$. Gerko is led to this notion in his study of Foxby's semidualizing modules [5] and Christensen's semidualizing complexes [3]. In particular, Gerko [6, Theorem 4.5] proves that if R is artinian and possesses a sequence of strongly Tor-independent modules of length n, then $\mathfrak{m}_R^n \neq 0$. (Note that Gerko's result only assumes the modules are finitely generated and strongly Tor-independent, not necessarily semidualizing.) This generalizes readily from artinian rings to Cohen–Macaulay rings; see Proposition 5.1 below.

Our goal in this paper is to prove the following non-Cohen–Macaulay complement to Gerko's result.

Theorem 1.1 *Assume* $(R, \mathfrak{m}_R)$ *is a local ring. If* $N_1, \ldots, N_n$ *are non-free, strongly Tor-independent* R*-modules, then* $n \leqslant \operatorname{ecodepth}(R)$.

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C. Miller et al. (eds.), *Women in Commutative Algebra*, Association for Women in Mathematics Series 29, https://doi.org/10.1007/978-3-030-91986-3_1

Here $\operatorname{ecodepth}(R) = \beta_0^R(\mathfrak{m}_R) - \operatorname{depth}(R)$ is the *embedding codepth of* R, where $\beta_0^R(\mathfrak{m}_R)$ is the minimal number of generators of $\mathfrak{m}_R$. Note that our result does not recover Gerko's, but compliments it. Our proof is the subject of Sect. 5 below.

Part of the proof of our result is modeled on Gerko's proof with one crucial difference: where Gerko works over an artinian ring, we work over a finite dimensional DG algebra. See Sects. 2 and 3 for background material and foundational results, including our DG version of Gerko's notion of strong Tor-independence. Theorem 4.7 is our main result in the DG context, which is the culmination of Sect. 4. Our proof relies on a DG syzygy construction of Avramov et al. [2].

2 DG Homological Algebra

Let R be a nonzero commutative Noetherian ring with identity. We work with R-complexes indexed homologically, so for us an R-complex X has differential $\partial_i^X : X_i \to X_{i-1}$. The *supremum* and *infimum* of X are respectively

$$\sup(X) = \sup\{i \in \mathbb{Z} \mid X_i \neq 0\} \qquad \inf(X) = \inf\{i \in \mathbb{Z} \mid X_i \neq 0\}.$$

The *amplitude* of X is $\operatorname{amp}(X) = \sup(X) - \inf(X)$. Frequently we consider these invariants applied to the total homology $H(X)$, e.g., as $\sup(H(X))$.

As we noted in the introduction, the proof of Theorem 1.1 uses DG techniques which we summarize next. See, e.g., [1, 4] for more details.

A *differential graded (DG) R-algebra* is an R-complex A equipped with an R-linear chain map $A \otimes_R A \to A$ denoted $a \otimes a' \mapsto aa'$ that is unital, associative, and graded commutative. We simply write *DG algebra* when $R = \mathbb{Z}$. The chain map condition here implies that this multiplication is also distributive and satisfies the *Leibniz Rule*: $\partial(aa') = \partial(a)a' + (-1)^{|a|}a\partial(a')$ where $|a|$ is the homological degree of a. We say that A is *positively graded* provided $A_i = 0$ for all $i < 0$. For example, the trivial R-complex R is a positively graded DG R-algebra, so too is every Koszul complex over R, using the wedge product. The *underlying algebra* associated to A is the R-algebra $A^\natural = \bigoplus_{i \in \mathbb{Z}} A_i$.

If R is local, then a positively graded DG R-algebra A is *local* provided $H_0(A)$ is Noetherian, each $H_0(A)$-module $H_i(A)$ is finitely generated for all $i \geqslant 0$, and the ring $H_0(A)$ is a local R-algebra.

Let A be a DG R-algebra. A *DG A-module* is an R-complex X equipped with an R-linear chain map $A \otimes_R X \to X$ denoted $a \otimes x \mapsto ax$ that is unital and associative. For instance DG R-modules are precisely R-complexes. We say that X is *homologically bounded* if $\operatorname{amp}(H(X)) < \infty$, and we say that X is *homologically finite* if $H(X)$ is finitely generated over $H_0(A)$. We write $\Sigma^n X$ for the *nth shift of* X obtained by $(\Sigma^n X)_i = X_{i-n}$ and $\partial_i^{\Sigma^n X} = (-1)^n \partial_{i-n}^X$. Quasiisomorphisms between R-complexes, i.e., chain maps that induce isomorphisms on the level of homology, are identified with the symbol $\simeq$.

Let A be positively graded and let X be a DG A-module such that $\inf(X) > -\infty$. We say that X is *semifree* if the underlying $A^\natural$-module $X^\natural$ is free. In this case a *semibasis* for X is a set of homogeneous elements of X that is a basis for $X^\natural$ over $A^\natural$. A *semifree resolution* of a DG A-module Y with $\inf(H(Y)) > -\infty$ is a quasiisomorphism $F \xrightarrow{\simeq} Y$ such that F is semifree. The derived tensor product of DG A-modules Y and Z is $Y \otimes_A^{\mathbf{L}} Z \simeq F \otimes_A Z$ where $F \xrightarrow{\simeq} Y$ is a semifree resolution of Y. We say that Y is *perfect* if it has a semifree resolution $F \xrightarrow{\simeq} Y$ such that F has a finite semibasis.

Let A be a local DG R-algebra, and let Y be a homogically finite DG A-module. By [2, Proposition B.7] Y has a *minimal semifree resolution*, i.e., a semifree resolution $F \xrightarrow{\simeq} Y$ such that the semibasis for F is finite in each homological degree and $\partial^F(F) \subseteq \mathfrak{m}_A F$.

3 Perfect DG Modules and Tensor Products

Throughout this section, let A be a positively graded commutative homologically bounded DG algebra, say $\operatorname{amp}(H(A)) = s$, and assume that $A \not\simeq 0$.

Most of this section focuses on four foundational results on perfect DG modules.

Lemma 3.1 *Let L be a non-zero semifree DG A-module with a semibasis B concentrated in a single degree n. Then $L \cong \Sigma^n A^{(B)}$. In particular,* $\inf(H(L)) = n$ *and* $\sup(H(L)) = s + n$ *and* $\operatorname{amp}(H(L)) = s$.

Proof It suffices to prove that $L \cong \Sigma^n A^{(B)}$. Apply an appropriate shift to assume without loss of generality that $n = 0$.

The semifree/semibasis assumptions tell us that every element $x \in L$ has the form $\sum_{e\in B}^{\text{finite}} a_e e$; the linear independence of the semibasis tells us that this representation is essentially unique. Since A is positively graded, we have $L_{-1} = 0$, so $\partial^L(e) = 0$ for all $e \in B$. Hence, the Leibniz rule for L implies that

$$\partial^L\left(\sum_{e\in B}^{\text{finite}} a_e e\right) = \sum_i^{\text{finite}} \partial^A(a_e)e + \sum_i^{\text{finite}} (-1)^{|a_e|} a_e \partial^L(e) = \sum_i^{\text{finite}} \partial^A(a_e)e.$$

From this, it follows that the map $A^{(B)} \to L$ given by the identity on B is an isomorphism. □

Proposition 3.2 *Let L be a non-zero semifree DG A-module with a semibasis B concentrated in degrees $n, n+1, \ldots, n+m$ where $n, m \in \mathbb{Z}$ and $m \geqslant 0$. Then* $\inf(H(L)) \geqslant n$ *and* $\sup(H(L)) \leqslant s + n + m$, *so* $\operatorname{amp}(H(L)) \leqslant s + m$.

