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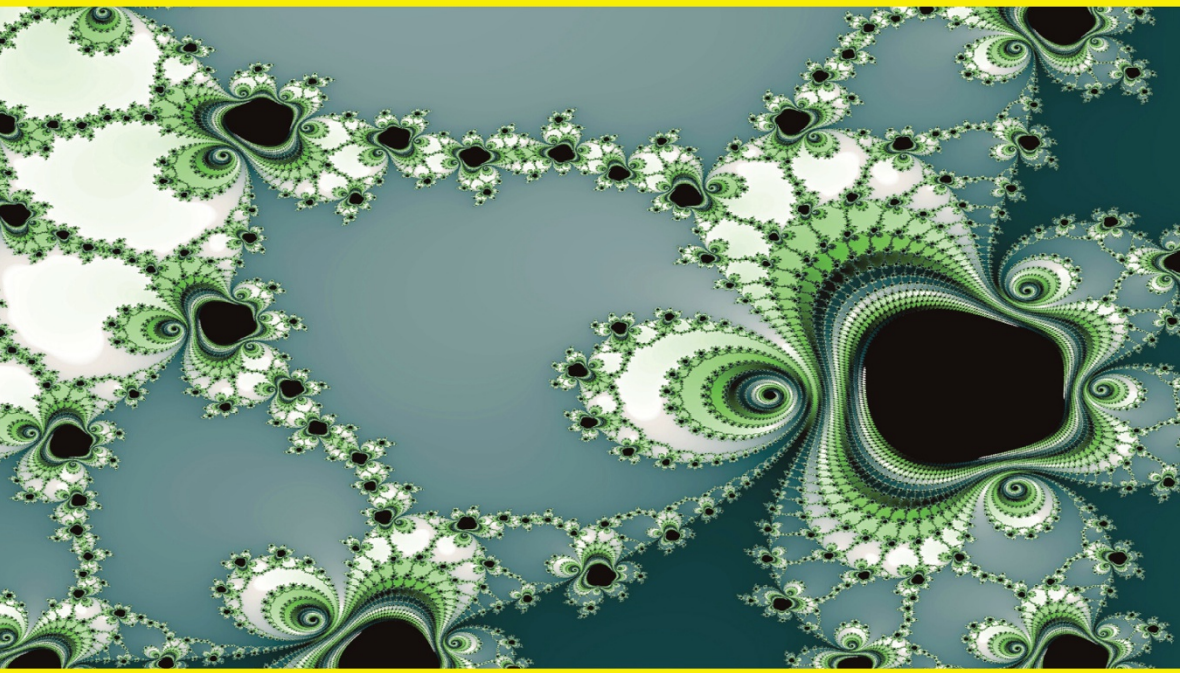
Random Evolutionary Systems

Asymptotic Properties and Large Deviations

Dmitri Koroliouk
Igor Samoilenko



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Random Evolutionary Systems

In memory of our teacher and mentor,
Vladimir Semenovich Korolyuk

Series Editor
Nikolaos Limnios

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Preface

This book examines random evolutionary systems and their asymptotic properties, as well as large deviations.

Our study of random evolutionary systems is based on the martingale characterization of their trajectories, the method of associated semigroups and generating operators, the method of inversion on the spectrum of reducible-invertible operators, as well as the singular perturbation problem and the phase merging method. A special role is played, not only by classical approximation schemes for random evolutionary systems such as, in particular, diffusion approximation, but also by new ones: the Poisson approximation and the Lévy approximation.

The diffusion and Poisson approximation of random evolutions plays a special role.

The study of random evolutionary systems in terms of operators constructed in the schemes of the diffusion and Poisson approximation allows us to obtain a number of limit theorems and asymptotic expansions of processes that model complex stochastic systems, both those that are autonomous and those dependent on an external random environment. In this case, various orders of scaling, both of processes and their time parameters, are used to obtain different limit results.

April 2021

Introduction

The theory of complex systems is a modern field of natural sciences, which considers systems that consist of a large number of interacting parts. There is not a large number of quality properties of a system that are observed in the study of its individual parts; they arise only as a result of their interactions.

