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Stability and Optimization of Structures

Generalized Sensitivity Analysis

Springer

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Series Preface

Mechanical engineering, and engineering discipline born of the needs of the industrial revolution, is once again asked to do its substantial share in the call for industrial renewal. The general call is urgent as we face profound issues of productivity and competitiveness that require engineering solutions, among others. The Mechanical Engineering Series is a series featuring graduate texts and research monographs intended to address the need for information in contemporary areas of mechanical engineering.

The series is conceived as a comprehensive one that covers a broad range of concentrations important to mechanical engineering graduate education and research. We are fortunate to have a distinguished roster of series editors, each an expert in one of the areas of concentration. The names of the series editors are listed on page v of this volume. The areas of concentration are applied mechanics, biomechanics, computational mechanics, dynamic systems and control, energetics, mechanics of materials, processing, thermal science, and tribology.

Preface

In our modern world, best structures with specified shape, stiffness, strength, stability, frequency, and so on, can be designed with the assistance of computer-aided methodologies including sensitivity analysis, reliability-based design, inverse engineering, optimization, and anti-optimization. Buckling is an extremely important design constraint for structures with slender members such as latticed domes and frames; buckling of geometrically nonlinear structures is a well developed field of research. Nevertheless, because of the complexity of nonlinear buckling behavior, optimization of nonlinear structures has come to be conducted only recently despite its importance.

It is possible to consider structural optimization as a straightforward application of mathematical programming and operations research, as well as heuristics and evolutionary approaches. Such, however, is not the case for optimization of structures that undergo buckling. As cautioned by the "danger of naive optimization [287]," optimized structures often become more imperfection-sensitive and their buckling loads are reduced sharply because of the inevitable presence of initial imperfections that arise from errors in manufacturing processes, material defects, and other causes. It is certainly ironic that dangerous structures are produced in the attempt to optimize their performance. In any search for the best structure, the imperfection sensitivity of optimized structures must be investigated.

This book offers an introduction to "optimization of geometrically nonlinear structures under stability constraint," which is an exciting and fast-growing branch of application of structural and mechanical engineering, and also necessarily involves applied mathematics. The premise of this book is that a thorough and profound knowledge of nonlinear buckling behaviors is crucial, via proper problem setting, as a step toward the successful design of the best structure.

Some optimized structures are shown to be safe, and readers are encouraged to carry out optimization-based design with confidence.

In Part I, design sensitivity analysis and imperfection sensitivity analysis are introduced as systematic tools to perform stability design of structures. The influence of design parameters on structural performance is to be expressed as parameter sensitivity. Design sensitivity analysis is implemented into the gradient-based algorithm for structural optimization in Part II. Imperfection sensitivity laws are introduced to evaluate the influence of initial imperfections on buckling loads quantitatively. In this book, design sensitivity analysis and imperfection sensitivity analysis, which have been addressed independently up to now, are described in a synthetic manner based on the general theory of elastic stability [166]. This theory, which once was an established means to describe the buckling of structures, is thus given a new role in the computer age. Part I is organized as follows.

- The overview of design sensitivity analysis and its theoretical backgrounds are presented in Chapter 1.
- Numerical methods of design sensitivity analysis are provided in Chapter 2.
- Imperfection sensitivity analysis is presented in the framework of modern stability theory in Chapter 3.

In Part II, based on the synthetic description of sensitivity analyses presented in Part I, we introduce state-of-the-art optimization methodologies of geometrically nonlinear finite-dimensional structures under stability constraints. These optimization methodologies are reinforced on the one hand by the stability theory and on the other hand by finite element method and mathematical programming with ever-increasing computing power. Design of compliant mechanisms is highlighted as an engineering application of shape and topology optimization with extensive utilization of snapthrough buckling. Part II is organized as follows.

- In Chapter 4, general formulation for optimization under stability constraints is provided. An optimized truss dome is shown to be less imperfection-sensitive than a non-optimal one.
- Optimal structures with snapthrough are investigated in Chapter 5 to pave the way for shape design of compliant mechanisms using snapthrough behavior in Chapter 6.
- Optimal frames with coincident buckling loads are investigated in Chapter 7.
- Imperfection sensitivity of hilltop branching points with simple, multiple, and degenerate bifurcation points are investigated in Chapters 8–10.

In Part III, in order to ensure the performance of optimized structures, we introduce two design methodologies:

- optimization via the worst imperfection, and
- probabilistic analysis via random imperfections.

In particular, imperfection sensitivity laws are extended to be applicable to many imperfection variables and, in turn, to deal with the probabilistic variation of the buckling loads of structures. Part III is organized as follows.

- The asymptotic theory on the worst imperfection is formulated in Chapter 11.
- An anti-optimization problem is formulated in Chapter 12 to minimize the lowest eigenvalue of the tangent stiffness matrix, and a design methodology is presented for a laterally braced frame.
- The worst imperfection is defined and investigated for a stable-symmetric bifurcation point in Chapter 13.
- The theory on random imperfections is presented in Chapter 14, and is applied to steel specimens with hilltop branching in Chapter 15.
- The theory is extended to the second-order imperfections in Chapter 16.

In the Appendix, derivations of several formulations and details of numerical examples are presented. In particular, the derivation of imperfection sensitivity laws by the power series expansion method is an important ingredient for readers who are interested in stability theory.