Proof It suffices to show that $\inf(H(L)) \geqslant n$ and $\sup(H(L)) \leqslant s + n + m$. We induct on m. The base case $m = 0$ follows from Lemma 3.1.

For the induction step, assume that $m \geqslant 1$ and that the result holds for semifree DG A-modules with semibasis concentrated in degrees $n, n+1, \ldots, n+m-1$. Set

$$B' = \{e \in B \mid |e| < n + m\}$$

and let L' denote the semifree submodule of L spanned over A by B'. (See the first paragraph of the proof of [2, Proposition 4.2] for further details.) Note that L' has semibasis B' concentrated in degrees $n, n+1, \ldots, n+m-1$. In particular, our induction assumption applies to L' to give $\inf(H(L')) \geqslant n$ and $\sup(H(L')) \leqslant s+n+m-1$.

If $L = L'$, then we are done by our induction assumption. So assume that $L \neq L'$. Then the quotient L/L' is semifree and non-zero with semibasis concentrated in degree $n+m$. So, Lemma 3.1 implies that $\inf(H(L/L')) = n+m$ and $\sup(H(L/L')) = s+n+m$. Now, consider the short exact sequence

$$0 \to L' \to L \to L/L' \to 0. \tag{1}$$

The desired conclusions for L follow from the associated long exact sequence in homology. □

Now, we use the preceding two results to analyze derived tensor products.

Lemma 3.3 *Let L be a non-zero semifree DG A-module with a semibasis B concentrated in a single degree, say n, and let Y be a homologically bounded DG A-module. Then $L \otimes^{\mathbf{L}}_A Y \simeq \Sigma^n Y^{(B)}$. In particular,* $\inf(H(L \otimes^{\mathbf{L}}_A Y)) = \inf(H(Y)) + n$ *and* $\sup(H(L \otimes^{\mathbf{L}}_A Y)) = \sup(H(Y)) + n$ *and* $\operatorname{amp}(H(L \otimes^{\mathbf{L}}_A Y)) = \operatorname{amp}(H(Y))$.

Proof Immediate from Lemma 3.1. □

Proposition 3.4 *Let L be a non-zero semifree DG A-module with a semibasis B concentrated in degrees $n, n+1, \ldots n+m$ where $n, m \in \mathbb{Z}$ and $m \geqslant 0$, and let Y be a homologically bounded DG A-module. Then* $\inf(H(L \otimes^{\mathbf{L}}_A Y)) \geqslant \inf(H(Y)) + n$ *and* $\sup(H(L \otimes^{\mathbf{L}}_A Y)) \leqslant \sup(H(Y)) + n + m$*, so* $\operatorname{amp}(H(L \otimes^{\mathbf{L}}_A Y)) \leqslant \operatorname{amp}(H(Y)) + m$.

Proof As in the proof of Proposition 3.2, we induct on m. The base case $m = 0$ follows from Lemma 3.3.

For the induction step, assume $m \geqslant 1$ and the result holds for semifree DG A-modules with semibasis concentrated in degrees $n, n+1, \ldots, n+m-1$ and $Y \in D_b(A)$. We work with the notation from the proof of Proposition 3.2, and we assume that $L \neq L'$. The exact sequence (1) of semi-free DG modules gives rise to the following distinguished triangle in $\mathcal{D}(A)$.

$$L' \otimes^{\mathbf{L}}_A Y \to L \otimes^{\mathbf{L}}_A Y \to (L/L') \otimes^{\mathbf{L}}_A Y \to$$

Another long exact sequence argument gives the desired conclusion. □

We close this section with our DG version of strongly Tor-independent modules.

Definition 3.5 The DG A-modules $K_1, \ldots, K_n$ are said to be *strongly Tor-independent* if for any subset $I \subset \{1, \ldots, n\}$ we have $\operatorname{amp}(H(\otimes^{\mathbf{L}}_{i \in I} K_i)) \leqslant s$.

Remark 3.6 It is worth noting that the definition of $K_1, \ldots, K_n$ being strongly Tor-independent includes $\operatorname{amp}(H(K_i))) \leqslant s$ for all $i = 1, \ldots, n$. Also, if $K_1, \ldots, K_n$

are strongly Tor-independent, then so is any reordering by the commutativity of tensor products.

4 Syzygies and Strongly Tor-independent DG Modules

Throughout this section, let $(A, \mathfrak{m}_A)$ be a local homologically bounded DG algebra, say $\operatorname{amp}(H(A)) = s$, and assume that $A \not\simeq 0$ and $\mathfrak{m}_A = A_+$. It follows that A_0 is a field.

The purpose of this section is to provide a DG version of part of a result of Gerko [6, Theorem 4.5]. Key to this is the following slight modification of the syzygy construction of Avramov et al. mentioned in the introduction.

Construction 4.1 Let K be a homologically finite DG A-module. Let $F \simeq K$ be a minimal semifree resolution of K, and let E be a semibasis for F. Let $F^{(p)}$ be the semifree DG A-submodule of F spanned by $E_{\leqslant p} := \cup_{m \leqslant p} E_m$.

Set $t = \sup(H(K))$, and consider the soft truncation $\widetilde{K} = \tau_{\leqslant r}(F)$ for a fixed integer $r \geqslant t$. Note that the natural morphism $F \to \widetilde{K}$ is a surjective quasiisomorphism of DG A-modules, so we have $\widetilde{K} \simeq F \simeq K$. Next, set $L = F^{(r)}$, which is semifree with a finite semibasis $E_{\leqslant r}$. Furthermore, the composition π of the natural morphisms $L = F^{(r)} \to F \to \widetilde{K}$ is surjective because the morphism $F \to \widetilde{K}$ is surjective, the morphism $L \to F$ is surjective in degrees $\leqslant r$, and $\widetilde{K}_i = 0$ for all $i > r$. Set $\operatorname{Syz}_r(K) = \ker(\pi) \subseteq L$ and let $\alpha\colon \operatorname{Syz}_r(K) \to L$ be the inclusion map.

Proposition 4.2 *Let K be a homologically finite DG A-module. With the notation of Construction 4.1, there is a short exact sequence of morphisms of DG A-modules*

$$0 \to \operatorname{Syz}_r(K) \xrightarrow{\alpha} L \xrightarrow{\pi} \widetilde{K} \to 0 \tag{2}$$

such that L is semifree with a finite semibasis and where $\widetilde{K} \simeq K$ and $\operatorname{Im}(\alpha) \subseteq A_+ L$.

Proof Argue as in the proof of [2, Proposition 4.2]. □

Our proof of Theorem 1.1 hinges on the behavior for syzygies documented in the following four results.