The theory of complex systems is based on methods of systems analysis as an applied scientific methodology, consisting, in particular, of mathematical methods, algorithmic software and computing tools, which provide the formation of holistic knowledge about the object, as a set of interconnected processes of different natures for further decision-making on its development and behavior, taking into account conflicting criteria, the presence of risk factors and inaccurate information. Due to the application of systems analysis methods, the theory of complex systems has been actively developing in recent decades and consists, in particular, of the development of methods for structuring, modeling, analysis and synthesis of deterministic and stochastic systems in problems of mathematical (including statistical) physics, applied questions of probability theory and fractal analysis and mathematical problems of biology, sociology, ecology, economics and medicine. The main motivation that unites all these diverse mathematical and natural sciences in a single discipline is closeness, as well as the kinship laws and methods of studying the collective behavior of such systems.

Mathematical problems associated with the theory of complex systems relate to the development of mathematical methods of system simplification, which can be very complex, even for computer analysis. In this case, the simplified system should be such that, first, its local characteristics are determined by simple enough functionals of the local characteristics of the original system, and, second, that the simplified model could be qualitatively analyzed by mathematical methods and its global characteristics are an effective approximation of the corresponding characteristics of the original system.

Typical examples of complex systems are evolutionary systems, which, in particular, are modeled using random evolutionary systems and random processes. For example, impulsive processes describe various models in the queuing theory, namely, network structures and production systems; processes with independent increments are used in modeling conflict systems, such as birth–death processes in ecological and biological systems, as well as in models of statistical physics; and stochastic evolutionary systems model problems that arise in reliability theory, control theory, financial mathematics and so on.

We provide an analysis of the asymptotic properties of such models, which are considered and studied from different points of view, using several approximation schemes. That is, the question of the convergence of evolutionary systems in Poisson approximation schemes is analyzed, the problem of large deviations for evolutionary systems in different approximation schemes is analyzed and some applications are discussed: the probability of leaving an interval, the conditions of equilibrium, stationarity, the classification of equilibria and so on.

The development of the theory of random evolutions began in the late 1960s, probably with the work of R. Griego and R. Hersh. They introduced the concept of random evolution in Griego and Hersh (1971).

Applications of this model follow from the work of R.Z. Khasminskii (1966a, 1966b), which were stimulated by the problems of stability of stochastic systems, in particular, by the works of R.L. Stratonovich (1967) on the problems of the nonlinear theory of oscillations in the presence of noise. In the 1960s and 1970s, problems related to the theory of random evolutions were actively studied by American mathematicians R. Hersh, M. Pinsky, G. Papanikolaou, T. Kurtz, R. Griego, L. Gorostiza (Pinsky 1968; Griego and Hersh 1971; Papanicolaou and Kellek 1971; Gorostiza 1972, 1973a, 1973b; Hersh and Papanicolaou 1972; Hersh and Pinsky 1972; Kurtz 1973; Hersh 1974; Papanicolaou 1975) and others. In particular, G. Papanicolaou, D. Strook and S. Varadhan proposed a martingale approach for proving limit theorems (Papanicolaou *et al.* 1977) using methods similar to solving the singular perturbation problem.

An effective method for proving the limit theorems in the theory of random evolutions is the theory of phase merging of complex systems, developed by V.S. Korolyuk and A.F. Turbin (1975, 1993). A.V. Skorokhod (2008) investigated several important problems of the theory of dynamic systems, denoted by stochastic differential equations. The problem of the stability of dynamic systems in the case of random perturbation of their parameters was investigated by R.Z. Khasminskii (2012) and V.S. Korolyuk (1991, 1998). Problems of stability and problems of stochastic approximation of evolutionary systems with switching were investigated by V.S. Korolyuk and Y.M. Chabanyuk (2007, 2007). V.S. Korolyuk and A.V. Swishchuk also developed a theory of semi-Markov random evolutions, based

on martingale theory (Korolyuk and Swishchuk 1995a, 1995b). Many applications of processes with switching to the analysis of network systems can be found in the works of V.V. Anisimov (2008).

Asymptotic methods are collected and described in detail in the book by V.S. Korolyuk and N. Limnios (2005), where these methods are mainly applied to models in the schemes of averaging and diffusion approximation. At increasing time intervals, the averaging scheme demonstrates the deterministic averaging behavior of the system, and the diffusion approximation scheme shows stochastic fluctuations around the deterministic averaged trajectory. These two schemes differ in the normalization of the switching process, namely, in the case of the averaging scheme, the acceleration of time by the parameter ε^{-1} is considered, while, in the case of diffusion, approximation by the parameter ε^{-2} or ε^{-3} is considered. An important element of the algorithm is the assumption of ergodicity of the switching (Markov or semi-Markov) process. In their book, V.S. Korolyuk and N. Limnios (2005) also considered several models for processes with independent increments and impulsive processes with switching in the schemes of Poisson and Lévy approximation. Here we generalize the models in the schemes of Poisson and Lévy approximation to the case of processes with locally independent increments, impulsive recurrent processes and so on.

