Mathematics Education in the Digital Era

Philippe R. Richard
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Mathematics Education in the Age of Artificial Intelligence

How Artificial Intelligence can Serve Mathematical Human Learning

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Mathematics Education in the Digital Era

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Foreword

AI *for* the Learning of Mathematics

It would have been good to have a precize and clear definition of Artificial Intelligence (AI) unanimously accepted. Unfortunately, it is not the case today as it was not the case formerly.[1] The common criterion, AI is a property of machines "exhibiting certain behaviours which strikes as intelligent",[2] reminds us that this is a judgement underpinned by a kind of human empathy. Looking closer, it appears that such a judgement assesses both the task which has been achieved by the machine and the way it has been achieved: a behaviour is striking because the task is acknowledged complex and/or the way in which it has been achieved looks *smart*.

Let's be back to the first *Smart mathematical machines*. In the middle of the seventeenth century, the Pascal's calculator performed four arithmetic operations, achieving tasks that only humans could perform until then. With no doubt, it looked striking as intelligent for some, but not to the eyes of Pascal himself who noticed that his invention lacks "willingness" which cannot be separated from "spirit".

The judgement of intelligence requires that the task considered is not merely achievable by a definite technique; a technique described by an algorithm fully specified and determinist, even if it is difficult to implement (and to learn). What I call a complex task is nothing but a *problem* which requires solving strategies and heuristics with the associated risk to fail. Then by AI, I mean that a machine has *knowledge* and *reasoning capabilities* to find *solutions to problems*. As mathematicians we are interested in what AI can do for mathematics, and as mathematics educators we are interested in what it can do for mathematics education. This book touches both, but it insists on the latter for which it opens original perspectives. Before coming to them, I will add a few words with an historical stance.

AI raised hopes in the 1970s with the mainstream research program on Intelligent Tutoring Systems (ITS) inspired by one-to-one tutoring model, that would

[1] e.g. Schank (1987); Wang (2008, 2019).

[2] Steven Van Vaerenbergh and Adrián Pérez-Suay, this book, Chap. 12 MS p. 2.

allow personalized learning for all and, as a consequence, possibly make the teacher unnecessary. The scientific program has evolved since then, but it keeps its genetic signature: individualization of learning and autonomy vis-a-vis human interventions. The methodological guideline was mimicking teachers' strategies and behaviours. This research in AI and Education really took off in the 1980s, and since then has been very active and productive. Projects improved with the progress of AI research and the advancement of cognitive sciences. Some ITS went to the mathematics classroom with evidence of success. Nevertheless, the dissemination of AI-based learning environments remained limited and still is, although with differences among the mathematical domains. Because of the prominence of their algorithmic dimension, teaching itself focusing on learning technics, Arithmetic and elementary Algebra received most of the attention. Geometry proved to be more challenging. In the early 1970s, building an artificial geometer seemed accessible but the dream fell short.[3] Modelling human reasoning even in a knowledge domain as well formalized as is mathematics encountered difficulties impossible to overcome without a drastic limitation of the problem-space.

The solutions to the problem of computational knowledge modelling, essentially rule based systems in the early period of ITS research, raised an epistemological problem that was pointed by Allan Newell who criticized the *de facto* AI confusion between knowledge and its representation. The ITS behaves as if learning a piece of knowledge were learning its representation—writing and grammar—and the associated skills—performing correctly procedures on and via representations. One must acknowledge that this reduction is more pragmatic than theoretical, several researchers saw in ITS research a theoretical stake. The classical Etienne Wenger survey of AI and ITS had the subtitle: "computational and cognitive approaches of communication of knowledge"[4]; the author chose not to define knowledge asserting that the objective of ITS research is to understand it.

To understand mathematical knowledge is an old and complex issue often considered as specific when compared with the same issue for other scientific knowledge domains; remember that the philosopher Karl Popper preferred to leave mathematics aside. In actuality, mathematics is the best example which backs Allan Newell's claim: "Knowledge remains forever abstract and can never be actually in hand".[5] Mathematics is forever *abstract*: mathematical knowledge has no referent in the material *world*: research in mathematics explores a world which is already mathematical.[6] Yet, smart mathematical machines bring to reality mathematical objects and offer to human perception physical manifestations of their properties. They open the way to experimental studies of mathematical objects and the exploration of their

[3] e.g. Balacheff & Boy de la Tour (2019).

[4] Wenger (1986).

[5] Newell (1982, p. 125).

[6] This does not mean that there are no tight relations between mathematic and reality, but these relations have a heuristic value for mathematics, as it is the case for the scientific disciplines which use tools that mathematics provides them for knowing—with their own concepts and their own methodologies—the world in which we live.

property. Paraphrasing Alan Newell, one could say that knowledge serves as a specification of what the representation systems should be able to display,[7] but these representations are *not* the knowledge it refers to. This is a classical semiotic tension that we need to understand and to overcome. I suggest that a solution is to bring back problems on stage as an epistemological solution.

