Sources and Studies in the History of Mathematics and Physical Sciences

Maria Hämeen-Anttila
Jan von Plato *Editors*

Kurt Gödel

The Princeton Lectures on Intuitionism

Springer

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Preface

Gödel's Princeton Lectures on Intuitionism of 1941 are preserved in two notebooks written in longhand English. They contain a detailed presentation of his famous functional interpretation of arithmetic and have been studied in connection with the editing of Gödel's *Collected Works*, in particular for the light they shed on a lecture on intuitionistic logic he gave at Yale. The writing is on the whole quite clear, with occasional additions and remarks in German shorthand, and a gap toward the end, at pages 89–106. It turned out in 2017 that the missing pages were inside an envelope in another place, ten reels apart in the microfilm edition of Gödel's manuscripts. That discovery was the starting point of the present edition. Gödel's *Arbeitshefte* or mathematical workbooks, especially number 9, have close connections to the Princeton Lectures. This source and others, including the *Resultate Grundlagen* notebook series, are described in the introduction written by the first editor.

The reader may ask why Gödel didn't publish his lectures at the time, or at least their main results. The answer should be that he failed to achieve his central aim, clearly indicated by the mentioned sources, namely to extend the functional interpretation to the transfinite to obtain a proof of the consistency of analysis.

Bill Howard generously shared his knowledge of Gödel's functional interpretation with us, and told about his encounters with Gödel, as reported in the introduction. We are very glad to dedicate this little volume to him.

Acknowledgements

The editing of Gödel's two notebooks that contain his Princeton Lectures on Intuitionism turned out to be a clearly more demanding enterprise than we first thought. There are lots of minute formal details, quite often incompletely and at places even erroneously given by Gödel himself. We thank Annika Kanckos and Tim Lethen, our research team members, for their support, as well as Fernando Ferreira and Bill Howard for sharing their expertise on Gödel's functional interpretation.

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Contents

Introduction: Gödel's functional interpretation in context

In the spring of 1941, Kurt Gödel held a lecture course on intuitionistic logic at the Institute for Advanced Study in Princeton. Two spiral notebooks labelled simply "Vorl." and two sets of loose notes contain handwritten notes for the lecture course. The lecture notes divide into two themes. The first part is an introduction to intuitionistic logic. The second part is a detailed presentation of Gödel's functional interpretation of Heyting Arithmetic and its applications.

The general aim of the lectures is to examine the constructivity of intuitionistic logic. In the first part of the lectures, Gödel focuses heavily on the interconnection between intuitionistic and classical logic. The standard *proof explanation* of the intuitionistic logic was, he believed, not adequate to show the constructive character of intuitionistic logic. By reinterpreting intuitionistic logic in a more precise way, Gödel wants to prove that Heyting Arithmetic is properly constructive in the sense that it has the existence property. This reinterpretation is Gödel's functional system Σ, and the Princeton course is the most detailed presentation of it.

The theme of the lectures was closely connected to Gödel's previous talks of 1933 and 1938, as well as a lecture given at Yale University in April 1941. In the lecture "The present situation in the foundations of mathematics" given in Cambridge, Massachusetts, in 1933, Gödel argues that intuitionistic logic is not an ideal basis for a constructive foundation of mathematics because of the nature of its logical operations and the proof explanation. In his "Zilsel lecture" of 1938, he mentions an alternative interpretation of the logical operations in terms of a system of primitive recursive functionals of higher types. Finally, the system is developed in detail in the Princeton course and the Yale lecture. These results – apart from the Princeton lectures – were published posthumously in Gödel's *Collected Works* in 1995; the first published article on the functional interpretation appeared 17 years after the Princeton course, in the journal *Dialectica* in 1958.

In what follows, I will give an overview of the lecture course, highlighting the features which are missing from the other works of the 1930s and early 1940s. Apart from higher level of detail, the new aspects include an alternative version of Gödel's negative translation between Peano and Heyting Arithmetic (Gödel 1933b), the "truth table theorem" that proves that classical and intuitionistic

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propositional logics coincide under the assumption of decidability of atomic formulas, and a presentation of applications of the functional system Σ only mentioned in the Yale lecture. However, even where Gödel considers themes already mentioned in the other works, we often gain new insight into his views on particular issues. In this sense, the Princeton lectures complement the shorter lectures and give a richer picture of Gödel's early views on intuitionism.