We also present another important area of research, namely, the solution of the large deviations problem, using the methods of asymptotic analysis of nonlinear exponential generators, associated with the singular perturbation problem. The large deviations problem arose as a method of solving statistical problems, related to estimating the probabilities of rare events. The first work in this direction was, obviously, the article by H. Cramér (1938), but this idea was finally fully formed in the work of G. Chernoff (1952).

The main method of studying such problems is the technique of substituting the measure and using the Chebyshev inequality to obtain estimates of the probabilities of rare events, and to calculate the rate functional. In this case, a new measure is introduced, in relation to which the studied events have a high probability, while the probabilities of these events relative to the initial measure are determined in terms of the Radon–Nicodemus derivative, which connects the two mentioned measures. An important role in determining the rate functional is also played by the Legendre transform, which links the rate functional and the cumulant of the process. The most famous works in this technique, which relate to Markov processes, are the works of M. Donsker and S. Varadan (1975a, 1975b, 1976) and M.Y. Freidlin and O.D. Wentzell (1976, 1978, 1979, 1990) (see also Deuschel and Stroock 1989; Dupuis and Ellis 1997; Dembo and Zeitouni 2010). These works contain comprehensive bibliographic references on this topic.

Another approach to solving the problem is related to the convergence and compactness of probabilistic measures. The works in which such methods are used include articles by A.A. Puhalsky (1991), G. O'Brien and W. Vervaat (1995) and A. de Acosta (1997).

Here, we use a method that has emerged and developed recently and is closely related to the control problem. The idea of such an approach was presented in Hopf (1950) and Çinlar *et al.* (1980) and was finally developed by Fleming and Suganidis (1986). The most developed and generalized version of this approach can be found in the book by J. Feng and T. Kurtz (2006), which also contains a relevant bibliography and historical review. It should be noted that the classical normalization scheme in which the large deviations problem is studied is the small diffusion scheme (see Feng and Kurtz (2006) and Freidlin and Ventzell (2012) and the corresponding bibliographic sections of these works). One of the rare models where another possible normalization for the process with independent increments was investigated is the work of A.A. Mogulsky (1993) (compare this to other works by A.A. Borovkov and A.A. Mogulsky (1992, 2012)).

We investigate the large deviations problem for processes with independent increments and impulsive processes with switching in the schemes of the Poisson and Lévy approximations. Models with phase merging are studied for the first time. This formulation of the problem is completely new and allows us to obtain some significant generalizations. For example, the presence of a diffusion component in the small diffusion scheme is embedded in the definition of the limit process. Conversely, the Poisson approximation scheme considers processes with independent increments with switching without a diffusion component, which occurs only after the limit transition. The Poisson approximation scheme, which takes into account rare large jumps in the process, is certainly a natural object for study, especially in terms of the problem of large deviations.

1

Basic Tools for Asymptotic Analysis

1.1. Basic concepts of operator asymptotic analysis

This first chapter introduces the known auxiliary concepts, which are actively used in the following sections. In particular, the concepts of the Poisson approximation and Lévy approximation, impulsive recurrent process, random evolutionary system with locally independent increments, compensating operator, Nisio semigroup, nonlinear exponential operator, martingale control problem, rate functional, compactness, etc. are introduced. The following results are new:

1. The previously known results were used for a detailed substantiation of the relationship between the Nisio semigroup, the martingale control problem, the Brick formula, the nonlinear exponential operator and the corresponding rate functional. Some comparative examples are given for the classical formulation of the problem of large deviations in the scheme of small diffusion.

2. The problems related to the study of compactness and exponential compactness of processes, as well as the application of the comparison condition to the limit nonlinear exponential generator are described. For the processes that will be studied further, the corresponding compactness conditions are tested using the methods from the stability theory, in particular the Lyapunov function. For the corresponding nonlinear exponential generator, the comparison condition is checked.

3. For the first time, the problem of studying the problem of large deviations in the Poisson approximation scheme is set, and the corresponding nonlinear exponential generators for the process with independent increments are determined.

Random Evolutionary Systems: Asymptotic Properties and Large Deviations,
First Edition. Dmitri Koroliouk and Igor Samoilenko.