Problems are the *raison d'être* of knowledge. The relations between problems and knowledge are dual and dialectic: problems are the source of knowledge and of its evolution, and conversely.[8] But, just as knowledge cannot be equated to a representation, problems cannot be identified to a statement, especially in the learning context because both knowledge and problematization are there under construction. It is the *role of situations*[9] to give birth to a problem by setting the scene for interactions between students and a material and social space for actions, and creating the circumstances to stimulate students' engagement. Such situations can be designed to make students experience the relevance and efficiency of a piece of mathematical knowledge by solving problems. How could AI contribute to designing learning situations? ITSs have not proved being appropriate to provide an answer to this question, but there is another line of research which is exactly addressing it.

In the beginning of the 1970s, Marvin Minsky coined the concept of *microworlds* when looking for a strategy to make breakthroughs in robotic. He extended it, in collaboration with Seymour Papert,[10] to nourish a proposal for the learning of mathematics inspired by a firm critic of the evolution of mathematics teaching at the time of the new math movement, and of educational models privileging instruction. They aimed at providing learners—not students—with a rich environment offering the possibility to explore a mathematical domain in a way not foreign to the concrete experience of the world, and evolving with the learners' knowledge. Logo, the seminal microworld they designed, had two faces: a programming language and the image of a turtle moving in *the space of the screen*.[11] A program driving the turtle could be turned into a tool to be used by a new program, that is a "procedure". We understand that a procedure reflects the knowledge constructed by the learner. This knowledge has multiple representations: at the symbolic level (a program) and at the phenomenal level (drawings on the screen). The feedback to learners is a consequence of the internal logical structure of the microworld and not from a tutoring decision. The learner has all the benefit of an open environment with epistemic characteristics favouring the evolution of his or her knowledge in a way coherent with his or her project. As a matter of fact, Logo was not an environment specifically thought for the learning of geometry, but it had interesting potential for this purpose. The gap between Logo's geometry and the geometry of the curricula hindered its full dissemination. But the screen as a field of experience[12] offering genuine mathematical discovery

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[7] Newell (1982, p. 100).

[8] Vergnaud (1990, 2009).

[9] Brousseau (1986, 1997, 1970).

[10] A tribute to Seymour Papert (Balacheff, 2017).

[11] There was also a version with a concrete turtle moving on the floor.

[12] Boero (1989, p. 65).

was there, the next step was to fill in the gap with school mathematics. It came in the 1980s with *Dynamic Geometry Environment* (DGE); *this* is the contemporaneous legacy of Logo. While Logo is primarily a programming language, a radical change occurred in 1985 when Jean-Marie Laborde designed the first DGE, Cabri-géomètre, introducing the revolution of direct access to manipulating and constructing objects on the screen as if they were real and not mere representations (Laborde, 1995). The behaviours of the objects are a consequence of their construction under the constraints of geometrical primitives. The perceived *visual invariants* on the screen, when messing up constructions, are *theorems*. Denying intelligence, one could claim that *a DGE does not know geometry but that it is its materialization* based on a computational model brought by analytic geometry. This is too quick a judgement.

In the seminal project, a microworld “is very schematic; it talks about a fairyland in which things are so simplified that almost every statement about them would be literally false if asserted about the real world.”[13] In effect, Minsky had the idea of microworld as a response to the complexity of the robotic problems he was working on; in short, he decided that if he could not find a comprehensive model for coupling the eye and the hand, then he had to look for solutions of these problems in “simplified worlds”. The membership of this approach to AI has been criticized by some, and sometimes it has been denied.[14] However, Logo is not a microworld in the original sense, it is a programming language which opens access to an unlimited universe starting from a few primitive actions which semantic is moving around in a flat land. If Logo is a “fairyland”, a land conducive of mind storms, this land has been designed so that the behaviour of the drawings the turtle “leaves” on the screen offers a terrain on which geometry could grow as could other mathematical or algorithmic concepts. The invention of DGEs goes a step further. What is special with them, is that it is the land where geometrical figures thrive: a drawing is more than what you see, it is what you get when manipulating it, that is all drawings which satisfy the constraints imposed by its construction. DGEs materialize a world whose inhabitants are geometrical figures, not only their shadows (which are drawings), whose laws conform Euclidean Geometry.

The DGE *materialization* of Geometry is more than *visualisation*. Borrowing the words of Allan Newell, it is a symbol system which encodes a body of knowledge. It is a semiotic instrument to grasp the objects of Geometry and to discover their properties in *a space ruled by a rational principle*: a visual invariant when manipulating directly a free object on the screen is a graphical representation of a theorem; this is a perfect translation of the *theorems in action*, as Gérard Vergnaud conceptualized them. Moreover, learners can make the DGE evolve in parallel to the evolution of their own knowledge by creating graphical procedures with specified inputs and outputs and giving them a name. A DGE is a smart mathematical machine opening a field of mathematical experiences, but… with no didactical agenda.

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[13] Minsky & Papert (1970, p. 36) section 5 of the report: “Why we are studying knowledge and learning”.

[14] Dreyfus & Haugel (1981).