Lemma 4.3 *Let K be a homologically finite DG A-module with* $\operatorname{amp}(H(K)) \leqslant s$ *and* $K' = \operatorname{Syz}_r(K)$ *where* $r \geqslant \sup(H(K))$. *Then* $\sup(H(K')) \leqslant s + r$ *and* $\inf(H(K')) \geqslant r$. *Therefore,* $\operatorname{amp}(H(K')) \leqslant s$.

Proof Use the notation from Construction 4.1. Then $\sup(H(L)) \leqslant s + r$ by Proposition 3.2. Also, by definition we have $\sup(H(\widetilde{K})) = \sup(H(K)) \leqslant r \leqslant r+s$. The long exact sequence in homology coming from (2) implies $\sup(H(K')) \leqslant s+r$. Also, $\inf(H(K')) \geqslant \inf(K') \geqslant r$ because π_i is an isomorphism for all $i < r$ by Construction 4.1. So, $\operatorname{amp}(H(K')) = \sup(H(K')) - \inf(H(K')) \leqslant s+r-r = s$. □

Proposition 4.4 *Let K be a homologically finite DG A-module and set $K' = \mathrm{Syz}_r(K)$ where $r \geqslant \sup(H(K))$. Let Y be a homologically bounded DG A-module and assume that K, Y are strongly Tor-independent. Then $\sup(H(K' \otimes^{\mathbf{L}}_A Y)) \leqslant \sup(H(Y)) + r$ and $\inf(H(K' \otimes^{\mathbf{L}}_A Y)) \geqslant \inf(H(Y)) + r$. So, $\mathrm{amp}(H(K' \otimes^{\mathbf{L}}_A Y)) \leqslant s$; in particular, K', Y are strongly Tor-independent.*

Proof Let $G \xrightarrow{\simeq} Y$ be a semifree resolution of Y. Let $\widetilde{K}$ and L be as in Construction 4.1. As K, Y are strongly Tor-independent we have $\sup(H(\widetilde{K} \otimes_A G)) \leqslant s$. Also, Proposition 3.4 implies $\sup(H(L \otimes_A G)) \leqslant \sup(H(Y)) + r$. To conclude the proof, consider the short exact sequence

$$0 \to K' \otimes_A G \to L \otimes_A G \to \widetilde{K} \otimes_A G \to 0 \tag{3}$$

and argue as in the proof of Lemma 4.3. □

Proposition 4.5 *Let $K_1, K_2, \ldots, K_n$ be strongly Tor-independent, homologically finite DG A-modules for $n \in \mathbb{Z}^+$ and $K_i' = \mathrm{Syz}_{r_i}(K_i)$ where $r_i \geqslant \sup(H(K_i))$. Then $K_1', \ldots, K_m', K_{m+1}, \ldots, K_n$ are strongly Tor-independent for all $m = 1, \ldots, n$.*

Proof Induct on m using Proposition 4.4. □

Proposition 4.6 *Let $K_1, K_2, \ldots, K_j$ be strongly Tor-independent DG A-modules, and set $K_i' = \mathrm{Syz}_{r_i}(K_i)$ where $r_i \geqslant \sup(H(K_i))$ for $i = 1, 2, \ldots, j$. If $\mathfrak{m}_A^n = 0$, then $\mathfrak{m}_{H(A)}^{n-j} H(\otimes^{\mathbf{L}}_{i=1,\ldots,j} K_i') = 0$.*

Proof Shift K_i if necessary to assume without loss of generality that $\inf(H(K_i)) = 0$ for $i = 1, \ldots, j$. For $i = 1, \ldots, j$ let $G_i \xrightarrow{\simeq} K_i'$ be semifree resolutions, and consider the following diagram with notation as in Construction 4.1.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i' & \xrightarrow{\subseteq} & L_i & \longrightarrow & \widetilde{K_i} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\simeq} & \nearrow_{\alpha_i} & & & & & \\ & & G_i & & & & & & \end{array} \tag{4}$$

Notice, $\mathrm{Im}(\alpha_i) \subseteq K_i' \subseteq \mathfrak{m}_A L_i$ for $i = 1, 2, \ldots, j$.

Set $\mathcal{G} = \otimes_{i=1,\ldots,j-1} G_i$ and consider the following commutative diagram

$$\begin{array}{ccc} \left(\otimes^{\mathbf{L}}_{i=1,\ldots,j-1} K_i'\right) \otimes^{\mathbf{L}}_A K_j' \simeq \mathcal{G} \otimes_A G_j & \xrightarrow{\quad\beta\quad} & \mathfrak{m}_A^j((\otimes_{i=1,\ldots,j-1} L_i) \otimes_A L_j) \\ \downarrow{\scriptstyle \mathcal{G}\otimes\alpha_j} & \searrow^{\theta\otimes\alpha_j} & \downarrow{\scriptstyle\subseteq} \\ \mathcal{G} \otimes_A L_j & \xrightarrow[\theta\otimes L_j]{} & (\otimes_{i=1,\ldots,j-1} L_i) \otimes_A L_j \end{array}$$

where $\theta = \otimes_{i=1,\ldots,j-1} \alpha_i$ and β is induced by $\theta \otimes \alpha_j$.

Claim: $H(\beta)$ is 1-1. Notice that $H_i(\mathcal{G} \otimes_A G_j) = 0$ for all $i < r_1 + \ldots + r_j$, so it suffices to show that $H_i(\beta)$ is 1-1 for all $i \geqslant r_1 + \ldots + r_j$. To this end it suffices to show $H_i(\mathcal{G} \otimes \alpha_j)$ and $H_i(\theta \otimes L_j)$ are 1-1 for all $i \geqslant r_1 + \ldots + r_j$. First we show this for $H_i(\mathcal{G} \otimes \alpha_j)$. Consider the short exact sequence

$$0 \to \mathcal{G} \otimes_A G_j \xrightarrow{\mathcal{G} \otimes \alpha_j} \mathcal{G} \otimes_A L_j \to \mathcal{G} \otimes_A \widetilde{K_j} \to 0. \tag{5}$$

Proposition 4.4 implies

$$\sup(H(\mathcal{G} \otimes_A \widetilde{K_j})) \leqslant r_1 + \ldots + r_{j-1} + \sup(H(\widetilde{K_j})) \leqslant r_1 + \ldots + r_j.$$

Thus, the long exact sequence in homology associated to (5) implies $H_i(\mathcal{G} \otimes_A \alpha_j)$ is 1-1 for all $i \geqslant r_1 + \ldots + r_j$ as desired.

Next, we show $H_i(\theta \otimes L_j)$ is 1-1 for $i \geqslant r_1 + \ldots + r_j$. Consider the exact sequence

$$0 \to \mathcal{G} \otimes_A L_j \xrightarrow{\theta \otimes L_j} (\otimes_{i=1,\ldots,j-1} L_i) \otimes_A L_j \to (\otimes_{i=1,\ldots,j-1} \widetilde{K}_i) \otimes_A L_j \to 0. \tag{6}$$

The first inequality in the next display follows from Proposition 3.4

$$\begin{aligned}\sup(H((\otimes_{i=1,\ldots,j-1} \widetilde{K}_i) \otimes_A L_j)) &\leqslant \sup(H(\otimes_{i=1,\ldots,j-1} \widetilde{K}_i)) + r_j \\ &\leqslant r_1 + \ldots + r_{j-1} + r_j.\end{aligned}$$

Thus, the long exact sequence in homology associated to (6) implies $H_i(\theta \otimes L_j)$ is 1-1 for all $i \geqslant r_1 + \ldots + r_j$. This establishes the claim.

