

SOLUTIONS MANUAL TO ACCOMPANY

AN INTRODUCTION TO NUMERICAL METHODS AND ANALYSIS

THIRD EDITION

JAMES F. EPPERSON

WILEY

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Preface to the Solutions Manual for the Third Edition

This manual is written for instructors, not students. It includes worked solutions for many (roughly 75%) of the problems in the text. For the computational exercises I have given the output generated by my program, or sometimes a program listing. Most of the programming was done in MATLAB, some in FORTRAN. (The author is well aware that FORTRAN is archaic, but there is a lot of “legacy code” in FORTRAN, and the author believes there is value in learning a new language, even an archaic one.) When the text has a series of exercises that are obviously similar and have similar solutions, then sometimes only one of these problems has a worked solution included. When computational results are asked for a series of similar functions or problems, only a subset of solutions are reported, largely for the sake of brevity. Some exercises that simply ask the student to perform a straightforward computation are skipped. Exercises that repeat the same computation but with a different method are also often skipped, as are exercises that ask the student to “verify” a straight-forward computation.

Some of the exercises were designed to be open-ended and almost “essay-like.” For these exercises, the only solution typically provided is a short hint or brief outline of the kind of discussion anticipated by the author.

In many exercises the student needs to construct an upper bound on a derivative of some function in order to determine how small a parameter has to be to achieve a desired level of accuracy. For many of the solutions this

was done using a computer algebra package and the details are not given.

Students who acquire a copy of this manual in order to obtain worked solutions to homework problems should be aware that none of the solutions are given in enough detail to earn full credit from an instructor.

The author freely admits the potential for error in any of these solutions, especially since many of the exercises were modified after the final version of the text was submitted to the publisher and because the ordering of the exercises was changed between editions. While we tried to make all the appropriate corrections, the possibility of error is still present, and undoubtedly the author's responsibility.

Because much of the manual was constructed by doing “copy-and-paste” from the files for the text, the enumeration of many tables and figures will be different. I have tried to note what the number is in the text, but certainly may have missed some instances.

Suggestions for new exercises and corrections to these solutions are very welcome. Contact the author at jfe@ams.org OR jfepperson@gmail.com .

Differences from the text The text itself went through a copy-editing process after this manual was completed. As was to be expected, the wording of several problems was slightly changed. None of these changes should affect the problem in terms of what is expected of students; the vast majority of the changes were to replace “previous problem” (a bad habit of mine) with “Problem X.Y” (which I should have done on my own, in the first place). Some punctuation was also changed. The point of adding this note is to explain the textual differences which might be noticed between the text and this manual. If something needs clarification, please contact me at the above email.

CHAPTER 1

INTRODUCTORY CONCEPTS AND CALCULUS REVIEW

1.1 BASIC TOOLS OF CALCULUS

Exercises:

1. Show that the third-order Taylor polynomial for $f(x) = (x + 1)^{-1}$, about $x_0 = 0$, is

$$p_3(x) = 1 - x + x^2 - x^3.$$

Solution: We have $f(0) = 1$ and

$$f'(x) = -\frac{1}{(x+1)^2}, \quad f''(x) = \frac{2}{(x+1)^3}, \quad f'''(x) = -\frac{6}{(x+1)^4},$$

so that $f'(0) = -1$, $f''(0) = 2$, $f'''(0) = -6$.
Therefore,

$$\begin{aligned} p_3(x) &= f(0) + xf'(0) + \frac{1}{2}x^2f''(0) + \frac{1}{6}x^3f'''(0) \\ &= 1 + x(-1) + \frac{1}{2}x^2(2) + \frac{1}{6}x^3(-6) \\ &= 1 - x + x^2 - x^3. \end{aligned}$$

2. What is the third-order Taylor polynomial for $f(x) = \sqrt{x+1}$, about $x_0 = 0$?

Solution: We have $f(x_0) = 1$ and

$$f'(x) = \frac{1}{2(x+1)^{1/2}}, \quad f''(x) = -\frac{1}{4(x+1)^{3/2}}, \quad f'''(x) = \frac{3}{8(x+1)^{5/2}},$$

so that $f'(0) = 1/2$, $f''(0) = -1/4$,
 $f''' = 3/8$. Therefore

$$\begin{aligned} p_3(x) &= f(0) + xf'(0) + \frac{1}{2}x^2f''(0) + \frac{1}{6}x^3f'''(x) \\ &= 1 + x(1/2) + \frac{1}{2}x^2(-1/4) + \frac{1}{6}x^3(3/8) \\ &= 1 - (1/2)x - (1/8)x^2 + (1/16)x^3. \end{aligned}$$

3. What is the sixth-order Taylor polynomial for $f(x) = \sqrt{1+x^2}$, using $x_0 = 0$? Hint: Consider the previous problem.