On the one hand, *AI machines*, the intelligent tutoring systems, have proved promising results for the acquisition of technical skills but they are limited when coming to problems, what limits their impact on developing an understanding of mathematics. On the other hand, *Smart mathematical machines* are efficient tools to design problem situations thanks to the possibility they offer to create fields of mathematical experience, but they have no didactical functions to direct students towards the intended teaching objective. Moreover, one may emphasize that these machines complexify the work of teachers. To get the best of both, the project QED-Tutrix ambitions to design a platform which provides students with a space to explore a problem and an artificial supervisor to feedback on the proof under construction to ultimately validate a solution. There is another route to achieve the same, here illustrated by the case of GeoGebra, which consists in augmenting a smart mathematical machine with reasoning competences and the capacity to provide didactical feedback. This is the project of designing an environment augmenting a DGE with automated reasoning tools (ART).

The combination of smart machines and AI systems is more than an addition, the properties of the whole emerge from the interactions among the parts. Interaction is also a keyword of human-centred computer science. One may remember this claim of Joseph Conrad: "One writes only half the book; the other half is with the reader." This applies nicely to AI *for* the learning of mathematics. The interactions between the humans and the systems, as well as among the humans, have to be considered from the beginning of the design of the learning environment. Hence, its educational relevance and efficiency is not a property of the technology itself but of the orchestration of its use and the dynamic of the multiple interactions and feedback loops.

Eventually, reading a book on Artificial Intelligence and Education at the time of a major pandemic cannot be done without a thought for problems which challenge education, that is teaching and learning. The obligation to shut down classes—or even schools—has put on the fore all expectations decision-makers and the society have on the role that Technology Enhanced Learning (TEL) environments can play. A large panel of technology is deployed since the beginning of 2020. It consists of video communication channels, digital teaching platforms and MOOCs, mail and web-based resources and software specific to the content to be taught. This unprepared intense use of TEL has a mitigated success, but this is no surprise given the sudden and urgent radical change in practice for both students and teachers, and families as well. I don't doubt that this is the start of a new era which will be marked by a refreshed vision of TEL, maybe a less revolutionary than formerly claimed but a more pragmatic and efficient one. A lesson we can already keep for the future is the need students have for social interactions with other students, including sharing the learning experience, its challenges and successes. Another lesson, not the least, is that *teaching is a profession* that parents, technology and internet resources cannot replace. As a profession, it requires specific knowledge, skills and attitudes. It needs instruments and tools as well. Each new knowledge technology since the invention of writing has inspired the design of new teaching tools and learning material. The same is true in the fully digital era in which we live nowadays, where technology not only improves representing, memorizing and communicating knowledge, but allows manipulating and treating it in a way which is not mechanical but *intelligent*. Here is the challenge for research on AI and mathematics education of the future. The book arranged by Philippe R. Richard, M. Pilar Vélez, Steven Van Vaerenbergh—and the team of editors they brought together for this project—will be the source of questions and inspirations to contribute to take up this challenge.

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General Introduction

The relationship between humans and machines is very old. As early as the Neolithic period, machines emerged as sophisticated artefacts that could transform one movement into another, for example, to enable a small group of people to move heavy loads and to set them up for a variety of uses. In English, the word machine comes from the Middle French "machine" which is itself a borrowing from the classical Latin "machina", meaning ingenious invention, device, trick. As technology has developed, machines have become increasingly sophisticated. They have become stronger, faster, more precise, more efficient and more tireless tools, capable of performing tasks that man could no longer perform without them. They were equipped with a memory and a program, usually computer-controlled, designed to perform several specific tasks autonomously and in a coordinated manner. Then, their field of application opened up infinitely, being able to perform complex tasks that were once thought to be reserved for human intelligence, even surpassing it with a capacity for calculation, prediction and learning that forces us to completely rethink the nature and effects of the relationship we have with machines. If there has already been man–machine for better or for worse, there is now artificial intelligence which extends the panorama.

In offering a book that straddles the line between artificial intelligence and mathematics education, another linguistic issue arises. In Latin, Germanic or Slavic languages, we have « didactique des mathématiques » and « informatique » as substantives of reference, but in standard English, *didactics* and *informatics* are adjectives. And if we speak, even partially, of « preuves » or « démonstration » in geometry, we know that English prefers the action-oriented gerund *proving* rather than the noun, which primarily evokes the idealized concept. What kind of artificial intelligence will we be able to talk about? We have no choice but to integrate various perspectives and warn the reader when we take some linguistic liberties. Before the term « informatique » came into use, the syntagm « cerveau électronique » was flirted with. In French, the *electronic brain* is still visible in books and media from the 1950s to 1960s: it only reappeared a little later, in the 1990s, in connection with the idea of computer intelligence. While the comparison between human and machine continues, it is clear that it is always posed in terms of the nature of things, their characterization and qualification, from the conceptual aspects to the processes that these engage.

Whether it is natural or artificial, whether it is addressed to a brain or a machine, the notion of intelligence is polysemous. We have identified two trends. The first is based on the principle that if we know what intelligence is in humans, for example, because we have done our homework in philosophy or neuropsychology, then we can characterize the intelligence of the machine by comparison with human intelligence. This is a well-known position that is based on an induction hypothesis, just as mathematicians do. The second is based on the ability of a living being to adapt to a new situation, to understand and solve certain difficulties, to make sense of the things around it, to act with discernment. It is intelligence in an ecosystem, or more precisely within an interaction that constitutes an evolving system in itself. Intelligence is then seen on the level of knowledge and on that of mutual or reciprocal adaptation, if it is a question of a human–machine interaction which aims at the same end, which can be the fact or proceed by imitation, ruptures, accommodation, decision, calculation, etc.

