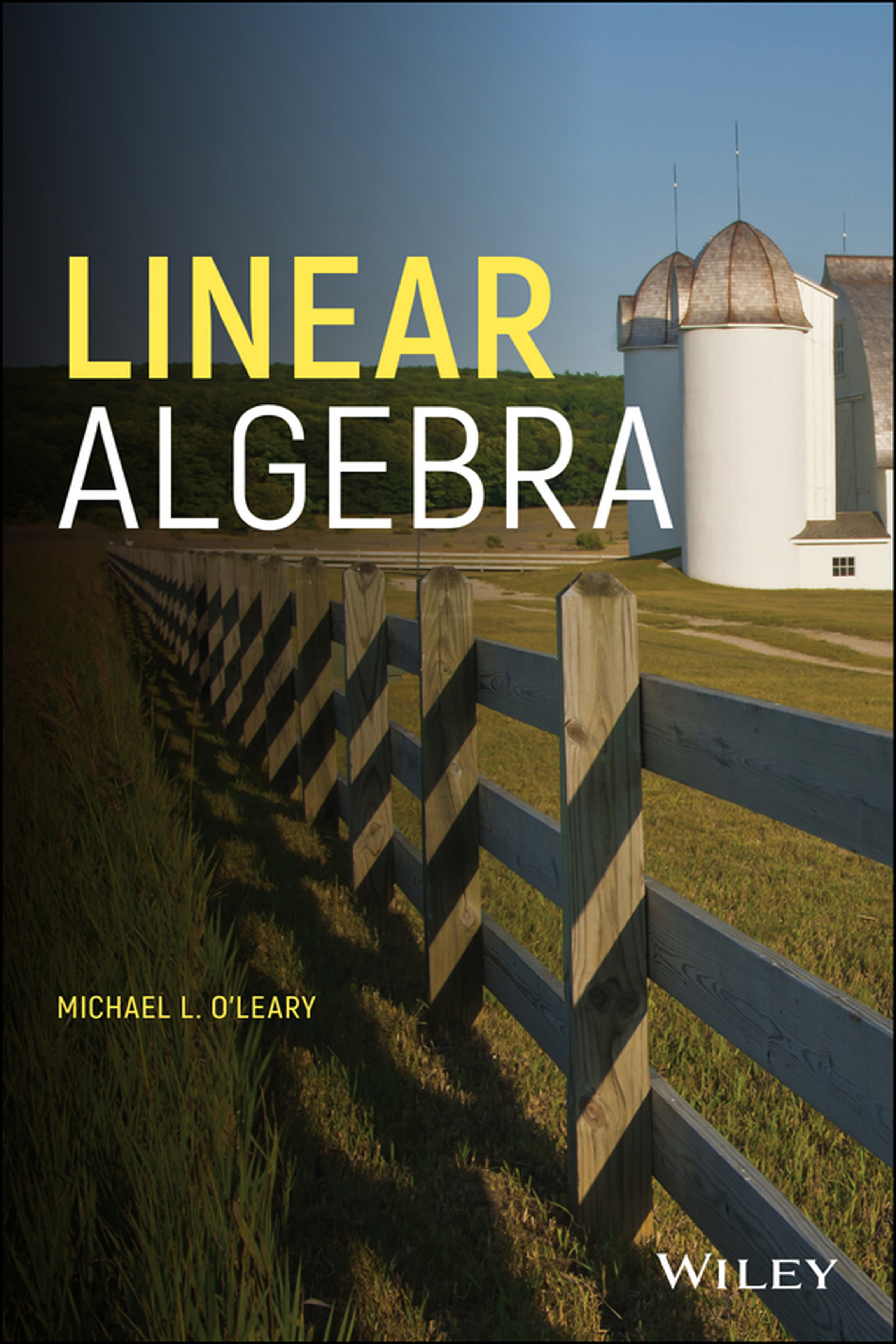

LINEAR ALGEBRA

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Linear Algebra

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For my niece,
Lindsay

Contents

Preface

This book is an introduction to linear algebra. Its goal is to develop the standard first topics of the subject. Although there are many computations in the sections, which is expected, the focus is on proving the results and learning how to do this. For this reason, the book starts with a chapter dedicated to basic logic, set theory, and proof-writing. Although linear algebra has many important applications ranging from electrical circuitry and quantum mechanics to cryptography and computer gaming, these topics will need to wait for another day. The goal here is to master the mathematics so that one is ready for a second course in the subject, either abstract or applied. This may go against current trends in mathematics education, but if any mathematical subject can stand on its own and be learned for its own sake, it is the amazing and beautiful linear algebra.