To complete the proof it remains to show $\mathfrak{m}_{H(A)}^{n-j} H((\otimes^{\mathbf{L}}_{i=1,\ldots,j-1} K'_i) \otimes^{\mathbf{L}}_A K'_j) = 0$. Since $H(\beta)$ is 1-1, we have $H((\otimes^{\mathbf{L}}_{i=1,\ldots,j-1} K'_i) \otimes^{\mathbf{L}}_A K'_j)$ isomorphic to a submodule of $H(\mathfrak{m}_A^j((\otimes_{i=1,\ldots,j-1} L_i) \otimes_A L_j))$. So it suffices to show that $\mathfrak{m}_{H(A)}^{n-j}$ annihilates $H(\mathfrak{m}_A^j((\otimes_{i=1,\ldots,j-1} L_i) \otimes_A L_j))$; this annihilation holds because $\mathfrak{m}_A^n = 0$. □

Here is the aforementioned version of part of [6, Theorem 4.5].

Theorem 4.7 *Let $K_1, \ldots, K_n$ be strongly Tor-independent non-perfect DG A-modules. Then $\mathfrak{m}_A^n \neq 0$, therefore, $n \leqslant s$.*

Proof Suppose $\mathfrak{m}_A^n = 0$. Proposition 4.6 implies that $0 = \mathfrak{m}_{H(A)}^0 H(\otimes^{\mathbf{L}}_{i=1,\ldots,n} K'_i) = H(\otimes^{\mathbf{L}}_{i=1,\ldots,n} K'_i)$. Since each K_i has a minimal resolution for $i = 1, \ldots, n$, we must have $H(K'_l) = 0$ for some l. Hence, K_l has a semifree basis concentrated in a finite number of degrees. This contradicts our assumption that K_i is not perfect for $i = 1, \ldots, n$. Therefore, $\mathfrak{m}_A^n \neq 0$.

Now we show $n \leqslant s$. Soft truncate A to get $A' \simeq A$ such that $\sup(A') = s$. Thus, $\mathfrak{m}_{A'}^{s+1} = 0$. The sequence of n strongly Tor-independent non-perfect DG A-modules gives rise to a sequence of n strongly Tor-independent non-perfect DG A'-modules. Since $\mathfrak{m}_{A'}^n \neq 0$ and $\mathfrak{m}_{A'}^{s+1} = 0$, we have $n \leqslant s$. □

5 Proof of Theorem 1.1

Induct on depth(R).

Base Case: depth$(R) = 0$. Let K denote the Koszul complex over R on a minimal generating sequence for $\mathfrak{m}_R$. The condition depth$(R) = 0$ implies

$$\operatorname{amp}(H(K)) = \operatorname{ecodepth}(R) = \operatorname{amp}(K). \tag{7}$$

Claim: The sequence $K \otimes_R^{\mathbf{L}} N_1, \ldots, K \otimes_R^{\mathbf{L}} N_n$ is a strongly Tor-independent sequence of DG K-modules. To establish the claim we compute derived tensor products where both $\bigotimes^{\mathbf{L}}$ are indexed by $i \in I$:

$$\bigotimes_K^{\mathbf{L}} (K \otimes_R^{\mathbf{L}} N_i) \simeq K \otimes_R^{\mathbf{L}} (\bigotimes_R^{\mathbf{L}} N_i).$$

From this we get the first equality in the next display.

$$\begin{aligned} \operatorname{amp}(H(\bigotimes_K^{\mathbf{L}} (K \otimes_R^{\mathbf{L}} N_i))) &= \operatorname{amp}(H(K \otimes_R^{\mathbf{L}} (\bigotimes_R^{\mathbf{L}} N_i))) \\ &= \operatorname{amp}(H(K \otimes_R (\textstyle\bigotimes_{i \in I} N_i))) \\ &\leqslant \operatorname{amp}(K \otimes_R (\textstyle\bigotimes_{i \in I} N_i)) \\ &= \operatorname{amp}(K) \\ &= \operatorname{amp}(H(K)) \end{aligned}$$

The second equality comes from the strong Tor-independence of the original sequence. The inequality and the third equality are routine, and the final equality is by (7). This establishes the claim.

A construction of Avramov provides a local homologically bounded DG algebra $(A, \mathfrak{m}_A)$ such that $A \simeq K \not\simeq 0$ and $\mathfrak{m}_A = A_+$; see [7, 8]. The strongly Tor-independent sequence $K \otimes_R^{\mathbf{L}} N_1, \ldots, K \otimes_R^{\mathbf{L}} N_n$ over K gives rise to a strongly Tor-independent sequence $M_1, \ldots, M_n$ over A. Now, Theorem 4.7 and (7) imply $n \leqslant \operatorname{amp}(H(A)) = \operatorname{amp}(H(K)) = \operatorname{ecodepth}(R)$. This concludes the proof of the Base Case.

Inductive Step: Assume depth$(R) > 0$ and the result holds for local rings S with depth$(S) =$ depth$(R) - 1$. For $i = 1, \ldots, n$ let N_i' be the first syzygy of N_i. Since the sequence $N_1, \ldots, N_n$ is strongly Tor-independent, so is the sequence $N_1', \ldots, N_n'$. Moreover, strong Tor-independence implies that $\bigotimes_{i \in I} N_i'$ is a submodule of a free R-module, for each subset $i \in \{1, \ldots, n\}$.

Use prime avoidance to find an R-regular element $x \in \mathfrak{m}_R - \mathfrak{m}_R^2$. Set $\overline{R} = R/xR$. Note that $\operatorname{depth}(\overline{R}) = \operatorname{depth}(R) - 1$ and $\operatorname{ecodepth}(\overline{R}) = \operatorname{ecodepth}(R)$. The fact that each $\bigotimes_{i \in I} N_i'$ is a submodule of a free R-module implies that x is also $\bigotimes_{i \in I} N_i'$-regular. It is straightforward to show that the sequence $\overline{R} \otimes_R^{\mathbf{L}} N_1', \ldots, \overline{R} \otimes_R^{\mathbf{L}} N_n'$ is strongly Tor-independent over $\overline{R}$. By our induction hypothesis we have

$$n \leqslant \operatorname{ecodepth}(\overline{R}) = \operatorname{ecodepth}(R)$$

as desired. □

We conclude with the generalization of Gerko's result [6, Theorem 4.5] from artinian rings to Cohen–Macaulay rings mentioned in the introduction. In preparation, recall that the *Loewy length* of a finite length R-module M is

$$\ell\ell_R(M) = \min\{i \geqslant 0 \mid \mathfrak{m}_R^i M = 0\}.$$

The *generalized Loewy length* of R is then

$$\ell\ell_R(R) = \min\{\ell\ell_R(R/\langle \mathbf{x} \rangle) \mid \mathbf{x} \text{ is a system of parameters of } R\}.$$

Notice that when R is artinian, i.e., when R has finite length as an R-module, the generalized Loewy length of R equals the Loewy length of R, so the symbol $\ell\ell_R(R)$ is unambiguous.

