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Raffaele Cerulli · Mauro Dell'Amico · Francesca Guerriero · Dario Pacciarelli · Antonio Sforza *Editors*

Optimization and Decision Science

ODS, Virtual Conference, November 19, 2020

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Preface

ODS2020, International Conference on "Optimization and Decision Science," organized by AIRO, the Italian Operations Research Society, was held online on November 19, 2020, due to the pandemic event, with Springer's support. In spite of the difficult situation, 60 talks were presented and almost 200 people participated in the conference.

In this volume, the reader will find the research papers submitted and accepted for publication after a peer-review process.

These 20 papers offer new and original contributions from both methodological and applied perspective, using models based on continuous and discrete optimization, graph theory, and network optimization, solved by heuristics, metaheuristics, and exact methods. A wide diversity of real-world applications is addressed. For this reason, although the book is aimed primarily at researchers and PhD students of the operations research community, the interdisciplinary content makes it interesting for scholars and researchers from other disciplines, including artificial intelligence, computer sciences, economics, mathematics, and engineering, as well as for practitioners facing complex decision-making problems in the aforementioned areas.

The 20 accepted papers are organized into 4 topical parts, listed in alphabetical order: Game Theory and Optimization; Healthcare; Scheduling and Planning; and Transportation and Logistics. In each part, the papers are listed alphabetically by the last name of the first author.

In the first part, Game Theory and Optimization, the reader will find the following chapters:

Integer Programming Reformulations in Interval Linear Programming, by Garajová et al. (2021) As known, interval linear programming provides a mathematical model for optimization problems affected by uncertainty, in which the uncertain data can be independently perturbed within the given lower and upper bounds. The authors explore the possibility of applying the existing integer programming techniques in tackling some of the problems arising in these operations.

On the Optimal Generalization Error for Weighted Least Squares Under Variable Individual Supervision Times, by Gnecco (2021) In this chapter, the trade-off between the number of labeled examples in linear regression and their precision of supervision is optimized, for the case where distinct examples can be associated with one among $M > 2$ different supervision times, and weighted least squares is used for learning.

On Braess' Paradox and Average Quality of Service in Transportation Network Cooperative Games, by Passacantando et al. (2021) In the theory of congestion games, the Braess' paradox shows that adding one resource to a network may sometimes worsen, rather than improve, the overall network performance. Here the paradox is investigated under a cooperative game theoretic setting, in contrast to the non-cooperative one typically adopted in the literature.

Optimal Improvement of Communication Network Congestion via Nonlinear Programming with Generalized Nash Equilibrium Constraints, by Passacantando and Raciti (2021) The chapter considers a popular model of congestion control in communication networks where each player/user sends their flow on a path of the network, with a cost function consisting of pricing and utility terms. The authors assume that the network system manager can invest a given amount of resource to improve the network by enhancing the capacity of a subset of links. The decision problem is modeled as a nonlinear knapsack problem with generalized Nash equilibrium constraints, giving some preliminary numerical results.

A Note on Network Games with Strategic Complements and the Katz–Bonacich Centrality Measure, by Raciti and Passacantando (2021) The paper investigates a class of network games by using the variational inequality approach. In the case where the Nash equilibrium of the game has some boundary components, they derive a formula which connects the equilibrium to the Katz-Bonacich centrality measure, thus generalizing the classical result for the interior solution case.

In the second part, Healthcare, the reader will find the following chapters:

An Optimization Model for Managing Reagents and Swab Testing During the COVID-19 Pandemic, by Colajanni et al. (2021) The authors affirm that COVID-19 pathology is characterized also by asymptomatic patients who could considerably spread the virus without being aware of it. Therefore, swab tests have to be used to diagnose positive cases. The paper presents a multi-period resource allocation model with the objective of maximizing the quantity of all analyzed swabs while minimizing the time required to obtain the swabs result, the costs due to increase the number of swabs analyzed per unit time, and the cost to transfer swabs between laboratories.

Modelling and Solving Patient Admission and Hospital Stay Problems, by Guido et al. (2021) Patient admission and patient-to-room assignment problems in a well-defined planning horizon are considered. The proposed optimization model is embedded in a matheuristic, tested by a set of benchmark instances characterized by real-world features. The experimental results show that the solution approach is effective and allows to obtain optimal/sub-optimal solutions in short computational times.