4. Given that

$$R(x) = \frac{|x|^6}{6!} e^{\xi}$$

for $x \in [-1, 1]$, where ξ is between x and 0, find an upper bound for $|R|$, valid for all $x \in [-1, 1]$, that is independent of x and ξ .

5. Repeat the above, but this time require that the upper bound be valid only for all $x \in [-\frac{1}{2}, \frac{1}{2}]$.

Solution: The only significant difference is the introduction of a factor of 2^6 in the denominator:

$$|R(x)| \leq \frac{\sqrt{e}}{2^6 \times 720} = 3.6 \times 10^{-5}.$$

6. Given that

$$R(x) = \frac{|x|^4}{4!} \left(\frac{-1}{1 + \xi} \right)$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$, where ξ is between x and 0, find an upper bound for $|R|$, valid for all $x \in [-\frac{1}{2}, \frac{1}{2}]$, that is independent of x and ξ .

7. Use a Taylor polynomial to find an approximate value for $\sqrt{e}$ that is accurate to within 10^{-3} .

Solution: There are two ways to do this. We can approximate $f(x) = e^x$ and use $x = 1/2$, or we can approximate $g(x) = \sqrt{x}$ and use $x = e$. In addition, we can be conventional and take $x_0 = 0$, or we can take $x_0 \neq 0$ in order to speed convergence.

The most straightforward approach (in my opinion) is to use a Taylor polynomial for e^x about $x_0 = 0$. The remainder after k terms is

$$R_k(x) = \frac{x^{k+1}}{(k+1)!} e^{\xi}.$$

We quickly have that

$$|R_k(x)| \leq \frac{e^{1/2}}{2^{k+1}(k+1)!}$$

and a little playing with a calculator shows that

$$|R_3(x)| \leq \frac{e^{1/2}}{16 \times 24} = 0.0043$$

but

$$|R_4(x)| \leq \frac{e^{1/2}}{32 \times 120} = 4.3 \times 10^{-4}.$$

So we would use

$$e^{1/2} \approx 1 + \frac{1}{2} + \frac{1}{2} \left(\frac{1}{2}\right)^2 + \frac{1}{6} \left(\frac{1}{2}\right)^3 + \frac{1}{24} \left(\frac{1}{2}\right)^4 = 1.6484375.$$

To fourteen digits, $\sqrt{e} = 1.64872127070013$, and the error is 2.84×10^{-4} , much smaller than required.

8. What is the fourth-order Taylor polynomial for $f(x) = 1/(x+1)$, about $x_0 = 0$?

Solution: We have $f(0) = 1$ and

$$f'(x) = -\frac{1}{(x+1)^2}, \quad f''(x) = \frac{2}{(x+1)^3}, \quad f'''(x) = -\frac{6}{(x+1)^4}, \quad f''''(x) = \frac{24}{(x+1)^5},$$

so that $f'(0) = -1$, $f''(0) = 2$,

$f''' = -6$, $f''''(0) = 24$. Thus,

$$p_4(x) = 1 + x(-1) + \frac{1}{2}x^2(2) + \frac{1}{6}x^3(-6) + \frac{1}{24}x^4(24) = 1 - x + x^2 - x^3 + x^4.$$

9. What is the fourth-order Taylor polynomial for $f(x) = 1/x$, about $x_0 = 1$?

10. Find the Taylor polynomial of third-order for $\sin x$, using:

a. $x_0 = \pi/6$.

Solution: We have

$$f(x_0) = \frac{1}{2}, \quad f'(x_0) = \frac{\sqrt{3}}{2}, \quad f''(x_0) = -\frac{1}{2}, \quad f'''(x_0) = -\frac{\sqrt{3}}{2},$$

so

$$p_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3;$$

b. $x_0 = \pi/4$;

c. $x_0 = \pi/2$.

11. For each function below construct the third-order Taylor polynomial approximation, using $x_0 = 0$, and then estimate the error by computing an upper bound on the remainder, over the given interval.

- a. $f(x) = e^{-x}, x \in [0, 1]$;
- b. $f(x) = \ln(1 + x), x \in [-1, 1]$;
- c. $f(x) = \sin x, x \in [0, \pi]$;
- d. $f(x) = \ln(1 + x), x \in [-1/2, 1/2]$;
- e. $f(x) = 1/(x + 1), x \in [-1/2, 1/2]$.