This book consequently offers a wide and profound insight into optimization-based and computer-assisted stability design of finite-dimensional structures in a readable and illustrative form for graduate students of engineering and applied mathematicians. General methodology is emphasized instead of studies of particular structures. Historical developments are outlined with many references to assist readers' further studies.

The authors are grateful to Dr. J. S. Arora for his support of an optimization program. The suggestion of Dr. K. K. Choi was vital for the publication of this book. The authors thank for the comments of Drs. K. Murota and Y. Kanno. For the realization of this book, the authors owe much to Drs. K. Uetani, K. Terada, S. Okazawa, S. Nishiwaki, J. Takagi, K. Oide, and Mr. J. Y. Zhang. The support of C. Simpson, E. Tham and K. Stanne was indispensable for the publication of this book. The authors conclude the preface with many thanks to Dr. I. Elishakoff for his encouragement.

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Part I:
Generalized Sensitivity of Nonlinear Elastic Systems

1

Introduction to Design Sensitivity Analysis

1.1 Introduction

Sensitivity analysis is conducted to evaluate the dependence of structural performances on design or imperfection parameters. As stated in the Preface, dependent on parameters to be employed, sensitivity analysis in structural stability can be classified as follows:

- In the design sensitivity analysis, employed as parameters are design variables, such as member stiffnesses and geometrical variables. The sensitivity (differential) coefficients of structural responses, such as displacements, stresses and buckling loads, with respect to these parameters are obtained. These coefficients, in turn, are put to use in gradient-based optimization algorithms [223, 227, 262].
- In the imperfection sensitivity analysis for structures subjected to buckling, employed as parameters are initial imperfections, such as errors in manufacturing process and material defects [20, 285].

Different notations and terminologies are used in the two sensitivity analyses presented above for the same physical properties. Because the design sensitivity and imperfection sensitivity are mathematically equivalent, especially for a limit point load, the two sensitivity analyses are formulated in Part I in a synthetic manner under the name of "parameter sensitivity." Part I serves as a theoretical foundation of the optimization methodologies presented in Part II.

Many branches of parameter sensitivity analysis exist, as shown in Fig. 1.1, and methods of sensitivity analysis to be employed vary with these branches. Sensitivity analysis is classified dependent on whether the governing equation is linear or nonlinear. Nonlinear sensitivity analysis is further classified according

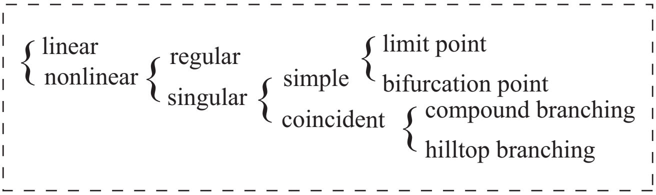

Fig. 1.1 Branches of parameter sensitivity analysis.

to whether the structure in question is in a regular state or in a singular state. Nonlinear sensitivity analysis for a singular state has sub-branches that are dependent on whether the singular state is associated with a limit point, a simple bifurcation point, a coincident critical point, and so on.

As a prologue to the main contents of this part, Chapters 2 and 3, we present in this chapter general framework of elastic stability to provide fundamental tools for design sensitivity analysis to compute the differential coefficients of structural responses with respect to the design parameters. We introduce sensitivity analysis for simple cases, including:

- linear elastic response,
- linear buckling load as a weakly nonlinear case,
- regular state for nonlinear response, and
- limit point load for nonlinear response.

This chapter is organized as follows. General framework of elastic stability is introduced in Section 1.2. Design parameterization is presented in Section 1.3. A simple introduction to design sensitivity analysis for linear and nonlinear responses is given in Sections 1.4 and 1.5, respectively. A historical review of studies on design sensitivity analysis is presented in Section 1.6.

1.2 General Framework of Elastic Stability

General framework of elastic stability of a conservative system that has the total potential energy is briefly introduced to define variables, basic equations, and critical points.

1.2.1 Governing equations and stability

Consider an elastic structure discretized by the finite element method and subjected to quasi-static nodal loads, which are parameterized by a load parameter or load factor Λ. The deformation of the structure is expressed in terms of an n-dimensional nodal displacement vector $\mathbf{U} = (U_i) \in \mathbb{R}^n$. The stationary condition of the total potential energy[1] $\Pi(\mathbf{U}, \Lambda)$ at a specified value of Λ leads to the following set of equilibrium equations:

$$S_{,i} = 0, \quad (i = 1, \ldots, n) \tag{1.1}$$

where

$$S_{,i} \equiv \frac{\partial \Pi}{\partial U_i}, \quad (i = 1, \ldots, n) \tag{1.2}$$

is the partial differentiation of Π with respect to U_i. The set of equilibrium points in $(\mathbf{U}, \Lambda)$-space is called an *equilibrium path*, and a path that contains the undeformed initial state is called the *fundamental equilibrium path* or the fundamental path.