In addition to the focus on proofs, linear transformations play a central role. For this reason, functions are introduced early, and once the important sets of $\mathbb{R}^n$ are defined in the second chapter, linear transformations are described in the third chapter and motivate the introduction of matrices and their operations. From there, invertible linear transformations and invertible matrices are encountered in the fourth chapter followed by a complete generalization of all previous topics in the fifth with the definition of abstract vector spaces. Geometries are added to the abstractions in the sixth chapter, and the book concludes with nice matrix representations. Therefore, the book's structure is as follows.

Logic and Set Theory Statements and truth tables are introduced. This includes logical equivalence so that the reader becomes familiar with the logic of statements. This is particularly important when dealing with implications and reasoning that involves De Morgan's laws. Sets and their operations follow with an introduction to quantification including how to negate both universal and existential sentences. Proof methods are next, including direct and indirect proof, and these are applied to proofs involving subsets. Mathematical induction is also presented. The chapter closes with an introduction to functions, including the concepts of one-to-one, onto, and binary operation.

Euclidean Space The definition of $\mathbb{R}^n$ is the focus of the second chapter with the main interpretation being that of arrows originating at the origin. Euclidean distance and length are defined, and these are followed by the dot and cross products. Applications include planes and lines, areas and volumes, and the orthogonal projection.

Transformations and Matrices Now that functions have been defined and interesting sets to serve as their domains and codomains have been given, linear transformations are introduced. After some basic properties, it is shown that these functions have nice representations as matrices. The matrix operations come next, their definitions being motivated by the definitions of the function operations. Linear operators on $\mathbb{R}^2$ and $\mathbb{R}^3$ serve as important examples of linear transformations. These include the reflections, rotations, contractions, dilations, and shears. The introduction of the kernel and the range is next. Issues with finding these sets motivate the need for easier techniques. Thus, Gauss–Jordan elimination and Gaussian elimination finally make their appearance.

Invertibility The fourth chapter introduces the idea of an invertible matrix and ties it to the invertible linear operator. The standard procedure of how to find an inverse is given using elementary matrices, and inverses are then used to solve certain systems of linear equations. The determinant with its basic properties is next. How the elementary row operations affect the determinant is explained and carefully proved using mathematical induction. The next section combines the inverse and the determinant, and important results concerning both are proved. The chapter concludes with some mathematical applications including orthogonal matrices, Cramer's Rule, and how the determinant can be used to compute the area or volume of the image of a polygon or a solid under a linear transformation.

Abstract Vectors Now that the concrete work has been done, it is time to generalize. Vector spaces lead the way as the generalization of $\mathbb{R}^n$, and these are quickly followed by linear transformations between these abstract vector spaces. The important topics of subspace, linear dependence and linear independence, and basis and dimension soon follow. The proof that every vector space has a basis is given for the sake of completion, but, other than for the result, the techniques are not pursued very far because this book is, after all, an introduction to the subject. Rank and nullity are defined, both in terms of linear transformations and in terms of matrices. The chapter then concludes with probably the most important topic of the book, isomorphism. Along with isomorphism, coordinates, coordinate maps, and change of basis matrices are presented. The section and chapter concludes with the discovery of the standard matrix of a linear transformation. Although there is more to come, a standing ovation for the standard matrix and its diagram would not be inappropriate.

Inner Product Spaces Although $\mathbb{R}^n$ is usually viewed as Cartesian space, it is technically just a set of $n \times 1$ matrices. Any geometry that it has was given to it in the second chapter, even though its geometry is a copy of the geometry of Cartesian space. A close examination reveals that the geometry of $\mathbb{R}^n$ is based on the dot product. Mimicking this, an abstract vector space is given its geometry with an inner product, which is a function defined so that it has the same basic properties as the dot product. The vector space then becomes an inner product space so that distances, lengths, and angles can be found using objects like matrices, polynomials, and functions. Other topics related to the inner product include a generalization of the orthogonal projection, orthonormal bases, direct sums, and the Gram–Schmidt process.

Matrix Theory The book concludes with an introduction to the powerful concepts of eigenvalues and eigenvectors. Both the characteristic polynomial and the minimal polynomial are defined and used throughout the chapter. Generalized eigenvectors are presented and used to write $\mathbb{R}^n$ as a direct sum of subspaces. The concept of similar matrices is given, and if a matrix does not have enough eigenvectors, it is proved that such matrices are similar to matrices with a nice form. This is where Schur's Lemma makes its appearance. However, if a matrix does have enough eigenvectors, the matrix is similar to a very nice diagonal matrix. This is the last section of the book, which includes orthogonal diagonalization, simultaneous diagonalization, and a quick introduction to quadratic forms and how to use eigenvalues to find an equation for a conic section without a middle term.