These two trends complement each other in the orientation of our book. Its spirit is radically different from today's 'machinocentric' viewpoint, which is mainly formulated by computer scientists or policy makers, the opposite of the anthropocentrism of yesteryear where humans were compared to machine systems. We often look at artificial intelligence (AI) in terms of the effects of computer science, but we do not question the causes that govern the choices made by decision-makers: humans are always users of AI proposed by computer scientists and they seem to be in thrall to the uses decided by third parties. However, our starting point is the teaching and learning of mathematics and it concerns the interaction between the human and the machine in both directions, as well as the emergence of the human–machine system to characterize instrumented mathematical work. This means that the machine we are interested in is the one that is a partner in the construction of knowledge, and for this to be the case, many questions need to be asked about the modelling of knowledge and the learning of the machine that aims to serve the human. In other words, if we were to undertake an ethical discourse on AI, we would have to start by asking questions about modelling and learning choices, and continue the questioning when the human is interacting with his or her partner.

At the same time, how can we not celebrate the revolution of machine intelligence to make the world a better place? When it comes to using machine learning algorithms in a traditional way or through deep learning, one surely thinks of automatic driving (autonomous vehicles), pattern recognition or computer pattern identification (speech analysis, image search, etc.). One also thinks of decision support in commerce or banking (intelligent assistants for routine work), measuring and predicting of ecological disasters (prevention and reduction of damage risks), content search engines (Google, WolframAlpha, etc.) or the possibility of asking more or less complex questions on one's mobile phone (Siri or equivalent). However, if a neural network simulates human functioning, it is by analogy within a mathematical model executed by a machine, and not a kind of laboratory reproduction of a reality controlled by data. When we take a modern definition of AI, such as that found in the Montreal declaration for a responsible development of artificial intelligence (2018):

> Artificial intelligence (AI) refers to the series of techniques which allow a machine to simulate human learning, namely to learn, predict, make decisions and perceive its surroundings. In the case of a computing system, artificial intelligence is applied to digital data.

It is clear that the human is a source of inspiration but does not seem to be a partner at all. Even if this is an entry that deserves to be highlighted, we must first ask ourselves: artificial intelligence by whom, for whom? The 'for whom' of course refers to possible uses, possibly in an instrumented perspective, while the 'by whom' cannot be limited to the world of industry and commerce. Despite their stratospheric means, which, like a space program, can trickle down useful techniques into other fields, their objectives are diametrically opposed to the public good and the development of the mind. Mathematics education is first and foremost a matter of generosity in a collegial dynamic. And it is up to us, authors and multidisciplinary colleagues, wearing different hats and accustomed to reflecting on the relationships between teacher/learner, trainer/student, administrator/colleague, politician/researcher and computer scientist/user, to put forward the principles of *mathematical human learning* and the vision of the mathematics education profession.

It must be said that the historical link between AI and didactics of mathematics is well established. One example is the book *Didactique et intelligence artificielle*, published almost 25 years ago at *Éditions de la pensée sauvage* in the series *Recherches en didactique des mathématiques*, which stated that advances in AI had paved the way for a vigorous stream of research into the development of computer environments for human learning and technology-enhanced learning. But after an initial period of enthusiasm, when we imagined that we would be able to do incredible things very quickly, we went through a somewhat more sombre period of disillusionment when we realized that we had somewhat underestimated the difficulties. Today, in a paradoxical turn of events, it seems that AI is joining didactics with its non-routine problem-solving approaches, which involve learning, modelling and prediction phases that evoke both mathematical work and the design of solutions by specialists. If AI has a role to play in fostering academic success and providing support for learning outcomes, any collaboration must begin with the consideration of didactics in AI models to understand the needs of the student and the teacher, and to be at their service. It is up to the system to adapt to the human and not the other way around.

From this follows a guiding principle: the creation of any digital artefact that respects didactic needs, while remaining epistemologically and cognitively sound, requires the establishment of an ongoing dialogue in human–machine interaction that values mathematical literacy. To be truly intelligent, these systems must first remain flexible enough to adapt to the natural evolution of mathematical work in the classroom, and then leave the parameterization to the teacher or trainer. Furthermore, if we know enough about the resources, goals and orientations of a class, we can come to understand, explain and model actions and decisions that seem unusual or abnormal to the external eye. Intelligence is then revealed in an iterative process of convergence between a priori and observed effects that is progressively refined in use, hopefully releasing the bright side of uncertainty. In mathematics education, it seems then that AI cannot be formulated solely in terms of the machine, but above all in terms of a finalized activity with humans. In a way, we are close to idoneism in the sense of Ferdinand Gonseth, in relation to a double concern of truth and reality.

Our book weaves together "machine thinking" and "human thinking" by using artificial intelligence in the continuation of the scientific work and achievements of the book's authors. The first part looks more at the machine and the third at the human, as if they were the two vectors of the same interaction. As for the second part, in a good intermediate section, it looks more at interaction as such, even if at times a vectorial view is imposed.