Proposition 5.1 *Assume that R is Cohen–Macaulay and that $K_1, \ldots, K_n$ are non-free, finitely generated, strongly Tor-independent R-modules. Then $n \leqslant \ell\ell_R(R)$.*

Proof We induct on $d = \dim(R)$. In the base case $d = 0$, the ring R is artinian, so Gerko's result [6, Theorem 4.5] says that $\mathfrak{m}_R^n \neq 0$. By the definition of Loewy length, this is exactly the desired conclusion.

For the induction step, assume that $d \geqslant 1$, and that our result holds for Cohen–Macaulay local rings of dimension $d - 1$. Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters of R such that $\ell\ell_R(R) = \ell\ell_{R/\langle \mathbf{x} \rangle}(R/\langle \mathbf{x} \rangle)$. Since R is Cohen–Macaulay, this is a maximal R-regular sequence. Furthermore, the definition of generalized Loewy length implies that $\ell\ell_{R/\langle x_1 \rangle}(R/\langle x_1 \rangle) \geqslant \ell\ell_{R/\langle \mathbf{x} \rangle}(R/\langle \mathbf{x} \rangle) = \ell\ell_R(R)$.

Replace the modules K_i with their first syzygies if necessary to assume without loss of generality that x_1 is K_i-regular for $i = 1, \ldots, n$. From this, it is straightforward to use the assumptions on the K_i to conclude that $K_1/x_1K_1, \ldots, K_n/x_1K_n$ are non-free, finitely generated, strongly Tor-independent $R/\langle x_1 \rangle$-modules. Thus, our induction hypothesis implies that $n \geqslant \ell\ell_{R/\langle x_1 \rangle}(R/\langle x_1 \rangle) \geqslant \ell\ell_R(R)$, as desired. □

Acknowledgement We are grateful to the anonymous referee for their thoughtful comments.

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Properties of the Toric Rings of a Chordal Bipartite Family of Graphs

Laura Ballard

1 Introduction

In recent decades, there has been a growing interest in the investigation of algebraic invariants associated to combinatorial structures. Toric ideals of graphs (and the associated edge rings), a special case of the classical notion of a toric ideal, have been studied by various authors with regard to invariants such as depth, dimension, projective dimension, regularity, graded Betti numbers, Hilbert series, and multiplicity, usually for particular families of graphs (see for example [2, 3, 5, 7–10, 12, 14, 16–19, 22, 23, 26]). We note in Remarks 2.6 and 2.14 that the family we consider does not overlap at all or for large n with those considered in [5, 8, 9], and [23]; it is more obviously distinct from other families that have been studied. We think it fitting to mention that the recent book by Herzog et al. [15] also investigates toric ideals of graphs as well as binomial ideals coming from other combinatorial structures.

In this work, we consider a family of graphs with iterated subfamilies and develop algebraic properties of the toric rings associated to the family which depend only on the number of vertices (equivalently, the number of edges) in the associated graphs. In the development of this project, we were particularly inspired by the work of Jennifer Biermann, Augustine O'Keefe, and Adam Van Tuyl in [3], where they establish a lower bound for the regularity of the toric ideal of any finite simple graph and an upper bound for the regularity of the toric ideal of a chordal bipartite graph. Our goal is to construct as "simple" a family of graphs as possible that still yields

The author was partially supported by the National Science Foundation (DMS-1003384).

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C. Miller et al. (eds.), *Women in Commutative Algebra*, Association for Women in Mathematics Series 29, https://doi.org/10.1007/978-3-030-91986-3_2

interesting toric ideals. It is our hope that our process and results will lead to further generalizations of properties of toric ideals for other (perhaps broader) families of graphs, or for graphs containing or arising from such graphs.

Herein, we introduce the infinite family $\mathcal{F}$ of chordal bipartite graphs G_n^t, where n determines the number of edges and vertices and t determines the structure of the graph, and establish some algebraic properties of the toric rings $R(n, t)$ associated to the graphs G_n^t. The use of bipartite graphs makes each $R(n, t)$ normal and Cohen–Macaulay by [25] and [15]; we use the latter in Sect. 3. Our main results prove to be independent of t and depend only on n.

In Sect. 2, we construct the family $\mathcal{F}$ of graphs G_n^t from a family of ladder-like structures L_n^t so that the toric ideals of the G_n^t are generalized determinantal ideals of the L_n^t. The ladder-like structures associated to a subfamily $\mathcal{F}_1 \subset \mathcal{F}$, introduced in Example 2.4, are in fact two-sided ladders (for large n), so that the family of rings $R(n, t)$ is a generalization of the family of ladder determinantal rings coming from $\mathcal{F}_1$. While the rings arising from $\mathcal{F}_1$ come from a distributive lattice and have easily derived properties (see for example [15]), we show that the rings associated to $\mathcal{F}$ do not naturally arise from any lattice in general, and merit closer study.

In Sect. 3, we establish some algebraic properties of the $R(n, t)$, particularly Krull dimension, projective dimension, multiplicity, and regularity. To do so, we prove that the determinantal generators of the defining ideal $I_{G_n^t}$ are a Gröbner basis (it follows immediately from [15] that $R(n, t)$ is Koszul) and work with the initial ideal $\text{in}_> I_{G_n^t}$. We also develop a system of parameters $\overline{X_n}$ that allows us to work with Artinian reductions in part of our treatment, and their Hilbert series.

Our first result gives an alternate proof for the Krull dimension of the toric ring $R(n, t) = S(n)/I_{G_n^t}$, already known due to a result of Villarreal for bipartite graphs [27, Prop 3.2]. Here, the ring $S(n) = k[x_0, x_2, x_3, \ldots, x_{2n+3}, x_{2n+4}]$ is the polynomial ring over the edges of G_n^t and $I_{G_n^t}$ is the toric ideal of G_n^t.

Theorem 1.1 (Theorem 3.4) *The dimension of $R(n, t)$ is*

$$\dim R(n, t) = n + 3.$$

As a corollary, since $R(n, t)$ comes from a bipartite graph and is hence Cohen–Macaulay (Corollary 2.16), we obtain the projective dimension of $R(n, t)$.

Corollary 1.2 (Corollary 3.5) *The projective dimension of $R(n, t)$ over $S(n)$ is*

$$\text{pd}_{S(n)} R(n, t) = n + 1.$$

We then develop a linear system of parameters for $R(n, t)$, using differences of elements on antidiagonals of the ladder-like structure L_n^t.

Proposition 1.3 (Proposition 3.10) *Let $R(n, t) = S(n)/I_{G_n^t}$. Then the image of*

$$X_n = x_0, x_2 - x_3, x_4 - x_5, \ldots, x_{2n} - x_{2n+1}, x_{2n+2} - x_{2n+3}, x_{2n+4}$$

in $R(n,t)$ is a system of parameters for $R(n,t)$.

Since $R(n,t)$ is Cohen–Macaulay, the linear system of parameters above is a regular sequence (Corollary 3.12).