As with any textbook, where the course is taught influences how the book is used. Many universities and colleges have an introduction to proof course. Because such courses serve as a prerequisite for any proof-intensive mathematics course, the first chapter of this book can be passed over at these institutions and used only as a reference. If there is no such prerequisite, the first chapter serves as a detailed introduction to proof-writing that is short enough not to infringe too much on the time spent on purely linear algebra topics. Wherever the book finds itself, the course outline can easily be adjusted with any excluded topics serving as bonus reading for the eager student.

Now for some technical comments. Theorems, definitions, and examples are numbered sequentially as a group in the now common chapter.section.number format. Although some proofs find their way into the text, most start with **Proof**, end with ■, and are indented. Examples, on the other hand, are simply indented. Some equations are numbered as (chapter.number) and are referred to simply using (chapter.number). Most if not all of the mathematical notation should be clear. It was decided to represent vectors as columns. This leads to some interesting typesetting, but the clarity and consistency probably more than makes up for any formatting issues. Vectors are boldface, such as $\mathbf{u}$ and $\mathbf{v}$, and scalars are not. Most sums are written like $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k$. There is a similar notation for products. However, there are times when summation and product notation must be used. Therefore, if $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ are vectors,

$$\sum_{i=1}^{k} \mathbf{u}_i = \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k \quad \text{and} \quad \sum_{i \neq 2} \mathbf{u}_i = \mathbf{u}_1 + \mathbf{u}_3 + \cdots + \mathbf{u}_k,$$

and if $r_1, r_2, \dots, r_k$ are real numbers,

$$\prod_{i=1}^{k} r_i = r_1 r_2 \cdots r_k \quad \text{and} \quad \prod_{i \neq 2} r_i = r_1 r_3 \cdots r_k.$$

Each section ends with a list of exercises. Some are computations, some are verifications where the job is to make a computation that illustrates a theorem from the section, and some involve proving results where remembering one's logic and set theory and how to prove sentences will go a long way.

Solution manuals, one for students and one for instructors, are available. See the book's page at *wiley.com*.

Lastly, this book was typeset using LaTeX from the free software distribution of TeX Live running in Arch Linux with the KDE Plasma desktop. The diagrams were created using LibreOffice Draw.

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Glen Ellyn, Illinois
September, 2020

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About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/oleary/linearalgebra

The website includes the solutions manual and will be live in the fall of 2021.

CHAPTER 1

Logic and Set Theory

1.1 Statements

A sentence that is true or false but not both is called a **statement**. Here are some sentences, some of which are statements.

- Please read the linear algebra book.
  — This is not a statement because it is a request. It is neither true nor false.
- All quadrilaterals have 4 sides.
  — This is a true statement.
- Some triangles have 5 sides.
  — This is a false statement.
- $x + y = y + x$.
  — This is not a statement since the variables have not been assigned values.
- $x + y = y + x$ for all integers x and y.
  — This is a true statement.
- $x + y = 10$ for all real numbers y.
  — This is not a statement because x has not been assigned a value.

Linear Algebra, First Edition. Michael L. O'Leary.

Companion Website: www.wiley.com/go/o'leary/linearalgebra

Connectives

Let p and q represent sentences. These variables combined with the **(logical) connectives**, which are $\sim$ (**not**), $\wedge$ (**and**), $\vee$ (**or**), $\rightarrow$ (**if...then**), $\leftrightarrow$ (**if and only if**), can be used to represent **compound sentences**. To illustrate the meanings of the connectives, let p be the statement, *lines intersect in a point*, and q be the sentence, *planes intersect in a line*. These sentences can be combined using the connectives:

$\sim p$	Lines do not intersect in a point.
$p \wedge q$	Lines intersect in a point, and planes intersect in a line.
$p \vee q$	Lines intersect in a point, or planes intersect in a line.
$p \rightarrow q$	If lines intersect in a point, then planes intersect in a line.
$p \leftrightarrow q$	Lines intersect in a point if and only if planes intersect in a line.

Let p and q be statements. This means that the **truth value** of p is either true (T) or false (F). The same can be said of q. Joining p and q with a connective yields a statement. The truth value of the resulting statement depends on the truth values of p and q. A **truth table** is used to identify all of the statement's possible truth values.