More specifically, the first part introduces the reader to "machine thinking" through different novel AI systems, always with a focus on the human use of them. Scientific calculators, Computer Algebra Systems (CAS) and Dynamic Geometry Systems (DGS) are widely extended and used in mathematics teaching and learning, most of the time, as tools to perform arithmetic or algebraic calculations or to visualize mathematical concepts or properties. Nevertheless, their potential extension to become AI systems for "smart learning" has been developed by researchers with different purposes but oriented to Mathematics Education. This part presents some of these systems and their integration in the classroom, mainly in cooperation or symbiosis with CAS or DGS. For instance, DGS provided with reasoning capabilities to decide if a theorem is true or false, derive conclusions or discover properties starting from a geometric configuration in a mathematically rigorous way; a tutorial system to guide students in their solving problems performance; a system where AI allows discovering the hidden mathematics in a monument and the relation with its architectural style; or a system of knowledge organization for the Mathematics curriculum.

The second part deals with new ways of building mathematical knowledge through the interaction between the learner and different AI systems. The didactic implications and the development of digital competences when an AI system intervenes are approached from different points of view: the design of a system based on the experience of teachers and learners, the impact on the mathematical activity and on the learning process or the learner–machine interaction in the development of activities and problem-solving. It also analyses the knowledge and skills required for this transition towards the empowerment of "human thinking" in relation to "machine thinking".

The third part focuses on the human point of view when integrating technology into learning spaces or generating digital *milieus* inspired by classic manipulative resources. The reader can find some several years of studies such as a tertiary level experience in technology integration through the use of CAS, teacher training using expert systems able to simulate classroom situations and mentor–teacher interaction, or classroom use of DGS from paper/pencil constructions to tasks especially designed to take advantage of DGS potentialities. Furthermore, Virtual Reality (VR) and educational approach to learning that uses Science, Technology, Engineering, the Arts and Mathematics (STEAM) are present: an empirical study on the potentialities for visualization and manipulation using a VR system in the classroom, and a STEAM project carried out in cooperation with mathematics and technology teachers who use 3D printing tools or programming environments.

The theme of the book started from an incessant questioning of the authors of the natural links between computer science, mathematics and their didactics at the age of artificial intelligence. But the real trigger for the book came during several informal discussions during the Applications of Computer Algebra conference, held in Montreal in July 2019. We would like to thank Michel Beaudin for providing us with a room and resources at the École de technologie supérieure (ÉTS Montréal) for an off-program meeting where ideas, partnerships and first drafts of the project were initiated. In February 2020 on the other side of the Atlantic Ocean, just before the pandemic restrictions, the International Centre for Mathematical Encounters (CIEM) of the Universidad de Cantabria, in Castro Urdiales, hosted us so that we could crystallize the editorial project with consensual and effective guidelines. We would like to thank Springer Nature and, in particular, the Mathematics Education in the Digital Era series editors, Dragana Martinovic and Viktor Freiman, for their unstinting support from the beginning.

Although the conception and planning of our work was carried out in face-to-face meetings, the editorial process, developed in parallel with the COVID-19 pandemic, has been online. Nevertheless, the digital coldness has been compensated by warm and friendly (online) constructive discussions between the editors, the coordinators of the sections and the emeritus contributors. Our thanks and acknowledgements go to the coordinators Pedro Quaresma, Jana Trgalová and Jean-baptiste Lagrange, who have ensured the quality and internal consistency of each section, as well as, to the emeritus contributors, Nicolas Balacheff and Tomás Recio, for sharing their extensive knowledge, experience and expertise. We are also certainly grateful that all of them kept up with us editors throughout the entire project despite our editorial extravagances, such as insisting to keep the communication between editors in Spanish, to conduct the coordination meetings in French and to organize all other correspondence in English.

Each chapter has undergone a review process by at least two experts, selected from among the authors and external specialists depending on the topic, in addition to a review by the section coordinators. Thanks also to the reviewers for their essential, generous and hidden work. To complement the spirit of collaboration, the introductions to each section were reread by the authors. In short, we would like to emphasize that this book is the result of a collective effort carried out by all these authors and collaborators in a dynamic where the research of some, the experience of others and the reflection of all have led us to the realization of a lasting work. We are very grateful.

Finally, we wish the reader, who has this book in front of her or his eyes, a pleasant and fruitful reading.

Philippe R. Richard
M. Pilar Vélez
Steven Van Vaerenbergh

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Creation of AI Milieus to Work on Mathematics

Introduction to Section 1 by the Coordinator Pedro Quaresma

In the past decade, artificial intelligence (AI) technologies have accomplished several breakthroughs in solving complex tasks, most notably in computer vision and the development of autonomous agents. These achievements are driven mainly by advances in machine learning and deep learning, and the availability of large computing power and extensive databases.

Currently, modern AI techniques are starting to find their way into several aspects of mathematical work and mathematics education. In interactive learning environments, for instance, AI can be used to extract mathematical knowledge from the real world to generate new methods of content creation. On a more abstract level, AI is a promising technology for automated learner modelling, motivated by results from current research in AI for abstract mathematical reasoning. These technologies are, furthermore, expected to contribute to more intelligent tutoring systems, as employed in online learning environments, which, at present, already use data mining techniques to extract quantifiable insights from the learner's actions.