1.1.1. *Reducibly invertible and potential operators*

We denote by $\mathbf{B}$ the Banach space of real-valued measurable functions, defined on a state space E of a random process, with $\sup$-norm

$$||\varphi|| = \sup_{x \in E} |\varphi(x)|, \varphi \in \mathbf{B}.$$

Let $Q : \mathbf{B} \to \mathbf{B}$ be a linear operator on $\mathbf{B}$. We introduce the following subspaces:

$$\mathcal{D}_Q := \{\varphi : \varphi \in \mathbf{B}, Q\varphi \in \mathbf{B}\} - \text{domain of definition } Q,$$

$$\mathcal{R}_Q := \{\psi : \psi = Q\varphi, \varphi \in \mathbf{B}\} - \text{subspace of values } Q,$$

$$\mathcal{N}_Q := \{\varphi : Q\varphi = 0, \varphi \in \mathbf{B}\} - \text{subspace of zeros } Q.$$

The operator Q will be called bounded if there exists a constant $C > 0$ such that $||Q\varphi|| \leq C||\varphi||, \varphi \in \mathcal{D}_Q$.

DEFINITION 1.1.– *A bounded linear operator Q is called reducibly invertible if the Banach space $\mathbf{B}$ can be represented as a direct sum of two subspaces*

$$\mathbf{B} = \mathcal{N}_Q \oplus \mathcal{R}_Q,$$

where the null-subspace has a non-trivial dimension

$$\dim \mathcal{N}_Q \geq 1.$$

More detailed information on the properties of reducibly invertible operators can be found in the works of Nashed (1976), Korolyuk and Turbin (1993) and Korolyuk and Limnios (2005); for possible generalizations in the Hilbert space, see Boichuk and Samoilenko (2004).

The following representation defines a projector on a subspace $\mathcal{N}_Q$:

$$\Pi\varphi := \begin{cases} \varphi, \varphi \in \mathcal{N}_Q, \\ 0, \varphi \in \mathcal{R}_Q. \end{cases}$$

Instead, the operator $I - \Pi$ is a projector on a subspace $\mathcal{R}_Q$

$$(I - \Pi)\varphi := \begin{cases} 0, \varphi \in \mathcal{N}_Q, \\ \varphi, \varphi \in \mathcal{R}_Q. \end{cases}$$

We can also define a projector by means of the resolvent

$$R_\lambda := [\lambda I - Q]^{-1}.$$

DEFINITION 1.2.– *A projector on a subspace $\mathcal{N}_Q$ is determined by the following ratio:*

$$\Pi := \lim_{\lambda \to 0+} \lambda R_\lambda.$$

When applied to the study of Markov processes, another way of defining the corresponding projector is important. It becomes possible in the presence of a Feller semigroup $P_t, t \geq 0$, which corresponds to the reducibly invertible operator Q (see, for example, Ethier and Kurtz (1986, p. 473)).

DEFINITION 1.3.– *The projector on a subspace $\mathcal{N}_Q$ is determined by the following ratio:*

$$\Pi\varphi := \lim_{\lambda \to 0+} \lambda \int_0^\infty e^{-\lambda t} P_t \varphi dt.$$

DEFINITION 1.4.– *The potential operator of a reducibly invertible operator Q is called the operator*

$$R_0 := \Pi - (Q + \Pi)^{-1} = (\Pi - Q)^{-1} - \Pi,$$

or in terms of resolvent

$$R_0 := \lim_{\lambda \to 0+} [R_\lambda - \Pi/\lambda].$$

For details, see Nashed (1976), Korolyuk and Turbin (1993) and Korolyuk and Limnios (2005).

If for a known semigroup $P_t, t \geq 0$, which corresponds to a reducibly invertible operator Q, the ergodicity condition

$$\lim_{t \to \infty} P_t = \Pi \neq 0,$$

verifies, then the potential operator R_0 is bounded and can be defined as

$$R_0 := \int_0^\infty (P_t - \Pi) dt.$$

The potential operator has the following properties:

$$QR_0 = R_0 Q = \Pi - I, \qquad [1.1]$$

$$\Pi R_0 = R_0 \Pi = 0. \qquad [1.2]$$

1.1.2. *Singular perturbation problem*

The singular perturbation problem (see Kurtz 1973; Papanicolaou *et al.* 1977; Ethier and Kurtz 1986; Korolyuk and Limnios 2005) for a reducibly invertible operator Q, corresponding to the switching process in the series scheme, with a small series parameter $\varepsilon \to 0 (\varepsilon > 0)$ and the perturbing operator Q_1 is solved in the following steps.