In the description of stability, we refer to the tangent stiffness matrix or the stability matrix

$$\mathbf{S} = [S_{,ij}] \tag{1.3}$$

where $S_{,ij} \equiv \partial \Pi^2 / \partial U_i \partial U_j$ $(i, j = 1, \ldots, n)$. Note that $\mathbf{S} \in \mathbb{R}^{n \times n}$ is symmetric owing to existence of potential, i.e.,

$$S_{,ij} = S_{,ji}, \quad (i, j = 1, \ldots, n) \tag{1.4}$$

The rth eigenvalue λ_r and the associated eigenvector $\mathbf{\Phi}_r = (\phi_{ri}) \in \mathbb{R}^n$ of $\mathbf{S}$ are defined by

$$\sum_{j=1}^{n} S_{,ij} \phi_{rj} = \lambda_r \phi_{ri}, \quad (i = 1, \ldots, n) \tag{1.5}$$

Since $\mathbf{S}$ is symmetric, all eigenvalues of $\mathbf{S}$ are real, and are ordered such that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \tag{1.6}$$

The eigenvectors $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n$ are ortho-normalized by

$$\sum_{i=1}^{n} \phi_{ri} \phi_{si} = \delta_{rs}, \quad (r, s = 1, \ldots, n) \tag{1.7}$$

where δ_{rs} is the Kronecker delta, being equal to 1 for $r = s$ and 0 for $r \neq s$.

The stability of an equilibrium state is classified as

$$\begin{cases} \textit{stable:} & \text{all eigenvalues of } \mathbf{S} \text{ are positive } (\lambda_1 > 0) \\ \textit{unstable:} & \text{at least one eigenvalue of } \mathbf{S} \text{ is negative } (\lambda_1 < 0) \end{cases} \tag{1.8}$$

Thus the stability depends on the sign of λ_1.

1.2.2 Critical state

A critical state is defined as an equilibrium state at which at least one eigenvalue is zero. The first critical state on the fundamental path that is defined by $\lambda_1 = 0$ is of most engineering interest as λ_1 defines the stability by (1.8).

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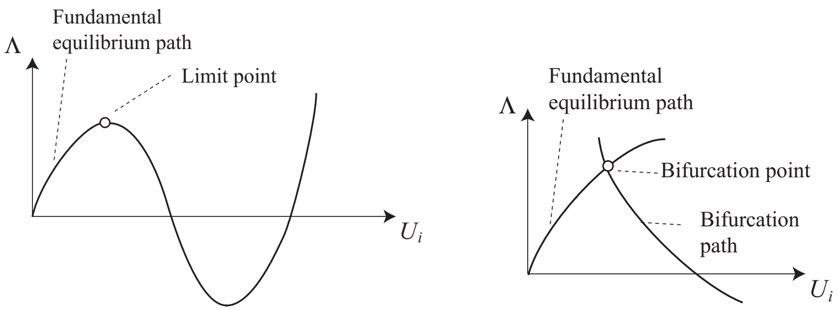
[1] The total potential energy function $\Pi(\mathbf{U}, \Lambda)$ is assumed to be sufficiently smooth. The contact problem, for example, is out of scope of this book.



(a) Limit point (b) Bifurcation point

Fig. 1.2 Equilibrium paths for simple critical points.

Load factor, eigenvalue and eigenvector at the critical state are denoted by Λ^{c}, λ_r^{c} and $\mathbf{\Phi}_r^{\mathrm{c}}$, respectively. The superscript $(\,\cdot\,)^{\mathrm{c}}$ indicates a value at a critical state throughout this book.

Critical points are classified as follows:

- **simple critical point**: a critical point is called *simple* or *distinct* if only one eigenvalue λ_r becomes zero. A simple critical point is classified to a *limit point* and a *bifurcation point.*
- **coincident critical point**: a critical point with m (≥ 2) zero eigenvalues is called *coincident*[2]. The coincident critical point consisting of several bifurcation points is called a *compound branching point.* The coincident point with bifurcation point(s) at a limit point is called a *hilltop branching point.*

A more detailed classification of critical points will be given in Section 3.3.

Equilibrium paths for simple critical points are illustrated in Fig. 1.2 in terms of the relation between a displacement U_i and Λ. At a limit point, Λ reaches a maximum or minimum in the $(\mathbf{U}, \Lambda)$-space, while the uniqueness of the equilibrium state is lost at a bifurcation point.

Remark 1.2.1 An equilibrium path can be traced by the load control method, for which Λ is chosen as the path parameter t. Namely, the equilibrium equations (1.1) are solved for a given $t = \Lambda$. However, the path near the limit point should be traced by the displacement increment method or the arc-length method [255], for which the path parameter $t = t(\mathbf{U}, \Lambda)$ is generally defined as a function of $\mathbf{U}$ and Λ. □

Remark 1.2.2 Many numerical methods for the computation of critical points are available [66, 90, 156, 311, 312]. For example, a simple critical point can be pinpointed accurately by iteratively solving the extended system formulated by (1.1) and (1.5) with (1.7) for $r = 1$, $\lambda_1 = 0$ considering $\mathbf{U}$, $\mathbf{\Phi}_1$ and Λ as variables. In numerical examples of this book, critical points are computed within

[2] *Coincident*, *compound*, *multiple* and *repeated* have a similar meaning, and *coincident* is mainly used in this book.

sufficiently good accuracy so as not to influence the accuracy of sensitivity coefficients. □