A Two-Stage Variational Inequality for Medical Supply in Emergency Management, by Scrimali and Fargetta (2021) The chapter proposes a stochastic approach to optimizing competition between healthcare institutions for medical supplies in emergency situations, caused by natural disasters. A scenario-based stochastic programming model in a generalized Nash equilibrium framework is proposed, providing the optimal amounts of medical supplies from warehouses to hospitals, in order to minimize both the purchasing cost and the transportation costs. A two-stage stochastic programming model is proposed, taking into account the unmet demand at the first stage and the consequent penalty. An alternative two-stage variational inequality formulation is also presented.

In the third part, Scheduling and Planning, the reader will find the following chapters:

The Value of the Stochastic Solution in a Two-Stage Assembly-to-Order Problem, by Brandimarte et al. (2012) The authors consider a simple assembly to order problem, where components must be manufactured under demand uncertainty and end items are assembled only after demand is realized. The problem can be cast as a two-stage stochastic linear program with recourse. The chapter investigates the conditions under which a stochastic programming approach yields significant advantages over a straightforward deterministic model based on the expected value of demand.

Robust Optimal Planning of Waste Sorting Operations, by Pinto et al. (2021) This chapter investigates the operations of waste recycling centers where materials are collected by a fleet of trucks and then sorted in order to be converted in secondary raw materials. The chapter proposes a mixed integer linear programming model for planning and scheduling the packaging waste recycling operations taking into consideration the stochastic nature of waste arrivals. Experiments are performed on instances taken from a real case in Italy and comparisons are made against different planning strategies.

Solution Approaches for the Capacitated Scheduling Problem with Conflict Jobs, by Tresoldi (2021) This chapter presents a new arc-based mathematical formulation and a heuristic algorithm for the capacitated scheduling problem with conflicting jobs. The effectiveness of the approach is tested through extensive computational experiments.

In the fourth part, Transportation and Logistics, the reader will find:

A Decision Model for Enhancing Driving Security, by Baldi et al. (2021) Intelligent Advance Driver Assistance Systems (ADAS) can improve vehicle control performance and, thus, driver and passenger safety. In particular, identification and prediction of driving intention are fundamental for avoiding collisions as they can provide useful information to drivers and vehicles in their vicinity. The paper proposes a lane change prediction model based on machine learning able to distinguish between left and right lane changes, a distinction that becomes particularly important when driving in a highway. Models have been trained and validated using a real dataset gathered online by using a high-tech demonstrator vehicle provided by Fiat Research Center within the European Project DESERVE. Two models based on Support Vector Machines and Random Forest are proposed.

A Two-Echelon Truck-and-Drone Distribution System: Formulation and Heuristic Approach, by Boccia et al. (2021) The authors study a two-echelon truck-and-drone distribution system where the first-echelon is composed by the depot and truck parking places, whereas the second-echelon is composed by the parking places and the final customers that are served by a fleet of drones. Starting from previous works, different truck-and-drone delivery systems have been proposed in literature, where the truck operates as a mobile depot for the drones. In this chapter, a mixed integer linear programming formulation and a two-stage heuristic that exploits the underlying structure of the problem is proposed. The approach is tested and validate on a set of instances up to 50 customers.

A Heuristic Approach for the Human Migration Problem, by Cappello et al. (2021) This chapter presents a network-based model for human migration in which a utility function is maximized. The resulting nonlinear optimization problem is characterized by a variational inequality formulation. Due to the high complexity of this problem, in order to efficiently solve realistic instances, a heuristic method is proposed. The presented algorithms are tested and compared over a number of randomly generated instances

In-store Picking Strategies for Online Orders in Grocery Retail Logistics, by Chou et al. (2021) Customers shifting from stationary to online grocery shopping and the decreasing mobility of an ageing population pose major challenges for the stationary grocery retailing sector. To fulfill the increasing demand for online grocery shopping, traditional bricks-and-mortar retailers use existing store networks to offer customers click-and-collect services. The current COVID-19 pandemic is accelerating the transition to such a mixed offline/online model, and companies are facing the need of a re-design of their business model. Currently, a majority of the operations to service online demand consists of in-store picker-to-parts order picking systems, where employees go around the shelves of the shop to pick up the articles of online orders. The paper proposes optimization ideas and solutions for these in-store operations. Experimental simulations on a real store with real online orders are performed.