You need to build a vector $\varphi^{\varepsilon} = \varphi + \varepsilon\varphi_1$ and a vector ψ, which satisfy the asymptotic representation

$$[\varepsilon^{-1}Q + Q_1]\varphi^{\varepsilon} = \psi + \varepsilon\theta^{\varepsilon} \qquad [1.3]$$

with a uniformly bounded vector θ^{ε}, such that

$$||\theta^{\varepsilon}|| \le C, \varepsilon \to 0.$$

The left part of the obtained equation can be rewritten as

$$[\varepsilon^{-1}Q + Q_1](\varphi + \varepsilon\varphi_1) = \varepsilon^{-1}Q\varphi + [Q\varphi_1 + Q_1\varphi] + \varepsilon Q_1\varphi_1.$$

Equating to the right part of [1.3], we have:

$$\begin{cases} Q\varphi = 0, \\ Q\varphi_1 + Q_1\varphi = \psi, \\ Q_1\varphi_1 = \theta^{\varepsilon}. \end{cases} \qquad [1.4]$$

The last equation shows that the function $\varphi_1 \in \mathcal{D}_{Q_1}$. In addition, from the first equation, we have function φ as an arbitrary function from the null-subspace of operator Q.

Thus, the main question remains a solution of equation

$$Q\varphi_1 = \psi - Q_1\varphi.$$

The solvability condition for the reducibly invertible operator Q has the form:

$$\Pi Q\Pi\varphi_1 = 0 = \Pi\psi - \Pi Q_1\Pi\varphi,$$

where we finally get

$$\Pi\psi = \Pi Q_1\Pi\varphi.$$

Note that the operator $\Pi Q_1\Pi$ acts in subspace $\mathcal{N}_Q$; therefore, we can consider the reduced operator $\widehat{Q}_1$ on the reduced subspace $\widehat{\mathcal{N}}_Q$:

$$\Pi Q_1\Pi = \widehat{Q}_1\Pi.$$

We also use $\widehat{\psi} := \widehat{\Pi\psi} \in \widehat{\mathcal{N}}_Q$.

Then, the last equation takes the form

$$\widehat{\psi} = \widehat{Q}_1 \widehat{\varphi}.$$

Since we have a relation in subspace $\widehat{\mathcal{N}}_Q$, we can solve the second equation of system [1.4] with respect to φ_1 (see [1.1]):

$$\varphi_1 = R_0(Q_1\varphi - \psi), \Pi\varphi_1 = 0.$$

So,

$$\varphi_1 = R_0\widetilde{Q}_1\varphi, \widetilde{Q}_1 := Q_1 - \widehat{Q}_1,$$

and finally we have

$$\theta^\varepsilon = Q_1\varphi_1 = Q_1 R_0 \widetilde{Q}_1 \varphi.$$

Thus, the explicit expressions for functions $\psi, \varphi_1, \theta^\varepsilon$, are obtained, which give the solution of the singular perturbation problem.

1.1.3. *Markov process*

Let $x(t), t \geq 0$ be a Markov process on a standard state space $(E, \mathcal{E})$ (where E is a Polish space and $\mathcal{E}$ is its Borel σ-algebra), defined by means of the generator

$$Q\varphi(x) = q(x) \int_{\mathbf{E}} [\varphi(y) - \varphi(x)] P(x, dy), \ x \in E, \varphi(u) \in \mathcal{B}_E. \qquad [1.5]$$

The semi-Markov kernel

$$Q(x, B, t) = P(x, B)(1 - e^{-q(x)t}), x \in E, B \in \mathcal{E}, t \geq 0,$$

determines the associated Markov renewal process (x_k, τ_k), $k \geq 0$, where $x_k, k \geq 0$, is called the embedded Markov chain, given by the stochastic kernel

$$P(x, B) = P(x_{k+1} \in B | x_k = x),$$

and $\tau_k, k \geq 0$, is the point process of jump moments, which is determined by the time distribution function of sojourn time $\theta_{k+1} = \tau_{k+1} - \tau_k, k \geq 0$:

$$P(\theta_{k+1} \leq t | x_k = x) = 1 - e^{-q(x)t}.$$

The corresponding counting process of jumps is

$$\nu(t) := \max\{k \geq 0 : \tau_k \leq t\}.$$

The main assumption about the Markov process is the following condition:

CM: The Markov process $x(t), t \geq 0$, is uniformly ergodic with a stationary distribution $\pi(A),\ A \in \mathcal{E}$.