■ Definition 1.1.1

The sentence $\sim p$ is the **negation** of p. If p is a statement, the truth table of $\sim p$ is:

p	$\sim p$
T	F
F	T

■ Definition 1.1.2

The sentence $p \wedge q$ is the **conjunction** of p and q, and the sentence $p \vee q$ is the **disjunction** of p and q. If p and q are statements, the truth tables of $p \wedge q$ and $p \vee q$ are:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

■ Definition 1.1.3

The sentence $p \rightarrow q$ is an **implication** or **conditional sentence**, where p is called the **antecedent** and q is called the **consequent**. The sentence $p \leftrightarrow q$ is a **biconditional**. If p and q are statements, the truth tables of $p \rightarrow q$ and $p \leftrightarrow q$ are:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

■ Example 1.1.4

The sentence $p \leftrightarrow (q \vee \sim p)$ is read by assuming that parentheses work as grouping symbols like in algebra and by attaching any $\sim$ to the first sentence to its immediate right. This implies that $p \leftrightarrow (q \vee \sim p)$ is interpreted by examining its sentences using an order starting with the variables:

$$p, q, \sim p, q \vee \sim p, p \leftrightarrow (q \vee \sim p),$$

which, if $p \leftrightarrow (q \vee \sim p)$ is a statement, produces the truth table:

p	q	$\sim p$	$q \vee \sim p$	$p \leftrightarrow (q \vee \sim p)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

Each column of the truth table on the right-hand side requires values from columns to its left. For example, evaluating $q \vee \sim p$ requires the truth values in the second and third columns.

Logical Equivalence

Two statements p and q are **(logically) equivalent** (written $p \equiv q$) means that they always have the same truth values. This is often proved using a truth table.

■ Example 1.1.5

The sentence $p \leftrightarrow q$ is the conjunction of two implications. Specifically, $p \leftrightarrow q$ means *p if q, and p only if q*. The first implication is $q \to p$, and the second implication is $p \to q$. Therefore, if p and q are statements,

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p).$$

This is confirmed with the truth table:

p	q	$\boldsymbol{p \leftrightarrow q}$	$p \to q$	$q \to p$	$\boldsymbol{(p \to q) \wedge (q \to p)}$
T	T	**T**	T	T	**T**
T	F	**F**	F	T	**F**
F	T	**F**	T	F	**F**
F	F	**T**	T	T	**T**

Example 1.1.6

Let p and q be statements.

- The **converse** of $p \to q$ is $q \to p$.
- The **contrapositive** of $p \to q$ is $\sim q \to \sim p$.

For example, given the implication

if lines intersect in a point, planes intersect in a line,

its converse is

if planes intersect in a line, lines intersect in a point,

and its contrapositive is

if planes do not intersect in a line, lines do not intersect in a point.

An implication and its converse are not logically equivalent, but an implication and its contrapositive are logically equivalent. The third and sixth columns (in boldface) of the truth table show $p \to q \equiv \sim q \to \sim p$, while the third and last columns show $p \to q \not\equiv q \to p$.

p	q	$\boldsymbol{p \to q}$	$\sim q$	$\sim p$	$\boldsymbol{\sim q \to \sim p}$	$q \to p$
T	T	**T**	F	F	**T**	T
T	F	**F**	T	F	**F**	T
F	T	**T**	F	T	**T**	F
F	F	**T**	T	T	**T**	T

Example 1.1.7

There are some equivalences that are quite famous. Observe that

$$p \to q \equiv \sim p \vee q. \tag{1.1}$$

This is seen by the truth table:

p	q	$\boldsymbol{p \to q}$	$\sim p$	$\boldsymbol{\sim p \vee q}$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

There are also **De Morgan's Laws**,

$$\begin{aligned} \sim(p \wedge q) &\equiv \sim p \vee \sim q, \\ \sim(p \vee q) &\equiv \sim p \wedge \sim q, \end{aligned} \tag{1.2}$$

where (1.2) is confirmed by the truth table:

p	q	$p \wedge q$	$\boldsymbol{\sim(p \wedge q)}$	$\sim p$	$\sim q$	$\boldsymbol{\sim p \vee \sim q}$
T	T	T	**F**	F	F	**F**
T	F	F	**T**	F	T	**T**
F	T	F	**T**	T	F	**T**
F	F	F	**T**	T	T	**T**

To illustrate how De Morgan's Law (1.2) works, use ≡ to assign

$$p \equiv 2 + 2 = 4,$$
$$q \equiv 3 + 5 = 10.$$

Then, (1.2) can be written as

it is false that both $2 + 2 = 4$ and $3 + 5 = 10$,

which is logically equivalent to

$$2 + 2 \neq 4 \text{ or } 3 + 5 \neq 10.$$

Another example is the **Double Negation Rule**,

$$\sim\sim p \equiv p, \tag{1.3}$$

which is proved by the simple truth table:

$\boldsymbol{p}$	$\sim p$	$\boldsymbol{\sim\sim p}$
T	F	**T**
T	F	**T**
F	T	**F**
F	T	**F**

■ **Example 1.1.8**

To understand the meaning of $p \to q$, notice that in addition to (1.1),

$$p \to q \equiv \sim(p \wedge \sim q). \tag{1.4}$$

This is proved by the truth table:

p	q	$\boldsymbol{p \to q}$	$\sim q$	$p \wedge \sim q$	$\boldsymbol{\sim(p \wedge \sim q)}$
T	T	**T**	F	F	**T**
T	F	**F**	T	T	**F**
F	T	**T**	F	F	**T**
F	F	**T**	T	F	**T**

The statement $\sim(p \wedge \sim q)$ claims that it is not the case that p is true but q is false. This is the exact meaning of $p \to q$. An alternate proof of (1.4) involves applying De Morgan's Law (1.2) and Double Negation (1.3) to obtain

$$\sim(p \wedge \sim q) \equiv \sim p \vee \sim\sim q \equiv \sim p \vee q. \tag{1.5}$$

Then, (1.5) combined with (1.1) gives (1.4). Also, Double Negation (1.3) with (1.4) implies that

$$\sim(p \to q) \equiv p \land \sim q, \tag{1.6}$$

so to show that an implication is false, it must be demonstrated that the antecedent can be true at the same time that the consequent is false.

Exercises

1. Determine the truth value of each sentence that is a statement.
   (a) $42 + 13 = 55$
   (b) For all real numbers x, $x < 5$.
   (c) Study linear algebra.
   (d) There are seven prime integers.
   (e) $x = 7$
   (f) This statement is false.
2. Define: $p \equiv$ The sum of two odd integers is an odd integer.
   $q \equiv$ The angle sum of a rectangle is 2π.
   $r \equiv$ The tangent function is differentiable everywhere.

   For each of the given statements, write the statement using p, q, or r and the appropriate logical connectives and find its truth value.
   (a) The tangent function is differentiable everywhere, and the angle sum of a rectangle is 2π.
   (b) The sum of two odd integers is an odd integer, or the sum of odd integers is an odd integer.
   (c) If the angle sum of a rectangle is 2π, the tangent function is not differentiable everywhere.
   (d) The sum of two odd integers is an odd integer if and only if it is true that the tangent function is differentiable everywhere.
   (e) The tangent function is differentiable everywhere if and only if the angle sum of a rectangle is 2π, and the sum of two odd integers is an even integer.
   (f) It is not the case that the angle sum of a rectangle is not 2π.
3. Write each sentence in the form *if p then q* and determine its truth value. Some words may need to be changed so that the answer is grammatically correct.
   (a) If a rectangle has adjacent congruent sides, a square has adjacent congruent sides.
   (b) Polynomials have at most two roots if quadratic polynomials have two complex roots.
   (c) Trigonometric functions are periodic only if polynomials are periodic.
   (d) The derivative of a constant function is zero.
   (e) A necessary condition for the opposite angles of a parallelogram to be congruent is that parallel lines intersect.
   (f) A sufficient condition for all systems of linear equations to have a solution is that some systems of linear equations have a solution.