An Optimization Model for the Evacuation Time in the Presence of Delay, by Daniele et al. (2021) The chapter addresses the issue of planning the emergency evacuation of occupants of a building after a disaster event like a landslide. A network model that minimizes both the travel time and the delay of evacuating is proposed, introducing also a measure of the physical difficulties of evacuees and a parameter associated with the severity of the disaster. The variational inequality formulation is derived and a numerical example is presented.

Additive Bounds for the Double Traveling Salesman Problem with Multiple Stacks, by Diedolo and Righini (2021) The Double TSP with Multiple Stacks is a challenging combinatorial optimization problem, asking for two Hamiltonian cycles on two weighted graphs, a pick-up graph, and a delivery graph. The two cycles originate from two given depots. They visit the vertices in an order that allows a single vehicle to collect the pick-up items in a given number of stacks and to deliver them according to a Last-In-First-Out policy for each stack. The paper investigates the use of the additive bounding procedure, starting from the Held-Karp lower bound, within a branch-and-bound algorithm. Computational results show that this method can provide tighter bounds than the Double TSP relaxation.

Crowd-Shipping and Occasional Depots in the Last Mile Delivery, by Di Puglia Pugliese et al. (2021) Crowd-shipping is a new delivery paradigm that is gaining success in the last-mile and same-day delivery process. In crowd-shipping, the deliveries are carried out by both regular company vehicles and some crowd-drivers, named occasional drivers (ODs). ODs are ordinary people available to make deliveries, for a small compensation. The paper considers a setting in which a company not only has ODs available to make deliveries, but they may also use the services of intermediate pickup and delivery points, named occasional depots. In order to optimize the use of these depots, it considers two distinct groups of ODs with different operative ranges. Occasional depots are activated only if it is necessary or convenient, implying an "activation cost," which is the main difference with respect to the classical problem with transshipments nodes. These depots should increase the flexibility of the system and they lead to a more efficient managing of the uncertain availability of ODs. This chapter presents a mixed integer linear programming model able to represent this framework. Computational experiments to validate it on small size instances are carried out.

As editors of the volume, we thank the Program Committee, composed by the AIRO Scientific Board, the invited lecturer, the authors, and the researchers who spent their time for the review process, thus contributing to improve the quality of the selected papers. Finally, we express gratitude to the Springer team for support and cooperation in publishing this volume, bringing it to a nice form.

Branch and Bound and Dynamic Programming Approaches for the Path Avoiding Forbidden Pairs Problem, by Ferone et al. (2021) The chapter proposes a branch and bound and a dynamic programming algorithm for the Path Avoiding Forbidden Pairs Problem. Given a network and a set of forbidden node pairs, the problem consists in finding the shortest path from a source node s to a target node t, avoiding to traverse both nodes of any of the forbidden pairs. The problem has been shown to be NP-complete. The paper describes the problem, its mathematical model and two exact algorithms, comparing their performances against those of a commercial solver on instances with fully random graphs and grid graphs.

Revenue Management Approach for Passenger Transport Service: An Italian case study, by Guerriero et al. (2021) The main aim of the revenue management (RM) techniques is to sell the right product to the right customer, at the right time and price, to optimize the sales. RM has been successfully applied in numerous kinds of services. Recently, bus passenger transport has been deregulated and liberalized, thus companies are free to vary their prices, timetable, and routes. The use of RM could represent a key factor in a highly competitive market. The paper considers the problem of a bus transport company which operates from a given set of origins to a given set of destinations on a given time horizon. A dynamic programming formulation and a linear approximation are proposed. The linear approximation, representing the seat-allocation problem, is tested with reference to an Italian bus company. The computational experiments reveal that the proposed model could help the bus transport company to control the capacity levels, to improve customer service and bus utilization, by maximizing the revenue.

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About This Book

This book collects selected contributions from the international conference "Optimization and Decision Science" (ODS2020), which was held online on November 19, 2020, and organized by AIRO, the Italian Operations Research Society.