With the aim of obtaining the multiplicity and regularity of $R(n,t)$, we form an Artinian quotient of $R(n,t)$ by the regular sequence above and call it $\widehat{R(n,t)}$. We note that $\widehat{R(n,t)}$ does not denote the completion, and explain the choice of notation in Definition 3.7.

Using a convenient vector space basis for $\widehat{R(n,t)}$ established in Lemma 3.13, we show the coefficients of the Hilbert series for $\widehat{R(n,t)}$.

Theorem 1.4 (Theorem 3.16) *If $R(n,t) = S(n)/I_{G_n^t}$ and $\widehat{R(n,t)} \cong R(n,t)/(\overline{X_n})$, we have*

$$\dim_k(\widehat{R(n,t)})_i = \begin{cases} 1 & i = 0 \\ \dfrac{1}{i!}\displaystyle\prod_{j=1}^{i}(n+j-2(i-1)) & 1 \leqslant i \leqslant n/2+1 \\ 0 & i > n/2+1. \end{cases}$$

As a corollary, we obtain the regularity of $R(n,t)$, which is equal to the top nonzero degree of $\widehat{R(n,t)}$.

Corollary 1.5 (Corollary 3.18) *For $G_n^t \in \mathcal{F}$,*

$$\operatorname{reg} R(n,t) = \lfloor n/2 \rfloor + 1.$$

We include an alternate graph-theoretic proof of the result above at the end of this work. Beginning with an upper bound from [3] (or equivalently for our purposes, one from [14]) and then identifying the initial ideal $\text{in}_> I_{G_n^t}$ with the edge ideal of a graph, we use results from [4] (allowing us to use $\text{in}_> I_{G_n^t}$ instead of $I_{G_n^t}$) and then [13] for a lower bound which agrees with our upper bound.

From a recursion established in Lemma 3.15, we go on to prove a Fibonacci relationship between the lengths of the Artinian rings $\widehat{R(n,t)}$ in Proposition 3.19, and obtain the multiplicity of $R(n,t)$ as a corollary. In the following, we drop t for convenience.

Corollary 1.6 (Corollary 3.21) *For $n \geqslant 2$, there is an equality of multiplicities*

$$e(R(n)) = e(R(n-1)) + e(R(n-2)).$$

In particular,

$$e(R(n)) = F(n+3) = \frac{(1+\sqrt{5})^{n+3} - (1-\sqrt{5})^{n+3}}{2^{n+3}\sqrt{5}}.$$

For more background, detail, and motivation, we refer the reader to [1], but note that different notation and indexing conventions have been employed in this work. Throughout, k is a field.

2 The Family of Toric Rings

In the following, we define a family of toric rings $R(n, t)$ coming from an iterative chordal bipartite family of graphs, $\mathcal{F}$. We show that although one subfamily of these rings comes from join-meet ideals of a (distributive) lattice and has some easily derived algebraic invariants, this is not true in general. The reader may find the definition of the toric ideal of a graph in Sect. 2.2, when it becomes relevant to the discussion. We recall for the reader that a *chordal bipartite* graph is a bipartite graph in which every cycle of length greater than or equal to six has a chord.

2.1 The Family $\mathcal{F}$ of Graphs

Below, we define the family $\mathcal{F}$ of chordal bipartite graphs iteratively from a family of ladder-like structures L_n^t. We note that the quantities involved in the following definition follow patterns as follows:

n	$\lfloor n/2 \rfloor + 2$	$\lceil n/2 \rceil + 2$
0	2	2
1	2	3
2	3	3
3	3	4
$\vdots$	$\vdots$	$\vdots$

Definition 2.1 For each $n \geqslant 0$ and each $t \in \mathbb{F}_2^{n+1}$, we construct a ladder-like structure L_n^t with $(\lfloor n/2 \rfloor + 2)$ rows and $(\lceil n/2 \rceil + 2)$ columns and with nonzero entries in the set $\{x_0, x_2, x_3, \dots, x_{2n+4}\}$. To do so, we use the notation $\widehat{t} \in \mathbb{F}_2^n$ for the first n entries of t, that is, all except the last entry. The construction is as follows, where throughout, indices of entries in L_n^t are strictly increasing from left to right in each row and from top to bottom in each column. We note that L_n^t does not depend on t for $n < 2$, but does for $n \geqslant 2$.

- For $n = 0$, the ladder-like structure $L_0^0 = L_0^1$ is

$$\begin{matrix} x_0 & x_2 \\ x_3 & x_4 \end{matrix}$$

- For $n = 1$, to create L_1^t (regardless of what t is in $\mathbb{F}_2^2$), we add another column with the entries x_5 and x_6 to the right of $L_0^{\widehat{t}}$ to obtain

$$\begin{array}{ccc} x_0 & x_2 & x_5 \\ x_3 & x_4 & x_6 \end{array}$$

- For $2 \leqslant n \equiv 0 \mod 2$ ($\equiv 1 \mod 2$), to create L_n^t, we add another row (column) with the entries x_{2n+3}, x_{2n+4} below (to the right of) $L_{n-1}^{\widehat{t}}$ in the following way:
  - The entry x_{2n+4} is in the ultimate row and column, row $\lfloor n/2 \rfloor + 2$ and column $\lceil n/2 \rceil + 2$.
  - The entry x_{2n+3} is in the new row (column) in a position directly below (to the right of) another nonzero entry in L_n^t.
    - If the last entry of t is 0, x_{2n+3} is directly beneath (to the right of) the first nonzero entry in the previous row (column).
    - If the last entry of t is 1, x_{2n+3} is directly beneath (to the right of) the second nonzero entry in the previous row (column).

In this way, the entries in t determine the choice at each stage for the placement of x_{2n+3}.

Remark 2.2 We note a few things about this construction for $n \equiv 0 \mod 2$ ($\equiv 1 \mod 2$), which may be examined in the examples below:

- We note that x_{2n+4} is directly beneath (to the right of) x_{2n+2}.
- We note that the only entries in row $\lfloor n/2 \rfloor + 1$ (column $\lceil n/2 \rceil + 1$) of $L_{n-1}^{\widehat{t}}$ are x_{2n-1}, x_{2n}, and x_{2n+2}, so that the choices listed for placement of x_{2n+3} are the only cases. In particular, $t_{n+1} = 0$ if and only if x_{2n+3} is directly beneath (to the right of) x_{2n-1}, and $t_{n+1} = 1$ if and only if x_{2n+3} is directly beneath (to the right of) x_{2n}.
- Finally, we note that the only entries in column $\lceil n/2 \rceil + 2$ (row $\lfloor n/2 \rfloor + 2$) of L_n^t are x_{2n+1}, x_{2n+2}, and x_{2n+4}, and that the only entries in row $\lfloor n/2 \rfloor + 2$ (column $\lceil n/2 \rceil + 2$) of L_n^t are x_{2n+3} and x_{2n+4}.