4. Write the converse and contrapositive for the implications in Exercise 3.
5. Without using a truth table, explain the meaning of De Morgan's Laws found in Example 1.1.7.
6. Let p and q be true statements but r and s be false statements. Find the truth values.
   (a) $(p \wedge q) \vee r$
   (b) $q \leftrightarrow (r \vee \sim q)$
   (c) $p \rightarrow (q \rightarrow [r \rightarrow s])$
   (d) $(\sim p \wedge q) \vee ([p \rightarrow q] \wedge \sim s)$
   (e) $([p \wedge q] \rightarrow q) \wedge (p \rightarrow q)$
   (f) $\sim\sim p \leftrightarrow (q \wedge r)$
   (g) $([p \rightarrow q] \vee [q \rightarrow r]) \vee s$
   (h) $(p \rightarrow q) \vee ([q \rightarrow r] \vee s)$
   (i) $(p \vee q) \wedge (q \vee r)$
   (j) $([p \vee q] \wedge q) \vee ([p \vee q] \wedge r)$
7. Write the truth table.
   (a) $\sim p \rightarrow p$
   (b) $p \rightarrow \sim q$
   (c) $(p \vee q) \wedge \sim(p \wedge q)$
   (d) $(p \rightarrow q) \vee (q \leftrightarrow p)$
   (e) $p \rightarrow (q \wedge \sim p)$
   (f) $(p \rightarrow q) \wedge \sim p$
   (g) $(\sim p \vee q) \wedge ([p \rightarrow q] \vee \sim p)$
   (h) $(p \wedge q) \vee r$
   (i) $p \wedge (q \vee r)$
   (j) $(p \vee q) \rightarrow r$
   (k) $p \vee (q \rightarrow r)$
   (l) $(p \rightarrow q) \wedge \sim(r \vee p)$
   (m) $(p \rightarrow q) \leftrightarrow (r \rightarrow s)$
   (n) $p \vee ([\sim q \leftrightarrow r] \wedge q)$
8. Prove the given famous logical equivalences.
   (a) Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
   $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   (b) Commutative Laws: $p \wedge q \equiv q \wedge p$
   $p \vee q \equiv q \vee p$
   (c) Distributive Laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
9. Prove using truth tables or by using logical equivalences as in Example 1.1.8.
   (a) $p \vee \sim p \equiv p \rightarrow p$
   (b) $p \wedge q \equiv (p \leftrightarrow q) \wedge (p \vee q)$
   (c) $p \vee \sim p \equiv (p \vee q) \vee \sim(p \wedge q)$
   (d) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$
   (e) $r \wedge (p \rightarrow q) \equiv r \wedge (\sim q \rightarrow \sim p)$
   (f) $(p \wedge q) \rightarrow r \equiv (p \wedge \sim r) \rightarrow \sim q$
   (g) $(p \vee q) \vee r \equiv (q \vee p) \vee r$
   (h) $\sim p \wedge \sim(q \vee r) \equiv \sim(p \vee [q \vee r])$
   (i) $\sim(p \vee q) \vee r \equiv (p \vee q) \rightarrow r$
   (j) $(p \wedge q) \wedge r \equiv r \wedge (p \wedge q)$

1.2 Sets and Quantification

A **set** is a collection of things called **elements**. Anything can be an element, but in linear algebra, elements are typically numbers, matrices, functions, or vectors. If a is an element of the set A, write $a \in A$. If both a and b are elements of A, write $a, b \in A$. If c is not an element of A, write $c \notin A$. If B is a set that has exactly the same elements as A, write $A = B$, which means that A is **equal** to B. If $A \neq B$, there is an element that is in one set but not the other. If A contains no elements, write $A = \varnothing$, where $\varnothing$ is the **empty set**, the set with no elements.

Some examples of famous sets, written in **roster form**, are the following:

- $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ = the set of integers from 1 to 10.
- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ = the set of **natural numbers**.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ = the set of **integers**.
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ = the set of **positive integers**.

This approach to writing sets has the elements of each set found within braces and uses **ellipses** (…) to represent a repeating pattern. In general, writing

$$A = \{a_1, a_2, \ldots, a_n\}$$

means that A has n distinct elements, $a_1, a_2, \ldots, a_n$, and writing

$$A = \{a_1, a_2, a_3, \ldots\}$$

means that A has infinitely many elements, $a_1, a_2, a_3, \ldots$, where $a_i \neq a_j$ if $i \neq j$.

The problem with roster form is that it is not good for describing most sets, like

$$\mathbb{R} = \text{ the set of real numbers.}$$

What is needed is the ability to write a condition that describes exactly when an element is in a set.

Universal Quantifiers

The statement

$$\text{for all } x \in \mathbb{R}, x + 42 = 42 + x, \tag{1.7}$$

or, equivalently, $x + 42 = 42 + x$ for every $x \in \mathbb{R}$, claims that $x + 42 = 42 + x$ is true for every substitution of a real number for x. Use the function notation $p(x)$ to represent $x + 42 = 42 + x$. Substitutions work and are denoted as expected. For example, $p(7) \equiv 7 + 42 = 42 + 7$. Letting $\forall$ represent "for all," write (1.7) as $(\forall x \in \mathbb{R})p(x)$. If x is assumed to be a real number, write (1.7) as $(\forall x)p(x)$, or denote the inclusion of x in $\mathbb{R}$ by writing $(\forall x)[x \in \mathbb{R} \to p(x)]$.

■ **Definition 1.2.1**

If A is a set, then $(\forall x \in A)p(x) \equiv (\forall x)[x \in A \to p(x)]$. Both sentences claim that $p(a)$ is true for every $a \in A$. The symbol $\forall$ is called the **universal quantifier**.

■ **Example 1.2.2**

The statement $(\forall x \in \varnothing)p(x)$ is true because $x \in \varnothing$ is false for all substitutions of x, from which follows that $x \in \varnothing \to p(x)$ is true for all substitutions of x by Definition 1.1.3.