The book offers new and original contributions on optimization, decisions science, and prescriptive analytics from both a methodological and applied perspective, using models and methods based on continuous and discrete optimization, graph theory and network optimization, analytics, multiple criteria decision-making, heuristics, metaheuristics, and exact methods.

In addition to more theoretical contributions, the book chapters describe models and methods for addressing a wide diversity of real-world applications spanning health, transportation, logistics, public sector, manufacturing, and emergency management.

Although the book is aimed primarily at researchers and PhD students in the operations research community, the interdisciplinary content makes it interesting for practitioners facing complex decision-making problems in the aforementioned areas, as well as for scholars and researchers from other disciplines, including artificial intelligence, computer sciences, economics, mathematics, and engineering.

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About the Editors

Raffaele Cerulli is Full Professor of Operations Research at the University of Salerno. His main research interests focus on combinatorial optimization problems: labeled graph problems, minimum spanning tree problem, problems of the traveling salesman, vehicle routing problems, wireless sensor network, and interval linear programming. He has organized national and international conferences/schools in these fields. He is the author of almost 80 papers on discrete optimization and related areas. He is director of the Department of Mathematics at the University of Salerno and member of the scientific committee of UMI (Italian Mathematical Union). He is member of the board of the Italian Association for Operations Research (AIRO). He is editor-in-chief of *Soft Computing* (Springer) and of *Advances in Computational Intelligence* (Springer). He has participated as principal investigator in many international funded research projects.

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Part I
Game Theory and Optimization

Integer Programming Reformulations in Interval Linear Programming

Elif Garajová, Miroslav Rada, and Milan Hladík

Abstract Interval linear programming provides a mathematical model for optimization problems affected by uncertainty, in which the uncertain data can be independently perturbed within the given lower and upper bounds. Many tasks in interval linear programming, such as describing the feasible set or computing the range of optimal values, can be solved by the orthant decomposition method, which reduces the interval problem to a set of linear-programming subproblems—one linear program over each orthant of the solution space. In this paper, we explore the possibility of utilizing the existing integer programming techniques in tackling some of these difficult problems by deriving a mixed-integer linear programming reformulation. Namely, we focus on the optimal value range problem, which is NP-hard for general interval linear programs. For this problem, we compare the obtained reformulation with the traditionally used orthant decomposition and also with the non-linear absolute-value formulation that serves as a basis for both of the former approaches.

Keywords Interval linear programming · Integer programming · Optimal value range

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1 Introduction

Optimization under uncertainty plays a crucial role in modeling and solving real-world problems with inexact input data. In this paper, we consider the approach of interval linear programming [9, 17], which provides a suitable model for problems with uncertain data that can be independently perturbed within the given lower and upper bounds. Throughout the last years, interval programming has been used as an uncertain model for various practical optimization problems, such as transportation problems with interval data [1, 4] or portfolio optimization [2] to mention some.

Several difficult tasks in interval linear programming can be solved by decomposing the problem at hand into an exponential number of classical linear programs. This is also the idea behind the frequently used orthant decomposition method, which exploits the fact that the feasible set of an interval linear program becomes a convex polyhedron when we restrict the solution space to a single orthant [7, 16].

Here, we propose and explore an alternative approach to solving such tasks by utilizing the powerful techniques of integer programming. To illustrate the idea, we derive a (mixed) integer programming reformulation for computing the best optimal value of an interval linear program based on a non-linear absolute-value formulation of the problem [8]. A similar approach can be beneficial in solving other related problems, such as describing the set of all optimal solutions of an interval linear program [5, 12]. We conduct a computational experiment to compare the absolute-value formulation and the derived mixed-integer programming reformulation for the optimal value range problem and show their efficiency against the traditional orthant decomposition [17].

2 Interval Linear Programming

Let us first review some of the notions and notation used throughout the paper. For a comprehensive introduction to interval linear programming see [9, 17] and references therein.

Given a vector $x \in \mathbb{R}^n$, we denote by $\operatorname{diag}(x)$ the diagonal matrix with entries $\operatorname{diag}(x)_{ii} = x_i$ for $i \in \{1, \dots, n\}$. The inequality relations on the set of matrices and vectors, as well as the absolute value operator $|\cdot|$, are understood element-wise.