1.2.3 Proportional loading

For a proportional loading, most common in structural analysis, the nodal load vector $\mathbf{P} \in \mathbb{R}^n$ is given as the product of the load factor Λ and the specified load pattern vector $\mathbf{p} = (p_i) \in \mathbb{R}^n$, namely,

$$\mathbf{P} = \Lambda \mathbf{p} \tag{1.9}$$

Then the total potential energy $\Pi(\mathbf{U}, \Lambda)$ is given as

$$\Pi(\mathbf{U}, \Lambda) = H(\mathbf{U}) - \Lambda \mathbf{p}^\top \mathbf{U} \tag{1.10}$$

where $H(\mathbf{U})$ denotes the strain energy corresponding to the deformation $\mathbf{U}$ and $(\,\cdot\,)^\top$ means the transpose of the associated vector or matrix. The equilibrium equations are written as

$$S_{,i} = H_{,i} - \Lambda p_i = 0, \quad (i = 1, \dots, n) \tag{1.11}$$

where $H_{,i}$ is the partial differentiation of H with respect to U_i, and expresses the equivalent nodal load in the direction of U_i. From (1.11), we have $\mathbf{S} = [H_{,ij}]$.

1.3 Design Parameterization

We define a design parameter in structural optimization (or an imperfection parameter in stability analysis). Let $\mathbf{v} \in \mathbb{R}^\nu$ denote a ν-dimensional vector representing the mechanical properties of the structure. The vector $\mathbf{v}$ may correspond, e.g., to

- sizing design parameter such as cross-sectional areas and plate thickness,
- shape design parameter such as nodal coordinates, and
- material property design parameter such as Young's modulus and Poisson's ratio.

It is possible to choose topology (member location) and external loads also as design parameters. Among many design parameters, we focus mainly on cross-sectional areas of members and nodal coordinates in this book. See [49] for more issues on design parameters.

Indicate by $\mathbf{v} = \mathbf{v}^0$ the reference (perfect) structure. The superscript $(\,\cdot\,)^0$ denotes a variable associated with the reference structure throughout this book. Consider a modified structure, being defined by *design variation vector* $\mathbf{d}_i \in \mathbb{R}^\nu$ $(i = 1, \dots, \rho)$ and the associated scaling *design parameter* ξ_i $(i = 1, \dots, \rho)$ as

$$\mathbf{v} = \mathbf{v}^0 + \sum_{i=1}^{\rho} \xi_i \mathbf{d}_i \tag{1.12}$$

Dependent on problem formulation, we often use a single scaling parameter ξ and employ a simplified expression

$$\mathbf{v} = \mathbf{v}^0 + \xi \sum_{i=1}^{\rho} \mathbf{d}_i \tag{1.13}$$

Remark 1.3.1 In stability analysis, $\mathbf{v}$ is called the *imperfection parameter* vector, $\mathbf{v}^0$ corresponds to the *perfect structure*, $\mathbf{d}_i$ is the *imperfection pattern vector* and ξ_i is the *imperfection parameter*. The use of multiple vectors $\mathbf{d}_i$ is vital generalization in the development of the *worst imperfection* in Chapter 11, and the *probabilistic variation of critical loads* in Chapters 14 and 15. □

We simply call ξ_i the *parameter* and present a unified formulation for design sensitivity and imperfection sensitivity. Our task of sensitivity analysis is to quantitatively evaluate the differential coefficients of responses with respect to ξ_i.

For simplicity, in the remainder of this chapter, we consider only a single pair of parameter ξ_1 and vector $\mathbf{d}_1$, and set $\xi_1 = \xi$, $\mathbf{d}_1 = \mathbf{d}$. Then (1.12) reduces to

$$\mathbf{v} = \mathbf{v}^0 + \xi \mathbf{d} \tag{1.14}$$

Accordingly, the total potential energy is written as $\Pi(\mathbf{U}, \Lambda, \xi)$.

In the sensitivity analysis, the *total differentiation* and *partial differentiation* with respect to ξ should be distinguished and denoted by $(\,\cdot\,)'$ and $(\,\cdot\,)_{,\xi}$, respectively. For example, in the total differentiation, $\mathbf{U}$ is to be conceived as a function of ξ, as $\mathbf{U}$ for a fixed value of Λ is obtained by solving the equilibrium equation (1.1). The total differentiation $\mathbf{U}'$ of $\mathbf{U}(\xi)$ with respect to ξ is found by total differentiation of the equilibrium equation. The argument ξ will be often suppressed in this chapter, for simplicity.