Example 2.3 We have

$$L_2^{(1,1,1)} = \begin{array}{ccc} x_0 & x_2 & x_5 \\ x_3 & x_4 & x_6 \\ & x_7 & x_8 \end{array} \qquad L_2^{(0,0,0)} = \begin{array}{ccc} x_0 & x_2 & x_5 \\ x_3 & x_4 & x_6 \\ x_7 & & x_8 \end{array}$$

In either of the cases above, we could go on to construct $L_3^{\widehat{t}}$ and L_4^t in the following way: For $n = 3$, place x_{10} to the right of x_8 and place x_9 to the right of either x_5 or x_6, depending whether the last entry of $\widehat{t}$ is 0 or 1, respectively. Then for $n = 4$, place x_{12} below x_{10} and place x_{11} below either x_7 or x_8, depending whether the last entry of t is 0 or 1, respectively.

Example 2.4 In fact, when the entries of t are all ones, we see that $L_n^{(1,1,\dots,1)}$ has a ladder shape (is a two-sided ladder for $n \geqslant 3$), shown below in the case when $2 \leqslant n \equiv 0 \mod 2$:

$$\begin{array}{cccccccc}
x_0 & x_2 & x_5 & & & & & \\
x_3 & x_4 & x_6 & x_9 & & & & \\
 & x_7 & x_8 & x_{10} & x_{13} & & & \\
 & & x_{11} & x_{12} & x_{14} & x_{17} & & \\
 & & & x_{15} & x_{16} & x_{18} & \ddots & \\
 & & & & x_{19} & x_{20} & \ddots & x_{2n+1} \\
 & & & & & \ddots & \ddots & x_{2n+2} \\
 & & & & & & x_{2n+3} & x_{2n+4}.
\end{array}$$

We denote the subfamily of graphs coming from $t = (1, 1, \dots, 1)$ by $\mathcal{F}_1 \subset \mathcal{F}$.

When the entries of t are all zeros, $L_n^{(0,0,\dots,0)}$ has the following structure, shown below in the case when $2 \leqslant n \equiv 0 \mod 2$:

$$\begin{array}{ccccccccc}
x_0 & x_2 & x_5 & x_9 & x_{13} & x_{17} & x_{21} & \cdots & x_{2n+1} \\
x_3 & x_4 & x_6 & & & & & & \\
x_7 & & x_8 & x_{10} & & & & & \\
x_{11} & & & x_{12} & x_{14} & & & & \\
x_{15} & & & & x_{16} & x_{18} & & & \\
x_{19} & & & & & x_{20} & x_{22} & & \\
x_{23} & & & & & & x_{24} & \ddots & \\
\vdots & & & & & & & \ddots & x_{2n+2} \\
x_{2n+3} & & & & & & & & x_{2n+4}.
\end{array}$$

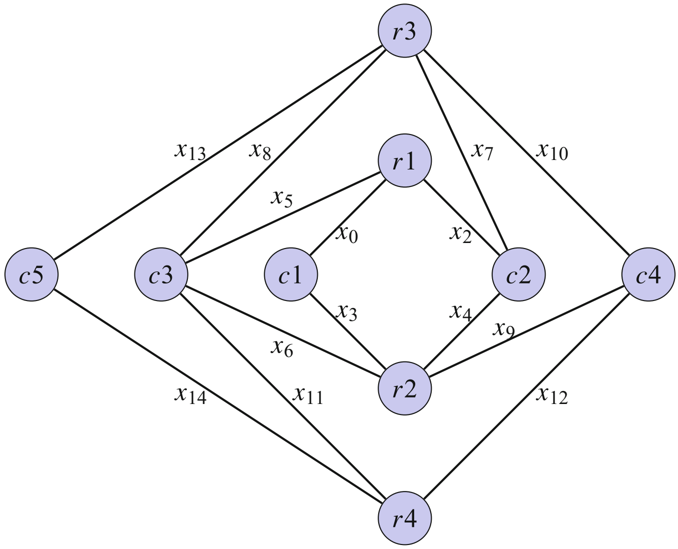
For a more varied example, we have $L_{16}^{(1,0,1,0,1,1,0,0,1,1,1,0,0,0,1,0,0)}$ below:

$$\begin{array}{cccccccccc}
x_0 & x_2 & x_5 & x_9 & & & & & & \\
x_3 & x_4 & x_6 & & & & & & & \\
 & x_7 & x_8 & x_{10} & x_{13} & x_{17} & & & & \\
 & & x_{11} & x_{12} & x_{14} & & & & & \\
 & & x_{15} & & x_{16} & x_{18} & x_{21} & x_{25} & x_{29} & x_{33} \\
 & & & & x_{19} & x_{20} & x_{22} & & & \\
 & & & & & x_{23} & x_{24} & x_{26} & & \\
 & & & & & x_{27} & & x_{28} & x_{30} & \\
 & & & & & & & x_{31} & x_{32} & x_{34} \\
 & & & & & & & x_{35} & & x_{36}.
\end{array}$$

Definition 2.5 If we associate a vertex to each row and each column and an edge to each nonzero entry of L_n^t, we have a finite simple connected bipartite graph G_n^t. The set V_r of vertices corresponding to rows and the set V_c of vertices corresponding to columns form a bipartition of the vertices of G_n^t. We say a graph G is in $\mathcal{F}$ if $G = G_n^t$ for some $n \geqslant 0$ and some $t \in \mathbb{F}_2^{n+1}$.

Remark 2.6 We note that by construction G_n^t has no vertices of degree one, since each row and each column of L_n^t has more than one nonzero entry. This ensures that for large n our family is distinct from that studied in [5], since a Ferrers graph with bipartitation V_1 and V_2 with no vertices of degree one must have at least two vertices in V_1 of degree $|V_2|$ and at least two vertices in V_2 of degree $|V_1|$, impossible for our graphs when $n \geqslant 3$, as the reader may verify. We also use the fact that G_n^t has no vertices of degree one for an alternate proof of the regularity of $R(n, t)$ at the end of this work.

Example 2.7 When $n = 5$, $G_5^{(1,1,\ldots,1)} \in \mathcal{F}_1$ is



We develop properties of the L_n^t which allow us to show in Sect. 2.2 that certain minors of the L_n^t are generators for the toric rings of the corresponding graphs G_n^t.

Definition 2.8 For this work, a *distinguished minor* of L_n^t is a 2-minor involving only (nonzero) entries of the ladder-like structure L_n^t, coming from a 2×2 subarray of L_n^t.

Proposition 2.9 *For each $i \geqslant 1$ and each $f \in \mathbb{F}_2^{i+1}$, the entry x_{2i+3} and the entry x_{2i+4} each appear in exactly two distinguished minors of L_i^f. For $i \equiv 0 \mod 2$ ($\equiv 1 \mod 2$), these minors are of the form*

$$s_{2i} := x_{2i+1}x_{2i+3} - x_{j_{2i}}x_{2i+4}$$

coming from the subarray

$$\begin{bmatrix} x_{j_{2i}} & x_{2i+1} \\ x_{2i+3} & x_{2i+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{2i}} & x_{2i+3} \\ x_{2i+1} & x_{2i+4} \end{bmatrix}\right)$$

for some $j_{2i} \in \{0, 2, 3, \ldots, 2i - 2\}$ *and*

$$s_{2i+1} := x_{2i+2}x_{2i+3} - x_{j_{2i+1}}x_{2i+4}$$

coming from the subarray

$$\begin{bmatrix} x_{j_{2i+1}} & x_{2i+2} \\ x_{2i+3} & x_{2i+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{2i+1}} & x_{2i+3} \\ x_{2i+2} & x_{2i+4} \end{bmatrix}\right)$$

for some $j_{2i+1} \in \{2i - 1, 2i\}$*, and the only distinguished minor of* L_n^t *with indices all less than* 5 *is* $s_1 := x_2x_3 - x_0x_4$.