Interval Data Let the symbol $\mathbb{IR}$ denote the set of all closed real intervals. Given two real matrices $\underline{A}, \overline{A} \in \mathbb{R}^{m \times n}$ satisfying $\underline{A} \leq \overline{A}$, we define an *interval matrix* $\mathbf{A} \in \mathbb{IR}^{m \times n}$ as the set

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} : \underline{A} \leq A \leq \overline{A}\}.$$

Alternatively, an interval matrix can also be determined by the *center* A_c and *radius* A_Δ, where

$$A_c = \frac{1}{2}(\overline{A} + \underline{A}), \qquad A_\Delta = \frac{1}{2}(\overline{A} - \underline{A}). \tag{1}$$

An *interval vector* $\mathbf{a} \in \mathbb{IR}^n$ can be defined analogously as an $n \times 1$ interval matrix. In the text, we denote all interval matrices and interval vectors by bold letters.

Interval Programming For an interval matrix $\mathbf{A} \in \mathbb{IR}^{m\times n}$ and interval vectors $\mathbf{b} \in \mathbb{IR}^m$, $\mathbf{c} \in \mathbb{IR}^n$, we define an *interval linear program* (abbreviated as ILP) as the set of all linear programs in the form

$$\min\ c^T x \text{ subject to } Ax \leq b, \tag{2}$$

with $A \in \mathbf{A}$, $b \in \mathbf{b}$ and $c \in \mathbf{c}$. For short, we also write an interval linear program determined by the triplet $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ as

$$\min\ \mathbf{c}^T x \text{ subject to } \mathbf{A}x \leq \mathbf{b}. \tag{3}$$

A particular linear program (2) is called a *scenario* of the interval linear program (3).

For the sake of simplicity, the formulation of an interval linear program introduced in (3) is not the most general one. Since the commonly used transformations in linear programming are not always applicable in the interval framework due to the so-called dependency problem (see e.g. [6]), different formulations of interval linear programs may have different properties. However, the approach presented in this paper can also be utilized for other types of interval linear programs in the same manner.

Feasibility and Optimality Several different concepts of feasible and optimal solutions of interval linear programs have been introduced in the literature. In this paper, we adopt the notion of weak feasibility and optimality.

A vector $x^* \in \mathbb{R}^n$ is called a *weakly feasible solution* of ILP (3), if it is a feasible solution of some scenario, i.e. if $Ax^* \leq b$ holds for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$. In general, the set of all weakly feasible solutions of an ILP forms a non-convex polyhedron, which is convex in each orthant [16]. By the Gerlach theorem for interval systems of inequalities [7], a vector $x \in \mathbb{R}^n$ is a weakly feasible solution of ILP (3) if and only if it solves the non-linear system

$$A_c x \leq A_\Delta |x| + \overline{b}. \tag{4}$$

Similarly, we say that a vector $x^* \in \mathbb{R}^n$ is a *weakly optimal solution* of the ILP, if it is an optimal solution of some scenario with $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$. Unless stated otherwise, we use the term "feasible/optimal solution" in the context of interval programming to refer to weakly feasible and weakly optimal solutions, respectively.

Optimal Values A common approach to computing optimal values of an interval linear program is to find the best and the worst value, which is optimal for some scenario of the program.

Let $f(A, b, c)$ denote the optimal value of the linear program (2), setting $f(A, b, c) = -\infty$ for unbounded programs and $f(A, b, c) = \infty$ for infeasible programs. Then, we define *optimal value range* of interval linear program (3) as the interval $[\underline{f}, \overline{f}]$, where the best optimal value $\underline{f}$ and the worst optimal value $\overline{f}$ are

$$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \min \{f(A, b, c) : A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\},$$

$$\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \max \{f(A, b, c) : A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\}.$$

The worst optimal value $\overline{f}$ of ILP (3) can be computed in polynomial time by solving a linear program (see [3, 15]). On the other hand, computing the best optimal value $\underline{f}$ of (3) is an NP-hard problem [17]. Since it might be difficult to compute the value exactly, methods providing a sufficiently tight approximation are also of interest [11, 13].