Solution:

- a. The polynomial is

$$p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3,$$

with remainder

$$R_3(x) = \frac{1}{24}x^4e^{-\xi}.$$

This can be bounded above, for all $x \in [0, 1]$, by

$$|R_3(x)| \leq \frac{1}{24}e$$

- b. The polynomial is

$$p_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3,$$

with remainder

$$R_3(x) = \frac{1}{4}x^4 \frac{1}{(1+\xi)^4}.$$

We *can't* bound this for all $x \in [-1, 1]$, because of the potential division by zero.

c. The polynomial is

$$p_3(x) = x - \frac{1}{6}x^3,$$

with remainder

$$R_3(x) = \frac{1}{120}x^5 \cos \xi.$$

This can be bounded above, for all $x \in [0, \pi]$, by

$$|R_3(x)| \leq \frac{\pi^5}{120}.$$

d. The polynomial is the same as in (b), of course,

$$p_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3,$$

with remainder

$$R_3(x) = \frac{1}{4}x^4 \frac{1}{(1+\xi)^4}.$$

For all $x \in [-1/2, 1/2]$ this can be bounded by

$$R_3(x) \leq \frac{1}{4}(1/2^4) \frac{1}{(1 - (1/2))^4} = \frac{1}{4}.$$

e. The polynomial is

$$p_3(x) = 1 - x + x^2 - x^3,$$

with remainder

$$R_3(x) = x^4 \frac{1}{(1+\xi)^5}.$$

This can be bounded above, for all $x \in [-1/2, 1/2]$, by

$$|R_3(x)| \leq (1/2)^4 \frac{1}{(1 - 1/2)^5} = 2.$$

Obviously, this is not an especially good approximation.

12. Construct a Taylor polynomial approximation that is accurate to within 10^{-3} , over the indicated interval, for each of the following functions, using $x_0 = 0$.

a. $f(x) = \sin x, x \in [0, \pi];$

b. $f(x) = e^{-x}, x \in [0, 1];$

- c. $f(x) = \ln(1 + x), x \in [-1/2, 1/2];$
- d. $f(x) = 1/(x + 1), x \in [-1/2, 1/2];$
- e. $f(x) = \ln(1 + x), x \in [-1, 1].$

Solution:

a. The remainder here is

$$R_n(x) = \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1} \cos c,$$

for $c \in [0, \pi]$. Therefore, we have

$$|R_n(x)| \leq \frac{1}{(2n+1)!} |\pi|^{2n+1} \leq \frac{\pi^{2n+1}}{(2n+1)!}.$$

Simple manipulations with a calculator then show that

$$\max_{x \in [0, \pi]} |R_6(x)| \leq 0.4663028067 \times 10^{-3}$$

but

$$\max_{x \in [0, \pi]} |R_5(x)| \leq 0.7370430958 \times 10^{-2}.$$

Therefore the desired Taylor polynomial is

$$p_{11}(x) = 1 - x + \frac{1}{6}x^3 - \frac{1}{120}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \frac{1}{11!}x^{11}.$$

b. The remainder here is

$$R_n(x) = \frac{(-1)^{n+1}}{(n+1)!} x^{n+1} e^{-c},$$

for $c \in [0, 1]$. Therefore, we have

$$|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1} \leq \frac{1}{(n+1)!}.$$

Simple manipulations with a calculator then show that

$$\max_{x \in [0,1]} |R_6(x)| \leq 0.0001984126984$$

but

$$\max_{x \in [0,1]} |R_5(x)| \leq 0.1388888889 \times 10^{-2}$$

Therefore the desired Taylor polynomial is

$$p_6(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6.$$

c. $f(x) = \ln(1+x), x \in [0, 3/4]$.

d. **Solution:** The remainder is now

$$|R_n(x)| \leq \frac{(1/2)^{n+1}}{(n+1)!},$$

and $n = 8$ makes the error small enough.

n	N	N_e	$T_1 = \text{Assembly time}$	$T_2 = \text{Solution time}$	Error
36	1,225	450	0.9990630000	0.0950365000	$2.13731255 \times 10^{-11}$
37	1,296	450	1.1342848000	0.0960442000	$2.80340473 \times 10^{-11}$
38	1,369	450	1.2022592000	0.1495144000	$1.37058143 \times 10^{-11}$
39	1,444	449	1.3825504000	0.1324822000	$3.25073787 \times 10^{-11}$
40	1,521	450	1.4657469000	0.1623935000	$2.54942734 \times 10^{-11}$
41	1,600	450	1.7576274000	0.1934468000	$4.89706053 \times 10^{-11}$
42	1,681	450	1.8699778000	0.2081700000	$3.11846104 \times 10^{-11}$
43	1,764	450	2.1324582000	0.2636492000	$2.29647412 \times 10^{-11}$

10.3 SPECTRAL METHODS FOR TIME-DEPENDENT PROBLEMS

Exercises:

1. Use spectral collocation with Crank-Nicolson time-stepping to solve the following PDE:

$$\begin{aligned}
 u_t &= u_{xx}, \\
 u(-1, t) &= 0, \\
 u(1, t) &= 0, \\
 u(x, 0) &= \cos \pi x / 2 - \sin 4\pi x.
 \end{aligned}$$

The exact solution is $u(x, t) = e^{-\pi^2 t / 4} \cos \pi x / 2 - e^{-16\pi^2 t} \sin 4\pi x$. Compute out to $t = 1$; use a sequence of values of N ; plot your approximation and the error for one of them at $t = 1$.