1.4 Design Sensitivity Analysis for Linear Response

Consider a linear response of a structure, and denote by $\mathbf{K}_{\mathrm{L}}(\xi) \in \mathbb{R}^{n \times n}$ the linear (infinitesimal) stiffness matrix, which is a function of ξ. The total potential energy is given as

$$\Pi(\mathbf{U}, \Lambda, \xi) = \frac{1}{2} \mathbf{U}^\top \mathbf{K}_{\mathrm{L}}(\xi) \mathbf{U} - \Lambda \mathbf{p}(\xi)^\top \mathbf{U} \tag{1.15}$$

Partial differentiation of $\Pi(\mathbf{U}, \Lambda, \xi)$ with respect to $\mathbf{U}$, with the use of symmetry of $\mathbf{K}_{\mathrm{L}}$, leads to the equilibrium equation

$$\mathbf{K}_{\mathrm{L}}(\xi) \mathbf{U} - \Lambda \mathbf{p}(\xi) = \mathbf{0} \tag{1.16}$$

Here $\mathbf{U}$ is to be obtained as a function of ξ as a solution to (1.16). Total differentiation of (1.16) with respect to ξ gives

$$\mathbf{K}_{\mathrm{L}} \mathbf{U}' + \mathbf{K}_{\mathrm{L}}' \mathbf{U} - \Lambda \mathbf{p}' = \mathbf{0} \quad \Longrightarrow \quad \mathbf{K}_{\mathrm{L}} \mathbf{U}' = -\mathbf{K}_{\mathrm{L}}' \mathbf{U} + \Lambda \mathbf{p}' \tag{1.17}$$

In the direct differentiation method [115], the sensitivity coefficient vector $\mathbf{U}'$ is directly computed by (1.17), as $\mathbf{U} = \Lambda (\mathbf{K}_{\mathrm{L}})^{-1} \mathbf{p}$ can be computed from (1.16) and $\mathbf{p}'$ and $\mathbf{K}_\mathrm{L}'$ can be easily obtained with explicit dependence of $\mathbf{p}$ and $\mathbf{K}_\mathrm{L}$ on ξ. For example, if $\mathbf{p}$ is the self-weight of a truss and ξ corresponds to the cross-sectional area, $\mathbf{p}$ and $\mathbf{K}_\mathrm{L}$ are explicit linear functions of ξ.

Remark 1.4.1 The second equation in (1.17) has the same form as (1.16) if the right-hand-side terms are regarded as nodal loads; therefore, it is computationally efficient to factorize $\mathbf{K}_\mathrm{L}$ in the process of solving (1.16) for $\mathbf{U}$. Note that the *adjoint variable method* is more computationally efficient if we have many design parameters and few response quantities, sensitivity coefficients for which are to be evaluated [49, 162]. □

Remark 1.4.2 Upon obtaining $\mathbf{U}$ and its sensitivity coefficient $\mathbf{U}'$, we can compute the sensitivity coefficients of stresses and strains in a straightforward manner. For example, for a truss with n^m members, the relation between $\mathbf{U}$ and the axial force vector $\mathbf{N} \in \mathbb{R}^{n^\mathrm{m}}$ can be written as

$$\mathbf{N} = \mathbf{D}\mathbf{U} \tag{1.18}$$

where $\mathbf{D} \in \mathbb{R}^{n^\mathrm{m} \times n}$ is generally a function of ξ. Accordingly, the sensitivity coefficients of $\mathbf{N}$ can be obtained from

$$\mathbf{N}' = \mathbf{D}'\mathbf{U} + \mathbf{D}\mathbf{U}' \tag{1.19}$$

□

1.5 Design Sensitivity Analyses for Nonlinear Responses

Design sensitivity analyses for nonlinear responses, including a linear buckling load and a limit point load, are presented.

1.5.1 Linear buckling load

We start with the sensitivity analysis of a linear buckling load, which is weakly nonlinear. Consider a structure subjected to the proportional load $\mathbf{P} = \Lambda\mathbf{p}$ of (1.9) with sufficiently small prebuckling deformation $\mathbf{U} \simeq \mathbf{0}$. The tangent stiffness matrix $\mathbf{S}$ is expressed as the sum of the linear stiffness matrix $\mathbf{K}_\mathrm{L}$, which does not depend on deformation, and the geometrical stiffness matrix $\mathbf{K}_\mathrm{G} \in \mathbb{R}^{n\times n}$, which is a function of $\mathbf{U}$ through the internal forces or stresses at the current (reference) state.

For the small deformation $\mathbf{U} \simeq \mathbf{0}$, the internal forces or stresses can be assumed to be proportional to Λ, and $\mathbf{K}_\mathrm{G}$ is given as the product of Λ and a constant matrix $\mathbf{K}_\mathrm{G0} \in \mathbb{R}^{n\times n}$, which is a function of stress under the given load $\mathbf{p}$, namely,

$$\mathbf{K}_\mathrm{G} = \Lambda\mathbf{K}_\mathrm{G0} \tag{1.20}$$

The tangent stiffness matrix $\mathbf{S}$ becomes

$$\mathbf{S} = \mathbf{K}_\mathrm{L} + \Lambda\mathbf{K}_\mathrm{G0} \tag{1.21}$$

In this case, the criticality condition for eigenvalue $\lambda_r = 0$ in (1.5) is expressed as

$$[\mathbf{K}_{\mathrm{L}} + \Lambda_{\mathrm{L}r}\mathbf{K}_{\mathrm{G0}}]\mathbf{\Phi}_r = \mathbf{0}, \quad (r = 1, \dots, n) \tag{1.22}$$

where $\Lambda_{\mathrm{L}r}$ is the rth linear buckling load factor, and eigenvector $\mathbf{\Phi}_r$ is normalized by $\mathbf{K}_{\mathrm{L}}$, which is positive definite, namely,

$$\mathbf{\Phi}_r^\top \mathbf{K}_{\mathrm{L}}\mathbf{\Phi}_r = 1, \quad (r = 1, \dots, n) \tag{1.23}$$