■ **Example 1.2.3**

$(\forall x \in \mathbb{R})x + 10 = 5$ is false because $1 + 10 \neq 5$ and $1 \in \mathbb{R}$. The statement

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})x + y = y + x$$

is true because

$$(\forall y \in \mathbb{R})a + y = y + a \tag{1.8}$$

is true for every $a \in \mathbb{R}$, and the reason that (1.8) is true is that the statement $a + b = b + a$ is true for every $b \in \mathbb{R}$.

Example 1.2.3 suggests a method for proving true a statement with a universal quantifier. Consider

$$(\forall x \in \mathbb{R})x + 0 = x. \tag{1.9}$$

To prove it, let $a \in \mathbb{R}$. Because the only property that a is assigned is that it is a real number, it is considered an arbitrary or randomly chosen real number. It is known that 0 has the property that $a + 0 = a$. Thus, because a is arbitrary, (1.9) is true.

Existential Quantifiers

The statement

$$\text{there exists } x \in \mathbb{R} \text{ such that } x + 27 = 42 \tag{1.10}$$

or, equivalently, $x + 27 = 42$ for some $x \in \mathbb{R}$, claims that $x + 27 = 42$ is true for at least one substitution of a real number for x. Letting $\exists$ represent "there exists" and $p(x)$ represent $x + 27 = 42$, (1.10) can be written as $(\exists x \in \mathbb{R})p(x)$. If x is assumed to be a real number, write (1.10) as $(\exists x)p(x)$, or denote the inclusion of x in $\mathbb{R}$ by writing $(\exists x)[x \in \mathbb{R} \wedge p(x)]$.

■ **Definition 1.2.4**

If A is a set, then $(\exists x \in A)p(x) \equiv (\exists x)[x \in A \wedge p(x)]$. Both statements claim that $p(a)$ is true for at least one $a \in A$. The symbol $\exists$ is called the **existential quantifier**.

■ **Example 1.2.5**

For any sentence $p(x)$, the statement $(\exists x \in \varnothing)p(x)$ is false. This is because $x \in \varnothing$ is false for all substitutions of x, from which follows $x \in \varnothing \wedge p(x)$ is false for all substitutions of x by Definition 1.1.2.

Example 1.2.6

$(\exists x \in \mathbb{R})x + 0 = x + 1$ is false because there is no real number a such that $a + 0 = a + 1$. The statement

$$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})x + y = 13$$

is true because

$$(\exists y \in \mathbb{R})5 + y = 13 \tag{1.11}$$

is true, and (1.11) is true because $5 + 8 = 13$ is true.

Example 1.2.6 suggests a method for proving true a statement with an existential quantifier. Consider $(\exists x \in \mathbb{R})x + 3 = 9$. To prove it, it is enough to find a real number a such that $a + 3 = 9$ is true. Taking $a = 6$ does it.

Negating Quantifiers

As observed in Example 1.2.3, the statement

$$(\forall x \in \mathbb{R})x + 10 = 5$$

is false because $1 + 10 \neq 5$ and $1 \in \mathbb{R}$. This means that

$$(\exists x \in \mathbb{R})x + 10 \neq 5$$

is true. Generalizing, $(\forall x \in A)p(x)$ is false if there exists $a \in A$ such that $p(a)$ is false, which means that $(\exists x \in A){\sim}p(x)$ is true. The element a that demonstrates that a statement starting with a universal quantifier is false is a **counterexample**.

Likewise, as noted in Example 1.2.6,

$$(\exists x \in \mathbb{R})x + 0 = x + 1$$

is false because $a + 0 \neq a + 1$ for every real number a. This means that

$$(\forall x \in \mathbb{R})x + 0 \neq x + 1$$

is true. Generalizing, $(\exists x \in A)p(x)$ is false if there is no $a \in A$ such that $p(a)$ is true, which means that $(\forall x \in A){\sim}p(x)$ is true. These two results are summarized in the next theorem.

Theorem 1.2.7

Let $p(x)$ be a sentence and let A be a set.

(a) ${\sim}(\forall x \in A)p(x) \equiv (\exists x \in A){\sim}p(x)$.

(b) ${\sim}(\exists x \in A)p(x) \equiv (\forall x \in A){\sim}p(x)$.