One of the first approaches to AI, in the 1950s, was the construction of automated theorem provers for geometry, combining the axiomatic theories and forward chaining deduction, with AI techniques. This combination was crucial to manage the complexity of the synthetic proofs, using the geometric constructions as models. In this way it was expected to be able to produce readable proofs, a goal with strong impact in education.

In the first three chapters the question of formal deduction in geometry is addressed. First, Quaresma gives us an account of the the history of automated deduction, from the early development of automated theorem provers for geometry, one of the early applications of AI, and from the emergence of the dynamic geometry systems, to the current status where different application systems combine dynamic geometry and automated deduction to create mathematical milieus where formal deduction tools help in the pursue of mathematical rigour. In the next two chapters, two such systems are described. Kovács et al. give us an account of the, recently included in *GeoGebra*, tools for the mathematically rigorous proof and discovery in geometry, and reflect on the potential educational impact of these new features. Font et al. describe *QED-Tutrix* intelligent tutor, a computational platform that includes an automated proof generator, allowing the students to solve high school geometry problems, tutored, with mathematical rigour, in any possible situation of their resolutions, rooted in a sound educational approach.

The next chapter continues this path, a clever combination of automated deductive and AI techniques at the service of education. Roanes-Lozano and Martínez-Zarzuelo show us a decision-making tool, inspired by a rule-based expert system verification approach, that, if an "official curriculum" exists, then it automatically checks the completeness and soundness of a given "development of the official curriculum". It will allow to produce better (error-free), instances of the curriculum, e.g. textbooks, project-based learning proposals.

The next two chapters deviate from this deductive tools path to show us more generic AI techniques, techniques that can be put at service in the construction of milieus to work on mathematics. First, Van Vaerenbergh and Pérez-Suay provide us with a bird's-eye view of AI systems for teaching and learning mathematics, proposing a classification that can help to choose the different "intelligent" components needed for a milieu construction.

Last, but not in any way, the least, the chapter by Martnez-Sevilla and Alonso Burgos describe a milieu where many AI techniques are in use to help its users to achieve a better understanding of mathematics behind our artistic and monumental heritage.

Evolution of Automated Deduction and Dynamic Constructions in Geometry

Pedro Quaresma

1 Introduction

> Logic appears in a *sacred* and in a *profane* form. The sacred form is dominant in proof theory, the profane form in model theory
>
> *D. van Dalen*, Logic and Structure (van Dalen, 1980)

Sacred Form

It can be said that the sacred form began circa 300 BC with the writing of Euclid's Elements. The Elements can be seen as a seminal work, establishing the basis for proof theory, with a collection of definitions, postulates, propositions (theorems and constructions) and mathematical proofs of the propositions. For centuries, it was included in the curriculum of the majority of the universities. For example, in 1692 Tirso de Molina, Superior General of the Jesuit Order wrote a letter with very specific orders to improve the teaching of Mathematics at the Portuguese province (i.e. Universities of Coimbra and Évora). One of the suggestions was the reproduction of the figures in the Elements in such a way that all the students could see those figures and discuss about the geometric properties behind those figures. The combination of that letter with the Portuguese tradition of tilling ("azulejos") gave rise to a collection of tiles with faithful representation of many of the figures in the *Elementa Geometriæ*

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Fig. 1 *Azulejos* with figures from *Elementa Geometriæ* by André Tacquet

by the Jesuit priest André Tacquet (1612–1660) (see Fig. 1) (Gabriel Silva, 2017; Simões, 2007).[1]

David Hilbert in his book *Grundlagen der Geometrie* (1899) established the foundation for a modern axiomatic treatment of Euclidean geometry (Hilbert, 1977). Later, Alfred Tarski built a decision method for elementary algebra and geometry (Tarski, 1951) allowing constructive and even automated approaches for geometry (Beeson, 2015; Quaife, 1989) and, more recently, (1995) Jan von Plato proposed another constructive approach to geometry (von Plato, 1995).

From a computational perspective, the history of geometry automated theorem provers (GATP) began with the early computers and the birth of Artificial Intelligence, in the 1960s. The different sets of axioms of Euclidean Geometry attracted researchers to an attempt to implement synthetic methods, such as the approaches by Gelernter (1995, 1960), Nevis (1975), Elcock, Greeno et al. (1977), Coelho and Pereira (1979, 1986), Chou et al. (1993, 1995). The difficulties found with the synthetic methods, where the need to find a suitable rule to be applied lead to a combinatorial explosion regarding all the possible choices. This resulted in the exploration of other approaches, algebraic, semi-synthetic and logical approaches.

The algebraic style approach is characterized by the translation of geometric problems to algebraic problems, and subsequent development of the proof by the application of algebraic manipulations. The characteristic set method, also known as Wu's method (Chou, 1985; Wu, 1984), the elimination method (Wang, 1995), the Gröbner bases method (Kapur, 1986a, b) and the Clifford algebra approach (Li, 2000) are examples of practical methods of this type. The algebraic approach led to efficient implementations, but, given that all the proofs are developed by algebraic means, the geometric meaning is lost, i.e. apart from a yes/no answer, it is not possible to have a correspondent geometric proof where the axioms of geometry are used. This led to the development of methods capable of, at least partially, combine the geometric readability of synthetics methods with the efficiency of algebraic methods.