Content of the lectures

If Gödel's lecture course had a specific title, it is not known to us: the IAS Bulletin of October 1941 tells only that "Dr. Gödel lectured on some results concerning intuitionistic logic," and that in the academic year 1941-1942, "he will continue his researches on this subject and its connection with the continuum problem." The course consisted of at least nine lectures, although the notes are not divided into sections. However, Gödel seems to have started each lecture with a review of the previous lecture's contents; there are, in total, nine of this kind of "last time..." summaries. At the Institute for Advanced Study, the Spring Semester lasted from 1st February to 1st May, and Gödel probably gave his course around this time. For the most part, the notes are clearly written and easy to understand, although toward the end more advanced themes are introduced. In a letter of 4th May 1941 to his brother, Gödel wrote that there were only three students left at the end of his course.[1] The wartime circumstances were probably one cause for the lack of attendance – and perhaps Gödel's rigorous yet terse presentation had scared away some of the listeners.

The lectures divide into two main parts. The first part, p. 1–47 of the lecture notes, introduces intuitionistic propositional and predicate logic and studies the interconnections between intuitionistic and classical logic. The second part, p. 48–117, concerns the functional interpretation of Heyting arithmetic. Moreover, Gödel's mathematical notebooks, the *Arbeitshefte*, contain early sketches of proofs featured in the lectures. The notebooks 7–10 (030025–030028)[2] probably date from early 1941; *Heft 7* (030025) is dated 1.1.1941 and in *Heft 9* (030027) we find the date "Feb 1941."[3] The earliest drafts of the functional system Σ in *Hefte* 7 and 9 are all titled "Gentzen" or "Gentzen Bew⟦eis⟧." This probably refers to Gentzen's first consistency proof of 1935 (Gentzen 1935/ 1974), which

[1] The letter is quoted in (Van Atten 2015, 201).

[2] The items in Gödel's *Papers* are referred to by their document code.

[3] Gödel was not in the habit of writing down dates of his notebook entries; he often only marked the change of the year.

he chose not to publish because of Gödel's and Paul Bernays' critique, involving a reduction procedure reminiscent of the "no-counterexample" interpretation of Gödel's Σ. The later proofs do not mention Gentzen.

It is beyond the scope of this introduction to consider Gödel's shorthand notebooks in depth. Unlike the lectures, the *Arbeitshefte* do not contain finished proofs ready for publication, there are many unfinished sketches, trial and error, and long computations.[4] However, in *Arbeitsheft 9*, p. 2–3, we find a numbered list written in shorthand and titled "Vorl. 1941 Sommer," which is clearly a plan for the Princeton lectures. The plan contains twelve points. Item number $2'$, a later addition on p. 2, summarizes Gödel's general agenda:

> On the basis of the intuitionistic axioms formulated by Heyting, criticism against them [especially the availability of negative universal statements.] *What is a properly intuitionistic* system [in particular, existential statements superfluous]. *Then also classical number theory derivable. This would perhaps be a reason against [[Heyting's logic]], but not correct, because the Brouwerian concepts are expressible in a system where no such unclarities occur. That is the goal of the lectures. It results also in a consistency proof for number theory.* First, however, the intuitionistic Heyting system and its properties.[5]

Although Gödel's goal is philosophically motivated, the lectures are mostly formal in nature. Nevertheless, each proof or formal explanation seems carefully planned to support the overarching goal of demonstrating the problems of intuitionistic logic and then giving an alternative interpretation in order to prove that intuitionistic logic (or, at least, arithmetic) is properly constructive. The lack of philosophical remarks is not surprising, as Gödel's early style was in gen-

[4] The other mathematical notebook series, *Resultate Grundlagen*, contains the finished proofs, but only two of them (the "constructive negation translation" discussed below and an inductive proof of computability of Σ-functionals; see (Hämeen-Anttila 2020, 98–102)) are directly related to the Princeton lectures.

[5] Aufgrund dieser intuit⟦uitionistischen⟧ Axiome formuliert [von Heyting] Kritik dagegen [insbesondere Vorhandensein der Negationen von Allaussagen]. *Was ist ein wirklich intuit⟦ionistisches⟧ ⟦System⟧?* [Insbesondere Existenzaussagen überflüssig]. Daher auch klassische Zahlentheorie ableitbar. *Das ⟦wäre⟧ vielleicht ein Grund dagegen, aber nicht richtig, denn die Brouwer'schen Begriffe ⟦sind⟧ ausdrückbar in einem System, in welchem keine solchen Unklarheiten vorkommen. Das ist der Zweck der Vorlesungen. Ergibt auch Widerspruchsfreiheitsbeweis für Zahlentheorie.* Zunächst aber intuit⟦ionistisches⟧ Heyt⟦ing'sches⟧ System und seine Eigenschaften.

eral very concise and rather formal.[6] It is only in the 1958 article in *Dialectica* where we find Gödel's – now more mature – philosophical views on constructivity fully laid out.