Then the following relationship holds from (1.22) and (1.23) at a stable initial state with $\Lambda_{\mathrm{L}r} \neq 0$:

$$\mathbf{\Phi}_r^\top \mathbf{K}_{\mathrm{G0}}\mathbf{\Phi}_r = -\frac{1}{\Lambda_{\mathrm{L}r}}, \quad (r = 1, \dots, n) \tag{1.24}$$

Eq. (1.22) defines a generalized eigenvalue problem, which yields n-eigenpairs of $\Lambda_{\mathrm{L}r}$ and $\mathbf{\Phi}_r$. The smallest positive eigenvalue[3] $\Lambda_{\mathrm{L}r}$ is called the *linear buckling load factor*, and the corresponding $\mathbf{\Phi}_r$ is called the *linear buckling mode*. The critical load factor can be approximated by the linear buckling load factor with good accuracy if the assumption $\mathbf{U} \simeq \mathbf{0}$ on prebuckling deformation is satisfied.

Total differentiation of (1.22) and (1.23) with respect to ξ gives, respectively,

$$\mathbf{K}_{\mathrm{L}}'\mathbf{\Phi}_r + \mathbf{K}_{\mathrm{L}}\mathbf{\Phi}_r' + \Lambda_{\mathrm{L}r}'\mathbf{K}_{\mathrm{G0}}\mathbf{\Phi}_r + \Lambda_{\mathrm{L}r}\mathbf{K}_{\mathrm{G0}}'\mathbf{\Phi}_r + \Lambda_{\mathrm{L}r}\mathbf{K}_{\mathrm{G0}}\mathbf{\Phi}_r' = \mathbf{0} \tag{1.25}$$

$$\mathbf{\Phi}_r^\top \mathbf{K}_{\mathrm{L}}'\mathbf{\Phi}_r + 2\mathbf{\Phi}_r^\top \mathbf{K}_{\mathrm{L}}\mathbf{\Phi}_r' = 0 \tag{1.26}$$

for $r = 1, \dots, n$. Sensitivity coefficients $\Lambda_{\mathrm{L}r}'$ and $\mathbf{\Phi}_r'$ can be obtained by solving the set of $n + 1$ simultaneous linear equations (1.25) and (1.26). Since $\mathbf{K}_{\mathrm{G0}}$ is a function of stresses under the load $\mathbf{p}$, the sensitivity coefficients of the stresses are required to compute $\mathbf{K}_{\mathrm{G0}}'$ in (1.25) (cf., Remark 1.5.1).

Premultiplying $\mathbf{\Phi}_r^\top$ to both sides of (1.25) and using symmetry of $\mathbf{K}_{\mathrm{L}}$ and $\mathbf{K}_{\mathrm{G0}}$, we obtain

$$\mathbf{\Phi}_r^\top [\mathbf{K}_{\mathrm{L}}' + \Lambda_{\mathrm{L}r}\mathbf{K}_{\mathrm{G0}}']\mathbf{\Phi}_r + \mathbf{\Phi}_r'^\top [\mathbf{K}_{\mathrm{L}} + \Lambda_{\mathrm{L}r}\mathbf{K}_{\mathrm{G0}}]\mathbf{\Phi}_r + \Lambda_{\mathrm{L}r}'\mathbf{\Phi}_r^\top \mathbf{K}_{\mathrm{G0}}\mathbf{\Phi}_r = 0 \tag{1.27}$$

By (1.22) and (1.24), (1.27) gives

$$\Lambda_{\mathrm{L}r}' = \Lambda_{\mathrm{L}r}\mathbf{\Phi}_r^\top [\mathbf{K}_{\mathrm{L}}' + \Lambda_{\mathrm{L}r}\mathbf{K}_{\mathrm{G0}}']\mathbf{\Phi}_r \tag{1.28}$$

Hence, $\Lambda_{\mathrm{L}r}'$ can be found from (1.28) by simple matrix computation if the sensitivity coefficient $\mathbf{\Phi}_r'$ of $\mathbf{\Phi}_r$ is not needed.

Remark 1.5.1 The evaluation of $\mathbf{K}_{\mathrm{G0}}'$ is the most difficult process in the computation of the sensitivity coefficients $\Lambda_{\mathrm{L}r}'$ in (1.28). For example, for a truss with n^{m} members with a parameter ξ defining the cross-sectional areas, $\mathbf{K}_{\mathrm{G0}}'$ is evaluated to

$$\mathbf{K}_{\mathrm{G0}}' = \sum_{j=1}^{n^{\mathrm{m}}} \frac{\partial \mathbf{K}_{\mathrm{G0}}}{\partial N_j} N_j' \tag{1.29}$$

[3]The negative eigenvalues correspond to the buckling loads against the proportional load in the opposite direction. However, in customary linear buckling analysis, the directions of the loads are fixed and Λ is assumed to be positive.