[1] Unfortunately most of the tiles were lost after the expelling of the Jesuits from Portugal by the Marquis of Pombal in 1759, the subsequent reform of the University of Coimbra and the construction of new buildings on the expense of the old ones.

The semi-synthetic methods use a set of specific *geometric quantities*, e.g. the *ratio of parallel directed segments* and *signed area*, to build an axiom system where the geometric relations and properties can be represented and the proofs developed using a set of geometric lemmas and simple algebraic manipulations. Examples of such methods are the area method (Chou et al., 1996a; Janičić et al., 2012) and the full-angle method (Chou et al., 1996b). These methods combine the readability of synthetic methods and the efficiency of algebraic methods, being able to prove many geometric theorems, efficiently and with geometric, readable, proofs.

More recently (2000–till present), new synthetic approaches are being proposed; the geometric deductive database method combines the full-angle axiom system with the techniques of deductive databases to develop an efficient GATP capable to prove a large set of geometric problems (Chou et al., 2000). Also tutorial systems like the *QED-Tutrix* (Gagnon et al., 2017; Tessier-Baillargeon et al., 2017) are proposed to address the problem in a more contained form, i.e. instead of trying to implement a generic GATP, the goal is to have an efficient and capable of readable geometric proofs GATP, to specific areas of geometry.

Also to be considered are the logical approaches, like the quantifier elimination method of Tarski (Collins, 1975; Tarski, 1951), or the use of axiom systems for geometry (e.g. Tarski, Quaife (1989)) and then using generic automated theorem provers (ATP) to develop the proofs. Many efficient and capable of proving many geometric conjectures ATP are available, but, like in the algebraic approach, the proof has no correspondence with any form of geometric reasoning. From the view point of a geometer, it is difficult to follow (geometrically) the formal proofs produced by the ATP.

Profane Form

The *Profane form* came with programs that allow to build and explore geometric figures. The 1988 Turing prize was awarded to Ivan Sutherland for his pioneering work in the area of computer graphics. The program Sketchpad changed the way people interacted with computers, from non-graphical to graphical (Sutherland, 1963, 2003). While the original aim was to make computers more accessible, introducing graphical manipulations, while retaining the powers of abstraction that are critical to programmers, the direct manipulation interfaces have since succeeded by reducing the levels of abstraction exposed to the user.

The program Sketchpad can be considered as the point of origin for today's computer-aided graphic design programs (CAD). Not detracting from CAD programs, they are of little interest for the geometry practitioner, they are very high-precision tools to draw figures, e.g. for architects, drawing building plans, but they miss the step from drawings (static object) to figures (geometric construction), i.e. a set of objects and geometric relations between then (dynamic object). Meanwhile, dynamic geometry systems (DGS) allow building geometric constructions from free objects and elementary constructions. It became possible to manipulate the free objects (objects universally quantified), preserving the geometric properties of the construction.

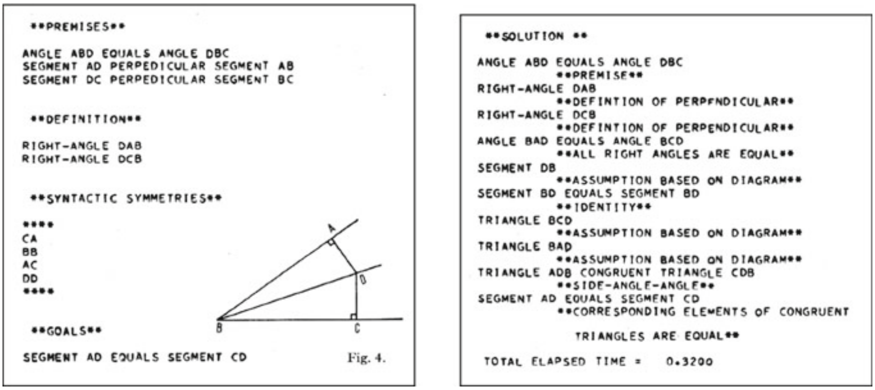

Fig. 2 Gelernter—Angle bisection

The first software packages that can be classified as dynamic geometry systems were Geometer's Sketchpad (Jackiw, 2001), which appeared first in 1989, and Cabri Géomètre (Laborde & Strässer, 1990), dating back to 1988, and they started another revolution: computers could be used in school for teaching geometry. Since then DGS become mature tools used by millions of users all over the world.

Dynamic geometry systems gave us the profane side of proofs in geometry. For example Fig. 2 (if done with a modern DGS) would have the points A, B and C as free points and point D as a constructed point (intersection of lines), moving the free points we can conjecture that the segments AD and CD are equal in length, i.e. we are exploring "all" possible configurations for a given geometric construction in the Cartesian model. Although those manipulations are not formal proofs because only a finite set of positions are considered and visualization can be misleading, they provide a first clue to the truthfulness of a given geometric conjecture.

The DGS and GATP are in a collision course and that is a good thing. From the development of GATP and DGS as completely separated tools, to the implementation of some GATP method in a DGS (e.g. *Cinderella*) or graphical components into a GATP (e.g. *GCLC* and *JGEX*) to the integration of GATP and DGS (e.g. *GeoGebra*). The fully integration of automated deduction components in other software is becoming a reality and it is expected that in a near future it will be possible to have those components broadly available.