Proof The last statement is clear by Definition 2.1; we prove the remaining statements by induction on i. For $i = 1$, we have the distinguished minors $s_2 = x_3x_5 - x_0x_6$ and $s_3 = x_4x_5 - x_2x_6$ coming from the subarrays

$$\begin{bmatrix} x_0 & x_5 \\ x_3 & x_6 \end{bmatrix}$$

and

$$\begin{bmatrix} x_2 & x_5 \\ x_4 & x_6 \end{bmatrix}$$

where $j_2 = 0 \in \{0\}$ and $j_3 = 2 \in \{1, 2\}$, so we have our base case. Now suppose the statement is true for i with $1 \leqslant i < n$, and let $n \equiv 0 \mod 2$ ($\equiv 1 \mod 2$) and $t \in \mathbb{F}_2^{n+1}$.

Case 1: If $t_{n+1} = 0$, then by Remark 2.2, x_{2n+3} is in the same column (row) as x_{2n-1}. By induction, we have the distinguished minor $s_{2n-2} = x_{2n-1}x_{2n+1} - x_{j_{2n-2}}x_{2n+2}$ coming from the subarray

$$\begin{bmatrix} x_{j_{2n-2}} & x_{2n+1} \\ x_{2n-1} & x_{2n+2} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{2n-2}} & x_{2n-1} \\ x_{2n+1} & x_{2n+2} \end{bmatrix}\right).$$

Then in fact we have a subarray of the form

$$\begin{bmatrix} x_{j_{2n-2}} & x_{2n+1} \\ x_{2n-1} & x_{2n+2} \\ x_{2n+3} & x_{2n+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{2n-2}} & x_{2n-1} & x_{2n+3} \\ x_{2n+1} & x_{2n+2} & x_{2n+4} \end{bmatrix}\right),$$

so that we have the distinguished minors

$$s_{2n} = x_{2n+1}x_{2n+3} - x_{j_{2n-2}}x_{2n+4}$$
$$s_{2n+1} = x_{2n+2}x_{2n+3} - x_{2n-1}x_{2n+4}$$

with

$$j_{2n} = j_{2n-2} \in \{0, 2, 3, \dots, 2n-4\} \subset \{0, 2, 3, \dots, 2n-2\}$$

by induction and with

$$j_{2n+1} = 2n-1 \in \{2n-1, 2n\}.$$

Since the only entries in row $\lfloor n/2 \rfloor + 2$ (column $\lceil n/2 \rceil + 2$) of L_n^t are x_{2n+3} and x_{2n+4} and since the only entries in column $\lceil n/2 \rceil + 2$ (row $\lfloor n/2 \rfloor + 2$) of L_n^t are x_{2n+1}, x_{2n+2}, and x_{2n+4} by Remark 2.2, these are the only distinguished minors of L_n^t containing either x_{2n+3} or x_{2n+4}, as desired.

Case 2 for $t_{n+1} = 1$ is analogous and yields

$$j_{2n} = j_{2n-1} \in \{2n-3, 2n-2\} \subset \{0, 2, 3, \dots, 2n-2\}$$

and

$$j_{2n+1} = 2n \in \{2n-1, 2n\}. \qquad \square$$

Definition 2.10 Define the integers j_{2i}, j_{2i+1} for $j_2, \dots, j_{2n+1}$ as in the statement of Proposition 2.9. We note in the remark below some properties of the j_k.

Remark 2.11 From the proof of Proposition 2.9, we note that $j_2 = 0$, $j_3 = 2$, and that for $i \geqslant 2$, we have the following:

$$t_{i+1} = 0 \iff j_{2i} = j_{2i-2} \iff j_{2i+1} = 2i - 1$$
$$t_{i+1} = 1 \iff j_{2i} = j_{2i-1} \iff j_{2i+1} = 2i.$$

For the sake of later proofs, we extend the notion of j_k naturally to $s_1 = x_2x_3 - x_0x_4$ and say that $j_1 = 0$, and note the following properties of the j_k for $1 \leqslant k \leqslant 2n+1$:

- We have $j_{2i} \in \{j_{2i-2}, j_{2i-1}\}$ and $j_{2i} \leqslant 2i - 2$. Indeed, for $i = 1$, $j_2 = j_1 = 0$, and for $i \geqslant 2$, this is clear from the statement above.
- We have $j_{2i+1} \in \{2i - 1, 2i\}$. Indeed, for $i = 0$, $j_1 = 0 \in \{-1, 0\}$, for $i = 1$, $j_3 = 2 \in \{1, 2\}$, and for $i \geqslant 2$, this follows from the statement above.
- The j_{2i} form a non-decreasing sequence. Indeed, for $i \geqslant 2$, either $j_{2i} = j_{2i-2}$ or $j_{2i} = j_{2i-1} \geqslant 2i - 3 > 2i - 4 \geqslant j_{2i-2}$.

Remark 2.12 We also note from the proof above that the following is a subarray of L_n^t for all $i \equiv 0 \mod 2$ ($\equiv 1 \mod 2$) such that $1 \leqslant i \leqslant n$, which we use in the proof of the proposition below:

$$\begin{bmatrix} x_{j_{2i}} & x_{2i+1} \\ x_{j_{2i+1}} & x_{2i+2} \\ x_{2i+3} & x_{2i+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{2i}} & x_{j_{2i+1}} & x_{2i+3} \\ x_{2i+1} & x_{2i+2} & x_{2i+4} \end{bmatrix}\right)$$

Proposition 2.13 *For $n \geqslant 0$, each graph $G_n^t \in \mathcal{F}$ is chordal bipartite with vertex bipartition $V_r \cup V_c$ of cardinalities*

$$|V_r| = \left\lfloor \frac{n}{2} \right\rfloor + 2$$
$$|V_c| = \left\lceil \frac{n}{2} \right\rceil + 2.$$

Proof We already know by Definition 2.5 that every graph G_n^t is bipartite for $n \geqslant 0$, with the bipartition above coming from the rows and columns of L_n^t. The cardinalities of the vertex sets follow from Remark 2.2. We prove the chordal bipartite property by induction on n. It is clear for $i = 0$ and $i = 1$ that G_i^f is chordal bipartite for $f \in \mathbb{F}_2^{i+1}$, since these graphs have fewer than six vertices. Now suppose G_i^f is chordal bipartite for i with $1 \leqslant i < n \equiv 0 \mod 2$ ($\equiv 1 \mod 2$), and consider G_n^t for $t \in \mathbb{F}_2^{n+1}$. We know that the following array (or its transpose) is a subarray of L_n^t by Remark 2.12, and we include for reference the corresponding subgraph of G_n^t with vertices labeled by row and column.

$$\begin{bmatrix} x_{j_{2n}} & x_{2n+1} \\ x_{j_{2n+1}} & x_{2n+2} \\ x_{2n+3} & x_{2n+4} \end{bmatrix}$$