■ **Example 1.2.8**

De Morgan's Law (1.2) and (1.6) implies

$$\begin{aligned}\sim(\forall x \in A)[p(x) \to q(x)] &\equiv (\exists x \in A)\sim[p(x) \to q(x)]\\ &\equiv (\exists x \in A)[p(x) \wedge \sim q(x)],\end{aligned}$$

and also by De Morgan's Law,

$$\begin{aligned}\sim(\exists x \in A)[p(x) \wedge q(x)] &\equiv (\forall x \in A)\sim[p(x) \wedge q(x)]\\ &\equiv (\forall x \in A)[\sim p(x) \vee \sim q(x)].\end{aligned}$$

A statement can have multiple quantifiers. Consider $2x - 7y = 1$. This line can be graphed by writing an x-y table. Values for y-coordinates are calculated based on the values chosen for x resulting in the points to be plotted. This process demonstrates that

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})2x - 7y = 1. \tag{1.12}$$

The statement (1.12) has a universal quantifier followed by the sentence

$$(\exists y \in \mathbb{R})2x - 7y = 1,$$

and since there is no quantifier on x, substitutions can be made for x. Conclude that (1.12) is true because whenever x is replaced with an arbitrary real number, a real number y can be found to satisfy $2x - 7y = 1$. Specifically, that real number is $y = (2x - 1)/7$.

■ **Example 1.2.9**

Find the negation of $(\exists x \in A)(\forall y \in A)[p(x) \to q(y)]$.

$$\begin{aligned}\sim(\exists x \in A)(\forall y \in A)[p(x) \to q(y)] &\equiv (\forall x \in A)\sim(\forall y \in A)[p(x) \to q(y)]\\ &\equiv (\forall x \in A)(\exists y \in A)\sim[p(x) \to q(y)]\\ &\equiv (\forall x \in A)(\exists y \in A)[p(x) \wedge \sim q(y)].\end{aligned}$$

■ **Example 1.2.10**

Prove or show false.

- $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})x + y = 3$
  — Because $7 + 10 \neq 3$, the numbers 7 and 10 are counterexamples, so the given statement is false.
- $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})x + y = 3$
  — Let $a \in \mathbb{R}$. Then, $(\exists y \in \mathbb{R})a + y = 3$ states that there exists a real number y such that $a + y = 3$, which is true because $a + (3 - a) = 3$. This means that the given statement is true.
- $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})x + y = 3$
  — The statement is false. To see this, take $a \in \mathbb{R}$. Then, $a + (2 - a) \neq 3$, proving $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})x + y \neq 3$ is true.
- $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})x + y = 3$
  — This is true since $3 + 0 = 3$.

Set-Builder Notation

Let A be a set. Consider a sentence $p(x)$ such that for every a,

$$a \in A \text{ if and only if } p(a) \text{ is true.}$$

Such a sentence serves as a condition that an element must satisfy in order to be in A. For example, let $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$ and $p(x)$ be the sentence

$$(\exists n \in \mathbb{Z})x = 2n. \tag{1.13}$$

Notice that $p(x)$ completely describes E because the even integers are exactly those elements a such that $p(a)$ is true. In particular, $p(0)$ and $p(-10)$ are true but $p(3)$ is false. Thus, 0 and -10 are elements of E, but 3 is not. Sentences like $p(x)$ can be used to define sets.

■ **Definition 1.2.11**

Let A be a set. If $p(x)$ is a sentence such that $a \in A$ if and only if $p(a)$, write

$$A = \{x : p(x)\}.$$

This is called **set-builder notation**. Read $\{x : p(x)\}$ as "the set of all x such that p of x."

Using Definition 1.2.11, write E with (1.13) using set-builder notation as

$$E = \{x : (\exists n \in \mathbb{Z})x = 2n\} = \{2n : n \in \mathbb{Z}\}.$$

■ **Example 1.2.12**

$A = \{-4, 4\}$ is the set of roots of the polynomial $x^2 - 16$. Using set-builder notation, A can be written as

$$A = \{x : x^2 - 16 = 0 \text{ and } x \in \mathbb{R}\} = \{x \in \mathbb{R} : (x+4)(x-4) = 0\}.$$

■ **Example 1.2.13**

- $\emptyset = \{x \in \mathbb{R} : x \neq x\}$.
- $\left\{\dots, -\frac{3}{5}, -\frac{2}{5}, -\frac{1}{5}, \frac{0}{5}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \dots\right\} = \left\{\frac{n}{5} : n \in \mathbb{Z}\right\}$.
- $\{\dots, x-4, x-2, x, x+2, x+4, \dots\} = \{x + 2n : n \in \mathbb{Z}\}$.

■ **Example 1.2.14**

- $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$ = the set of **rational numbers**.
- $\mathbb{R} = \left\{a : \lim_{n\to\infty} a_n = a \wedge (\forall i \in \mathbb{Z}^+)\, a_i \in \mathbb{Q}\right\}$ = the set of **real numbers**.
- $\mathbb{R}^+ = \{x : x \in \mathbb{R} \wedge x > 0\}$ = the set of **positive real numbers**.
- $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ = the set of **complex numbers**, where $i^2 = -1$.