2. Use your spectral code to solve the problem

$$\begin{aligned}
 u_t &= a u_{xx}, \\
 u(-1, t) &= 1, \\
 u(1, t) &= 1, \\
 u(x, 0) &= (x^2 - 1)^8.
 \end{aligned}$$

Assume $a = 1$ and compute out to $t = 1$, using a sequence of values of N . Plot the solution profile as the computation advances. Now vary a (you must keep it positive, of course) and investigate how this affects the solution.

Solution:

The initial condition approximates a “pulse” in the center of the interval. As the system evolves, the pulse will spread out and decay. Higher values of a (the diffusion coefficient) will speed up this process.

3. Now change the initial condition to $u(x, 0) = (x^4 - 1)^8$ and repeat the above problem.

Solution: This is essentially the same as the previous problem, but with a more sharply defined “pulse.”

4. Consider how to implement spectral collocation with variable coefficients. Construct the general linear system that would result from solving the problem:

$$\begin{aligned} u_t &= a(x)u_{xx}, \\ u(-1, t) &= 0, \\ u(1, t) &= 0, \\ u(x, 0) &= u_0(x). \end{aligned}$$

5. Apply your results from the above problem to approximate solutions to

$$\begin{aligned} u_t &= a(x)u_{xx}, \\ u(-1, t) &= 0, \\ u(1, t) &= 0, \\ u(x, 0) &= \cos \pi x / 2, \end{aligned}$$

for the following choices of a :

- $a(x) = (1 + x^2)$;
- $a(x) = e^{-x^2}$;
- $a(x) = (1 - x^2)$ (because a vanishes at the boundary, this problem is known as *degenerate* but you should be able to compute solutions).

6. Consider the nonlinear problem

$$\begin{aligned} u_t &= u_{xx} + Vuu_x, \\ u(-1, t) &= 0, \\ u(1, t) &= 0, \\ u_0(x) &= \sin \pi x. \end{aligned}$$

Use spectral collocation to attack this problem as suggested in Problem 12 of §9.1, by treating the nonlinearity explicitly (at time $t = t_n$) and the differential

equation implicitly (at time $t = t_{n+1}$). Compare your spectral solution to the explicit solution computed in Problem 10 of §9.1. Comment on your results.

10.4 CLENSHAW-CURTIS QUADRATURE

Exercises:

1. Use the appropriate change of variable to show how to apply Clenshaw-Curtis quadrature to an integral over an arbitrary interval $[a, b]$.

Solution: The change of variable is the same as obtained in §5.6 for Gaussian quadrature:

$$\int_a^b g(x)dx = \int_{-1}^1 f(z)dz,$$

for

$$f(z) = \frac{1}{2}(b-a)g\left(a + \frac{1}{2}(b-a)(z+1)\right).$$

2. Write a program to do Clenshaw-Curtis quadrature on each of the integrals in Problem 4 of §5.6. Compare your results to those obtained with Gaussian quadrature. Produce a log-log plot of the error as a function of N for each integral.

Solution: We will do (b) and (c). The results are summarized in [Table 10.2](#).

Table 10.2 Solution for 10.3.2(b) and (c).

N	10.3.2(b)	Error	10.3.2(c)	Error
2	0.65367552526213	-0.00702835762825	1.32175583998232	-0.02591897397799
4	0.64425803338082	0.00238913425305	1.29638987827887	-0.00055301227455
8	0.64664682519792	0.00000034243595	1.29583765211218	-0.00000078610785
16	0.64664716763387	0.00000000000000	1.29583686600986	-0.00000000000553
32	0.64664716763387	0.00000000000000	1.29583686600433	0.00000000000000

3. Looking at the plots in Fig. 10.24, we see that most of them show a very rapid decrease of the error, and a “rounding error plateau” is reached for most of the examples. The exception is the last one, where the integrand is given by $f(x) = \sqrt{1-x^2}$. Explain why this example is the one that displays this kind of sub-optimal performance.

Solution: The integrand is singular at the endpoints of the interval (the derivatives blow up) so this slows the convergence.

4. Let $C_N = \sum_{k=1}^N w_k^{(N)} f(\xi_k^{(N)}) \approx \int_{-1}^1 f(x) dx$ be the Clenshaw-Curtis quadrature operator. Show that C_{2N} uses some of the same function values as C_N . Why is this important?