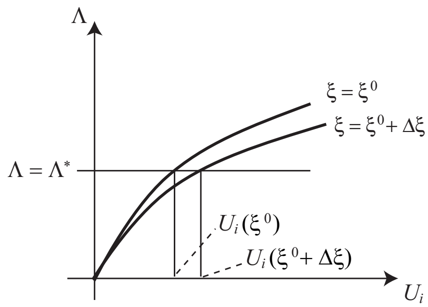


Fig. 1.3 Variation of an equilibrium path with respect to parameter modification $\xi = \xi^0 \longrightarrow \xi^0 + \Delta\xi$.

where N_j is the axial force of the jth member, and its sensitivity coefficient N_j' can be computed by (1.19) after obtaining $\mathbf{U}'$ by (1.17). □

1.5.2 Responses at a regular state

Consider a regular state of a nonlinear response. Since the critical load cannot be defined for the regular state, design sensitivity of displacements and stresses for a fixed load $\Lambda = \Lambda^*$, instead of the critical load, is investigated.

Consider an elastic structure satisfying

- the structure is in a regular state at a load level Λ^*, where the tangent stiffness matrix $\mathbf{S}$ is nonsingular, and
- the displacements and stresses are monotonically increasing functions of Λ.

Fig. 1.3 illustrates the variation of the equilibrium path with respect to the modification of the parameter ξ from ξ^0 to $\xi^0 + \Delta\xi$. In this case, sensitivity coefficients $\mathbf{U}'$ of $\mathbf{U}$ at $\xi = \xi^0$ can be obtained based only on the response at the load level $\Lambda = \Lambda^*$ without incrementally updating or accumulating $\mathbf{U}'$ along the fundamental path [45].

Since the total potential energy $\Pi(\mathbf{U}, \Lambda, \xi)$ is a function of $\mathbf{U}$, Λ and ξ, the total and partial differentiations of Π should be properly distinguished. The dependence of $\mathbf{U}$ on ξ is implicitly defined by the equilibrium equation (1.1). Therefore, the design sensitivity U_j' is to be found by total differentiation of (1.1) with respect to ξ for a fixed value of $\Lambda = \Lambda^*$ as

$$S_{,i}' \equiv \sum_{j=1}^{n} S_{,ij} U_j' + S_{,i\xi} = 0, \quad (i = 1, \ldots, n) \tag{1.30}$$

where in the partial differentiation

$$S_{,i\xi} = \frac{\partial S_{,i}(\mathbf{U}, \Lambda, \xi)}{\partial \xi} \tag{1.31}$$

the implicit dependence of $\mathbf{U}$ on ξ is not considered. Note that the sensitivity coefficient $\mathbf{U}'$ cannot be obtained from (1.30) at a critical state where $\mathbf{S} = [S_{,ij}]$ is singular.

For a proportionally loaded structure satisfying (1.11), (1.30) reduces to

$$\sum_{j=1}^{n} H_{,ij} U'_j = \Lambda p'_i - H_{,i\xi}, \quad (i = 1, \ldots, n) \tag{1.32}$$

Here $H = H(\mathbf{U}(\xi), \xi)$ implicitly depends on ξ through $\mathbf{U}$.

At the course of numerical path-tracing analysis, U'_j can be computed from (1.30) at minimal additional cost as follows (cf., [213, 262] for details):

- The tangent stiffness matrix $\mathbf{S} = [S_{,ij}]$ has already been computed during the path-tracing analysis.
- Since $S_{,i}$ has also been computed to obtain residual (unbalanced) forces and the dependence of $S_{,i}$ on ξ is known, $S_{,i\xi}$ can be computed with ease.
- Since $\mathbf{S}$ has been factorized at $\Lambda = \Lambda^*$, (1.30) can be solved with minimal additional computational cost.

Notice that the sensitivity equation (1.30) or (1.32) at a regular state has a similar form as (1.17) for the geometrically linear problem in Section 1.4; i.e., the linear stiffness matrix $\mathbf{K}_\mathrm{L}$ is to be replaced by the tangent stiffness matrix $\mathbf{S}$.

The sensitivity coefficients of eigenvalue λ_r and eigenvector $\mathbf{\Phi}_r$ of $\mathbf{S}$ are obtained from the following equations, which are derived by total differentiation of (1.5) and (1.7):

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{n} S_{,ijk} \phi_{rj} U'_k + S_{,ij\xi} \phi_{rj} + S_{,ij} \phi'_{rj} \right) = \lambda_r \phi'_{ri} + \lambda'_r \phi_{ri}, \quad (i = 1, \ldots, n) \tag{1.33}$$

$$\sum_{j=1}^{n} \phi_{rj} \phi'_{rj} = 0 \tag{1.34}$$

Note that we have $n+1$ equations for $n+1$ unknowns λ'_r and ϕ'_{ri} $(i = 1, \ldots, n)$ for each r. By multiplying ϕ_{ri} to (1.33), summing up by i and using (1.5) and (1.7), we obtain

$$\lambda'_r = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} S_{,ijk} \phi_{ri} \phi_{rj} U'_k + S_{,ij\xi} \phi_{ri} \phi_{rj} \right) \tag{1.35}$$

If ϕ'_{ri} is not needed, λ'_r can be obtained from (1.35) by arithmetical operation, after computing U'_i by (1.30).

1.5.3 Limit point load

Sensitivity coefficient $\Lambda^{c\prime}$ of a limit point load can be found only from the equilibrium equations (1.1) in a similar manner as the case of the regular state presented in Section 1.5.2.