Gödel's full plan (*Arbeitsheft 9*, p. 2–3) includes the following themes:

1. Definition of the logical connectives.
2. Basic intuitionistic logic, non-constructive existential statements and their origin, namely the axioms $A \vee \sim A$ and $\sim\sim A \supset A$.
3. The exclusion of these principles in intuitionistic logic and the definition of negation in terms of absurdity. The axioms concerning negation can thus be left out.
4. The intuitionistic predicate calculus.
5. Derivability and non-derivability in intuitionistic calculus; in particular, the addition of either of the two principles $A \vee \sim A$ and $\sim\sim A \supset A$ gives classical logic.
6. "System **S**"[7] has the properties of an intuitionistic system.
7. The interpretation Σ as well as the construction of existential statements.
8. Proof of the soundness of the intuitionistic axioms with respect to system Σ.
9. Consistency of number theory:
    (a) Formalization of classical number theory;
    (b) Interpretation of the aforementioned system;
    (c) The negative translation for the system **S**.
10. Proof of consistency of $\neg(p)(p \vee \neg p)$.

[6] Kreisel (1987, 144) describes the early works as "concise and cavalier, apparently scoffing [...] at the antics of the rhetoric." The later works, quite the contrary, are more sensitive to philosophical issues in particular.

[7] System Σ seems to refer to the quantified system $\overline{\Sigma}$ of the Princeton lectures here. **S**, on the other hand, probably refers to the quantifier-free version denoted by Σ in the Princeton lectures. At one place Σ is written as a mirror image, resembling number 3.

11. Computability of all functions in **S**.

12. Proof that consistency is not provable in any smaller system.

For the most part, the lectures proceed according to Gödel's plan; however, items 11 and 12 are not covered in the lectures. Of particular interest is the issue of computability of higher-type functions, which Gödel still thought he could prove successfully at this point. I will discuss this below in the section on the system Σ.

The more detailed overview of the Princeton lectures is divided into four themes. I will start with Gödel's presentation of intuitionistic logic and its properties, especially in relation to classical logic. The second theme is Gödel's criticism of intuitionism and the sources of this criticism. The third part discusses Gödel's presentation of the functional system and the features not covered in the Yale lecture of the same year. Finally, I will consider the last theme of Gödel's lecture, namely the applications of the quantified functional system $\overline{\Sigma}$.

Sources

The lecture notes can be found in two spiral notebooks (040407, 040408) and a dozen loose pages (040409) filed together in Gödel's papers. Elsewhere (030077) we can find an envelope with "Beweis d. Gültigkeit d. int. Ax" written on it which contains the soundness proof for the functional interpretation.[8] The original transcripts were made from microfilm copies of the original notes, which were later controlled against the originals at the Princeton University Library.

The pages in the envelope have originally been numbered from 1 to 16. The page numbers have then been erased and replaced by new ones continuing the page numbering in the second spiral notebook. The envelope also contains a slip explaining how the loose pages should be ordered.

The lecture notes are mainly written in longhand English, with some shorthand additions in German. Gödel was used to writing his personal notes in Gabelsberger shorthand; e.g., the *Arbeitshefte* are almost entirely written in this script. We have transcribed and translated these additions, and where there might be a possibility of misunderstanding or a longer shorthand passage, added the German transcription as well.