Orthant Decomposition As the set of all weakly feasible solutions of an interval linear program becomes a convex polyhedron when we restrict the solution space to a single orthant, we can utilize this property to solve various problems over the feasible set. This idea leads to the often used *orthant decomposition* method, which solves a given problem in interval programming by decomposing it into a set of linear programming subproblems, one for each orthant of the solution space.

Orthant decomposition can also be used to obtain the best optimal value $\underline{f}$ of ILP (3). Here, we can formulate a linear program to compute the minimum value of the objective function over the feasible set in a given orthant and then take the smallest of the computed values (see [17] for further details). An *orthant* of the solution space $\mathbb{R}^n$ can be described as the set

$$\{x \in \mathbb{R}^n : \operatorname{diag}(s)x \geq 0\}$$

for a particular sign vector $s \in \{\pm 1\}^n$. Therefore, we can compute $\underline{f}$ by solving the linear program

$$\begin{aligned} \text{minimize } & (c_c - \operatorname{diag}(s)c_\Delta)^T x \\ \text{subject to } & (A_c - A_\Delta \operatorname{diag}(s))x \leq \overline{b}, \\ & \operatorname{diag}(s)x \geq 0. \end{aligned} \tag{5}$$

for each $s \in \{\pm 1\}^n$. For a given vector s, denote by $f_s(\mathbf{A}, \mathbf{b}, \mathbf{c})$ the optimal value of program (5). Then, we obtain the best optimal value as

$$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \min \left\{ f_s(\mathbf{A}, \mathbf{b}, \mathbf{c}) : s \in \{\pm 1\}^n \right\}.$$

This amounts to solving (at most) 2^n linear programs to compute $\underline{f}$, with n denoting the number of variables of the ILP. Note that the number of orthants that have to be explored can be lowered, if some of the variables are known to be sign-restricted (non-negative or non-positive).

3 Integer Programming Reformulations

In this section, we build on the absolute-value characterization of the feasible set by Gerlach stated in (4). We derive a mixed-integer linear programming reformulation of the system in order to design an alternative method for computing the best optimal value of an interval linear program.

The aim is to utilize the available techniques and efficient algorithms of integer linear programming to tackle some of the difficult interval problems, such as the problem of computing the optimal value range.

Absolute-Value Formulation Instead of using the orthant decomposition, we can also restate the method for computing the best optimal value as an absolute-value program [8], which is derived from the Gerlach theorem for describing the weakly feasible set. By this result, we can compute $\underline{f}$ as the optimal value of the non-linear program

$$\begin{aligned} &\text{minimize} \quad c_c^T x - c_\Delta^T |x| \\ &\text{subject to} \quad A_c x - A_\Delta |x| \le \overline{b}. \end{aligned} \tag{6}$$

We can now attempt to solve formulation (6) directly as a non-linear program, or we can further linearize the program by modeling $|x|$ via binary variables and additional linear constraints as a mixed-integer linear program.

MIP Reformulation Now, we can use the absolute-value formulation (6) to derive a mixed-integer linear program for computing the best optimal value $\underline{f}$. To do this, we apply one of the traditional ways to model absolute values in integer programs using binary variables.

Here, we split the variable x into a positive and negative part as $x = x^+ - x^-$, using the lower and upper bound on x and auxiliary binary variables y_i. Then, we model the absolute value $|x|$ by introducing a new variable $z = x^+ + x^-$, leading to the formulation

$$
\begin{aligned}
\text{minimize} \quad & c_c^T x - c_\Delta^T z \\
\text{subject to} \quad & A_c x - A_\Delta z \leq \overline{b}, \\
& x = x^+ - x^-, \\
& z = x^+ + x^-, \\
& 0 \leq x_i^+ \leq |\overline{x_i}| y_i, && \forall i \in \{1, \dots, n\}, \\
& 0 \leq x_i^- \leq |\underline{x_i}|(1 - y_i), && \forall i \in \{1, \dots, n\}, \\
& y \in \{0, 1\}^n.
\end{aligned} \tag{7}
$$