By considering $\Lambda^{\rm c}$ as a function of ξ, and by differentiating the equilibrium equations (1.1) with respect to ξ at the limit point load $\Lambda = \Lambda^{\rm c}$, we obtain

$$\sum_{j=1}^{n} S_{,ij} U_j' + S_{,i\Lambda} \Lambda^{\rm c\prime} + S_{,i\xi} = 0, \quad (i = 1, \ldots, n) \tag{1.36}$$

where $(\,\cdot\,)_{,\Lambda}$ denotes partial differentiation with respect to Λ.

Premultiplying the critical eigenmode $\phi_{1i}^{\rm c}$ to the both sides of (1.36), taking summation over i, and using $\lambda_1 = 0$ for (1.5), we can obtain the sensitivity coefficient of the limit point load

$$\Lambda^{\rm c\prime} = -\sum_{i=1}^{n} S_{,i\xi} \phi_{1i}^{\rm c} \Big/ \sum_{i=1}^{n} S_{,i\Lambda} \phi_{1i}^{\rm c} \tag{1.37}$$

By (1.37), the sensitivity coefficient $\Lambda^{\rm c\prime}$ can be found directly without resort to the sensitivity coefficients $\mathbf{U}'$ of displacements and $\boldsymbol{\Phi}_1^{\rm c\prime}$ of the eigenmode. Since $\Lambda^{\rm c\prime}$ in (1.37) does not depend on ξ, the critical load $\Lambda^{\rm c}(\xi)$ of an imperfect system is written as a linear function of ξ as

$$\Lambda^{\rm c}(\xi) = \Lambda^{\rm c}(0) + \Lambda^{\rm c\prime} \xi \tag{1.38}$$

Remark 1.5.2 Eq. (1.37) in principle agrees with preexisting results of stability analysis of elastic conservative systems based on the derivatives with respect to the displacement in the direction of $\boldsymbol{\Phi}_1^{\rm c}$ (cf., [283, 285] and Section 3.5.2). However, we prefer formulations based on physical coordinates, as employed in (1.37), so as to be consistent with the conventional finite element analysis. □

Remark 1.5.3 The denominator of (1.37) vanishes at a bifurcation point (cf., (3.25) and (3.27) in Section 3.3); therefore, (1.37) is not extendable to a bifurcation point [239]. It should also be noted that the sensitivity equation presented in this section does not depend on the symmetry of the imperfection unlike the case of bifurcation loads (cf., Section 2.2). □

Remark 1.5.4 For a proportional loading, (1.36) and (1.37) are to be rewritten by using the following formulas (cf., (1.11)):

$$S_{,i\Lambda} = -p_i, \quad S_{,i\xi} = H_{,i\xi} - \Lambda p_i', \quad S_{,ij} = H_{,ij} \tag{1.39}$$

□

1.6 Historical Development

Design sensitivity analysis and shape sensitivity analysis were initiated mainly in the field of structural optimization [49, 115], and established works on design sensitivity analysis of linear elastic responses, eigenvalues of vibration, and so on, are at hand. In the 1980's, geometrically nonlinear formulations were developed for optimization against buckling of simple structures [150, 158, 249, 316].

For structures exhibiting limit point instability, the algorithm for design sensitivity analysis was developed [239, 314, 316], and the optimum design for a specified nonlinear buckling load factor was conducted [150, 158, 179, 249, 316]. Mathematical equivalence between design sensitivity and imperfection sensitivity was recognized [225]. The formulas for imperfection sensitivity coefficients were implemented into optimality conditions for problems under constraint on a limit point load, and the optimization for nonlinear buckling was studied [217, 232, 233, 254]. Thus general theory of elastic stability can effectively be used in design sensitivity analysis for structural optimization.

A semi-analytical approach for sensitivity analysis of critical loads, and an optimization algorithm implementing design modifications were presented [253]. Sensitivity analysis of critical loads based on asymptotic approach was also presented, where computation of the third-order derivatives of the total potential energy was necessitated for bifurcation loads [202]. Examples of optimum design with limit points were studied [203]. An approach considering postbuckling states was presented [97].

Several numerical approaches to design sensitivity analyses for bifurcation loads were developed in the framework of finite element analysis. For example, sensitivity analyses of bifurcation loads were conducted by an interpolation approach [227], and an approach using linear eigenvalue analysis [214, 316].

Another branch of sensitivity analysis of nonlinear response is found in elastoplastic continuum mechanics, and formulations suitable for computational implementation were presented [162, 177, 214, 262, 305]. However, there are only a few studies on design sensitivity analysis of elastoplastic critical loads [215].

1.7 Summary

In this chapter,

- general frameworks of elastic stability of conservative systems have been presented, and
- design sensitivity analyses for linear and nonlinear responses have been classified and briefly been introduced.

The major findings of this chapter are as follows.

- The design sensitivity analyses presented herein are expressed by simple formulas. Yet these formulas contain the essence of design and imperfection sensitivity analyses of buckling loads to be introduced in the following chapters.
- Design sensitivity analysis for regular states can be carried out easily based on the response quantities at the final load level. However, sensitivity coefficients at the critical states cannot be obtained similarly, because the tangent stiffness matrix is singular at the critical point.
- The sensitivity coefficient of a limit point load can be obtained simply by differentiation of the equilibrium equations.