Overview of the chapter In Sect. 2, the evolution of automated deduction in geometry is presented and in Sect. 3 the integration of GATP and DGS is discussed. In Sect. 4 other lines of research are presented and in Sect. 5 conclusions are drawn.

2 Automated Deduction in Geometry

For the last five decades, automated deduction in geometry, the sacred form, has been an important field in the area of automated reasoning. Various methods and techniques have been studied and developed for automatically proving and discovering geometric theorems (Chou, 1987; Chou & Gao, 2001; Chou et al., 1994).

2.1 Synthetic Methods

Adapting general-purpose reasoning approaches developed in the field of artificial intelligence (in the 60s of the twentieth century), synthetic methods, such as the approaches by Gelernter (1995, 1960, Nevis (1975), Elcock (1977), Greeno et al. (1979), Coelho and Pereira (1979, 1986), Zhang et al. (1995), were dedicated to automating traditional proving processes (Chou & Gao, 2001). Making use of axiomatic systems close to the ones used in secondary schools these systems tried to provide readable (by students and teachers) proofs. See Fig. 2 for an example of such proofs, from the GATP by Gelernter.

In many of these first attempts, the diagrams where used as a model (Coelho & Pereira, 1986):

- the diagram as a filter (acting as a counter-example);
- the diagram as a guide (acting as an example, suggesting eventual conclusions).

As a filter the diagram permits to test the non-provability of a candidate sub-goal, pruning the proof tree.

As a guide the diagram can be used as a positive indication. Quoting from Coelho et al. (1986) Coelho and Pereira (1986) (see Fig. 3):

> We want to prove two equal segments $UV = XY$, by congruent triangles. Suppose triangle XYZ exists, and our purpose is to find a triangle UVW on UV to compare to triangle XYZ. We need to search for existing or generated triangles on UV. The first thing is to find a convenient third point W, which must be different from U and V. The possible coordinates of the sought point W are computed from the coordinates of X, Y, Z, U and V, and a check is made in the diagram to see if a point with such coordinates exists. The diagram is used in a positive way for computing the possible coordinates for W.

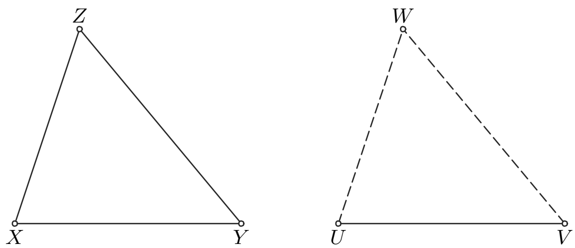

Fig. 3 Coelho et al. (1986)—Diagram as a guide

The possibility of having geometric proofs, with natural language and (eventually) visual renderings, is a key aspect of this approach. Unfortunately, the combinatorial explosion while applying postulates implied the use of suitable heuristics that narrow the scope of the GATP and prevent the development of a general-purpose efficient GATP.

New synthetic approaches are being proposed. The geometric deductive database method combines the full-angle axiom system with the techniques of deductive databases, to develop an efficient GATP capable of proving a large set of geometric problems (Chou et al., 2000). A coherent logic[2] based GATP, *ArgoCLP*, is being developed which can be used to generate both readable and formal (machine verifiable) proofs in various theories, primarily geometry. The possibility of, using a top-down approach (from the conjecture to the conclusion), producing natural language proofs is a positive point, but, efficiency considerations are still a major concern (Stojanović et al., 2011).

2.2 Algebraic Methods

A different approach is given by the algebraic style methods, given by the translation of geometric problems to algebraic problems and the subsequent development of the proof by the application of algebraic manipulations. The characteristic set method, also known as Wu's method (Chou, 1985; Wu, 1984), the elimination method (Wang, 1995), the Gröbner bases method (Kapur, 1986a, b) and the Clifford algebra approach (Li, 2000) are examples of practical methods based on the algebraic approach.

Let us consider, for example, the Euler's Line theorem.

Theorem 1 (Euler's Line Theorem) *In any given triangle, the orthocentre, the centroid and the centre of the circumscribed circle are collinear (Fig. 4).*

Transcribing it to algebraic form we get (GATP: *JGEX*, Wu's method):

```
The Algebraic Form:
A: (0,0)  B: (0,x4)  C: (x5,x6)  D: (x7,x8)  E: (0,x10)
F: (x11,x12)  G: (x13,x14)  H: (x15,x16)  I: (0,x18)
J: (x19,x20)  K: (x21,x22)  L: (x23,x24)  M: (x25,x26)

The Equational Hypotheses:
1: D : midpoint(BC)  2x8 - x6 - x4 = 0   2x7 - x5 = 0
2: E : midpoint(BA)  2x10 - x4 = 0
3: F : midpoint(AC)  2x12 - x6 = 0   2x11 - x5 = 0
⋮
11: LF ⊥ CA  x6x24 + x5x23 - x6x12 - x5x11 = 0
12: M : on line EL  x23x26 + (-x24 + x10)x25 - x10x23 = 0
```

[2] Coherent logic is a fragment of (finitary) first-order logic which allows only the connectives and quantifiers ∧ (and), ∨ (or), ⊤ (true), ⊥ (false), ∃ (existential quantifier).