Because the Gabelsberger system is language-specific and Gödel was lecturing in English, he had to write, for the most part, in longhand. However, even

[8] As far as I know, these missing pages were first discovered by Van Atten (2015).

his longhand writing retains many characteristics common to shorthand writing. These include the frequent use of abbreviations and the lack of punctuation or capital letters, and occasionally, a shorthand German word can be found in the middle of an English sentence. E.g., a passage on p. 66 of Gödel's notes reads:

> to be more exact if T_i should contain some var diff from $x_1 \dots x_n$ we form first terms T_i' by repl the *überflüssige* var by arb. const. and then these are correct Df. with T_i' inst of T_i For $n = 0$ we obtain the following special case $A(\overline{u_1}\_\_\overline{u_n}\,\underline{y_1} \dots \underline{y_r})$ is dem in $\overline{\Sigma}$ if and only if there are const $\alpha_1 \dots \alpha_n$ such that $A(\alpha_1 \dots \alpha_n\, y_1 \dots y_r)$ is dem in Σ

For someone accustomed to stenographic writing, the slow pace of longhand writing is surely frustrating, and this is probably one reason for Gödel's frequent use of abbreviations. To maintain readability, we have not indicated where an abbreviation has been completed or a comma or a full stop added. Only in cases where the interpretation is not completely straightforward have we indicated the completion of a word. For the most part, however, we felt that Gödel's (occasionally non-idiomatic) style of writing should be respected, and have avoided editing the text beyond those small completions and corrections, even where Gödel's grammar or choice of words could seem somewhat awkward.

Gödel's formal notation is not entirely uniform, and in this case, we have chosen to edit it more heavily. E.g., Gödel uses both brackets and dots to indicate order in formulas, so the formula $(A \to B) \to C$ might sometimes be written $A \to B\,.\to .\,C$. We have chosen to use the former notation which is easier to read. Gödel uses both $\cdot$ and . for conjunction, and sometimes he leaves the conjunction out altogether, so that $A \cdot B$ becomes AB. Here, too, we have opted for the symbol $\cdot$ which occurs most often in the original text. Gödel employs, as Heyting did in his 1930s works, two different sets of connectives for intuitionistic and classical logic: $\{\neg, \&, \mathrm{v}, \to, \leftrightarrows\}$ and $\{\sim, \cdot, \vee, \supset, \equiv\}$, respectively. (The quantifiers have no special symbols in intuitionistic logic.) These we have, of course, left untouched.

Gödel denotes arbitrary formulas by upper case $A, B, C \dots$ and occasionally with P, Q; however, he sometimes uses what is known as Sütterlin-Schrift instead of Latin letters. For formulas, where Gödel alternates between the two notations, we have chosen to use latin letters. However, Gödel consistently denotes sequences of variables by Sütterlin letters $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \dots$ and individual variables by lowercase Latin $x, y, z \ldots$. A printer would have typeset the Sütterlin letters in Fraktur, and this is the convention we have adopted in this case.

As mentioned, Gödel did not divide the notes into sections. The start of a new lecture is indicated only by Gödel's "last time …" summaries. These have been indicated in bold.

The intuitionistic viewpoint

Gödel starts with the question, "what is constructive reasoning in mathematics?" He first shows some examples of *non*-constructive reasoning, which is here defined as those ways of inference of classical mathematics which allow for non-constructive existence proofs, i.e., proofs of existential statements $(\exists x)\varphi(x)$ without a corresponding instance $\varphi(a)$. The task, then, is to formalize mathematics in a way that avoids these undesirable consequences. This means that we need to avoid the two principles known to lead to such non-constructive existence statements, namely the Principle of Excluded Middle $A \vee \sim A$ and the Double Negation Elimination $\sim\sim A \supset A$. Of course, there might be other axioms or rules that have the same effect, so we need to be careful in choosing the right axioms.

The principle by which the intuitionists have chosen their axioms, Gödel remarks, is that they are taken as primitive and based simply on evidence (p. 7). Gödel makes it clear that there is room for improvement, and indeed, giving a formal as opposed to an intuitive interpretation of the logical operations is his main objective in the second part of the lectures. For now, however, he simply introduces what is today known as the proof explanation or the BHK (Brouwer-Heyting-Kolmogorov) interpretation of the intuitionistic operators.

He then presents the rules of intuitionistic propositional logic, which he attributes to two sources: Gerhard Gentzen's "Untersuchungen über das logische Schliessen" (Gentzen 1934-35) and Arend Heyting's "Die formalen Regeln der intuitionistischen Logik" and "Die formalen Regeln der intuitionistischen Mathematik" (Heyting 1930a,b). Although Gödel's view of deduction was, as opposed to Gentzen's, axiomatic in nature, his axioms and rules resemble more closely Gentzen's simple system than Heyting's 1930 formalism, which has eleven axioms but rules only for Modus Ponens, propositional substitution, and conjunction introduction. The same holds for Gödel's formulation of intuitionistic predicate logic.

The interrelation between classical and intuitionistic logic is of particular