REMARK 1.1.– *Let* Π *be a projector at null-subspace of a reducibly invertible operator* Q *(defined in [1.5]):*

$$\Pi\varphi(x) = \int_E \pi(dx)\varphi(x).$$

The Markov process $x(t), t \geq 0$, is supposed to be *uniformly ergodic*, if for a semigroup P_t, determined by such a process, exists

$$\lim_{t\to\infty} P_t = \Pi \neq 0$$

in uniform operator topology. The main property of uniform ergodicity is the exponential speed of this convergence:

$$||P_t - \Pi|| \leq Me^{-\alpha t}, t > 0$$

for some $M > 1, \alpha > 0$. For details, see Doob (1990) and Korolyuk and Turbin (1993).

We should note that in this case the Markov process $x(t), t \geq 0$, has a stationary distribution $\pi(x)$, the embedded Markov chain $x_n, n \geq 1$ has a stationary distribution $\rho(x)$ and the following relationships are found:

$$\pi(dx)q(x) = q\rho(dx), q := \int_E \pi(dx)q(x).$$

1.1.4. *Semi-Markov process*

We call the Markov renewal process a two-component Markov chain $x_n, \tau_n, n \geq 0$ on $(E \times \mathbf{R}_+, \mathcal{E} \oplus \mathcal{B}_+)$, where $\tau_0 \leq ... \leq \tau_n \leq ...$ are renewal moments. Such a process is homogeneous by the second component, and its transient probabilities are determined by the semi-Markov kernel

$$Q(x, B, t) = P(x, B)F_x(t), x \in E, B \in \mathcal{E}, t \geq 0$$

through equality

$$\begin{aligned} Q(x, B, t) &= P(x_{n+1} \in B, \theta_{n+1} \leq t | x_n = x) \\ &= P(x_{n+1} \in B | x_n = x)P(\theta_{n+1} \leq t | x_n = x), \theta_{n+1} := \tau_{n+1} - \tau_n. \end{aligned}$$

Introduce the counting process $\nu(t), t \geq 0$:

$$\nu(t) = \sup\{n \geq 0 : \tau_n \leq t\},$$

which counts the number of renewal moments of the Markov renewal process over time $(0, t]$.

We call the semi-Markov process $x(t), t \geq 0$, associated with the Markov renewal process $x_n, \tau_n, n \geq 0$ a random process

$$x(t) := x_{\nu(t)}, t \geq 0.$$

Denote by Q the generator of the associated Markov process:

$$Q = q(x)(P - I),$$

where the transition probability operator P is defined as

$$Pf(x) = \int_E P(x, dy) f(y), x \in E,$$

for all bounded measurable real-function functions f, defined on E, $q(x)$ is determined as

$$q(x) := 1/m_1(x), m_1(x) := E\theta_x = \int_0^\infty \overline{F}_x(t)dt,$$

or $m_k(x) = \int_0^\infty s^k F_x(ds)$.

The main assumption about the semi-Markov process is the following condition:

CSM: The semi-Markov process $x(t), t \geq 0$ is uniformly ergodic with a stationary distribution

$$\pi(dx)q(x) = q\rho(dx), q := 1/m, m := \int_E \rho(dx)m(x),$$

$$\rho(B) = \int_E \rho(dx)P(x, B), \rho(E) = 1.$$

Let us denote a projector Π :

$$\mathcal{N}_Q := \Pi\mathcal{B}(E), \mathcal{R}_Q := (I - \Pi)\mathcal{B}(E);$$

$$\Pi\varphi(x) := \widehat{\varphi}\mathbf{1}, \widehat{\varphi} := \int_E \varphi(x)\pi(dx).$$

1.1.5. *Phase merging*

A Markov process $x^{\varepsilon}(t), t \geq 0$ is determined on a standard phase space $(E, \mathcal{E})$ with splitting

$$E = \bigcup_{k=1}^{N} E_k, E_k \cap E_{k'} = \emptyset, k \neq k'$$

in the series scheme, with a small series parameter $\varepsilon \to 0, \varepsilon > 0$.