Note that we can also reduce the number of variables in the model by simply substituting the expressions in terms of x^+ and x^- for the variable x and its absolute value z. Using the definition of the center and the radius of an interval matrix stated in (1), we obtain the simplified mixed-integer linear program

$$
\begin{aligned}
\text{minimize} \quad & \underline{c}^T x^+ - \overline{c}^T x^- \\
\text{subject to} \quad & \underline{A} x^+ - \overline{A} x^- \leq \overline{b}, \\
& 0 \leq x_i^+ \leq |\overline{x_i}| y_i, && \forall i \in \{1, \dots, n\}, \\
& 0 \leq x_i^- \leq |\underline{x_i}|(1 - y_i), && \forall i \in \{1, \dots, n\}, \\
& y \in \{0, 1\}^n.
\end{aligned} \tag{8}
$$

Further Applications Apart from computing the optimal value range, integer programming reformulations can also prove useful in solving other difficult problems in interval linear programming. A description of many important characteristics and properties of an interval linear program can be derived from the Gerlach and the Oettli–Prager theorems [7, 16], which describe the weakly feasible set via a system of absolute-value inequalities.

For example, the set of all weakly optimal solutions of ILP (3) can be described by primal feasibility, dual feasibility and strong duality as the set of x-solutions of the system

$$
\begin{aligned}
& Ax \leq b, \\
& A^T y = c, \ \ y \leq 0, \\
& c^T x = b^T y, \\
& A \in \mathbf{A}, \ \ b \in \mathbf{b}, \ \ c \in \mathbf{c}.
\end{aligned} \tag{9}
$$

Note that this is a parametric system, since there are dependencies between the two occurrences of the interval parameters that cannot be captured by a simple interval linear system (e.g. the two occurrences of the matrix $A \in \mathbf{A}$ should represent the same matrix in any considered scenario). However, we can relax these dependencies to obtain an interval linear system (see also [5, 10] and references therein), which provides an outer approximation of the optimal solution set:

$$\mathbf{A}x \le \mathbf{b}, \quad \mathbf{A}^T y = \mathbf{c},\ y \le 0, \quad \mathbf{c}^T x = \mathbf{b}^T y. \tag{10}$$

Here, we assume that the two occurrences of the interval parameters **A**, **b** and **c** are independent and in a particular scenario of the system, different values from the respective interval matrices and vectors can be chosen for them. System (10) is a classical interval linear system, so we can use the description of the weakly feasible set provided by the Gerlach and the Oettli–Prager theorems, leading to the absolute-value system

$$\begin{aligned} &A_c x \le A_\Delta |x| + \overline{b}, \\ &\overline{A}^T y \le \overline{c},\ \underline{A}^T y \ge \underline{c},\ y \le 0, \\ &|c_c^T x - b_c^T y| \le c_\Delta^T |x| - b_\Delta^T y. \end{aligned} \tag{11}$$

For system (11), we can formulate a mixed-integer linear program in a similar way as in the problem of computing the best optimal value. The program can then be used to compute an interval enclosure of the optimal set by finding the minimal/maximal value of each x_i over (11). We can also apply various integer programming relaxations and heuristics to derive more efficient approximation techniques for the optimal set. A tight approximation of the optimal set is also essential in solving the recently proposed outcome range problem [14], which generalizes the optimal value range by introducing an additional linear outcome function to the program.

4 Computational Experiment

We conducted a computational experiment to compare the derived integer programming reformulation with the traditionally used orthant decomposition method and the non-linear absolute-value formulation for the problem of finding the best optimal value $\underline{f}$ of ILP (3). Since all of these techniques are used to compute the value $\underline{f}$ exactly, the main criterion for comparison is the elapsed computation time.

Instances We compared the different programs for computing the best optimal value on a set of (pseudo-)randomly generated feasible instances. Since the best optimal value $\underline{f}$ can always be achieved for the upper bound $\overline{b}$ of the interval right-hand-side vector **b**, we only generated interval data for the constraint matrix and the objective vector. Thus, each instance is described by an interval matrix $\mathbf{A} \in \mathbb{IR}^{m \times n}$, a fixed right-hand-side vector $b \in \mathbb{R}^m$ and an interval objective vector $\mathbf{c} \in \mathbb{IR}^n$.