Its Markov kernel has the form

$$Q^{\varepsilon}(x, B, t) = P^{\varepsilon}(x, B)[1 - e^{-q(x)t}],\ x \in E, B \in \mathcal{E}, t \geq 0.$$

The following conditions should also be fulfilled:

ME1: A kernel describing the transient probabilities of the embedded Markov chain $x_n^{\varepsilon}, n \geq 0$, has the following representation:

$$P^{\varepsilon}(x, B) = P(x, B) + \varepsilon P_1(x, B).$$

The stochastic kernel $P(x, B)$ on the split phase space is defined as follows:

$$P(x, E_k) = \mathbf{1}_k(x) := \begin{cases} 1, x \in E_k, \\ 0, x \notin E_k. \end{cases}$$

The stochastic kernel $P(x, B)$ defines the accompanying Markov chain $x_n, n \geq 0$ on the classes $E_k, 1 \leq k \leq N$. The perturbing kernel $P_1(x, B)$ also satisfies the condition

$$P_1(x, E) = 0,$$

which is a direct consequence of the equality $P^{\varepsilon}(x, E) = P(x, E) = 1$.

ME2: The associated Markov process $x^0(t), t \geq 0$, set by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)],$$

is uniformly ergodic inside each of the classes $E_k, 1 \leq k \leq N$, with stationary distributions $\pi_k(dx), 1 \leq k \leq N$, which satisfy the ratio:

$$\pi_k(dx)q(x) = q_k \rho_k(dx),\ q_k := \int_{E_k} \pi_k(dx)q(x).$$

ME3: The average exit probabilities

$$\widehat{p}_k := \int_{E_k} \rho_k(dx) P_1(x, E\backslash E_k) > 0, 1 \le k \le N.$$

Thus, the perturbing kernel $P_1(x, B)$ determines the transition probabilities between classes $E_k, 1 \le k \le N$. So, the equality $P^\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B)$ means that the embedded Markov chain $x_n^\varepsilon, n \ge 0$ spends a long time in each of the classes E_k and jumps between the classes with small probabilities $\varepsilon P_1(x, E\backslash E_k)$.

Under conditions **ME1–ME3,** there is a weak convergence (see Korolyuk and Limnios (2005, Chapter 4))

$$v(x^\varepsilon(t)) \Rightarrow \widehat{x}(t), \varepsilon \to 0,\ v(x) = k \in \widehat{E} = \{1, ..., N\}, x \in E_k.$$

The limit Markov process $\widehat{x}(t), t \ge 0$ on a merging phase space $\widehat{E} = \{1, ..., N\}$ is determined by the generating matrix

$$\widehat{Q}_1 = (\widehat{q}_{kr}, 1 \le k, r \le N),$$

where:

$$\widehat{q}_{kr} = \widehat{q}_k \widehat{p}_{kr}, k \ne r, \widehat{q}_k = q_k \widehat{p}_k, 1 \le k \le N.$$

$$\widehat{p}_{kr} = p_{kr}/\widehat{p}_k, p_{kr} = \int_{E_k} \rho_k(dx) P_1(x, E_r), 1 \le k, r \le N, k \ne r,$$

$$\widehat{p}_k = -\int_{E_k} \rho_k(dx) P_1(x, E_k).$$

ME4: The merged Markov process $\widehat{x}(t), t \ge 0$ is ergodic, with the stationary distribution $\widehat{\pi} = (\pi_k, k \in \widehat{E})$.

Thus, the operator Q^ε can be represented as:

$$Q^\varepsilon = Q + \varepsilon Q_1,\ Q_1(x) = q(x) \int_E P_1(x, dy)\varphi(y).$$

REMARK 1.2.– *Generalizations of this approach can be found in Yin and Zhang (1998, 2005), where the operator* $Q^\varepsilon = Q + \varepsilon Q_1$,

$$Q = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)],\ Q_1(x) = q_1(x) \int_E P_1(x, dy)\varphi(y).$$

Let Π be a projector on the null-subspace of a reducibly invertible operator Q. Its action on test functions φ is defined as follows:

$$\Pi\varphi(x) = \sum_{k=1}^{N} \widehat{\varphi}_k \mathbf{1}_k(x), \widehat{\varphi}_k := \int_{E_k} \pi_k(dx)\varphi(x).$$

Denote the reduced operator $\widehat{Q}_1$ using the ratio

$$\widehat{Q}_1\Pi = \Pi Q_1 \Pi.$$

Let $\widehat{\Pi}$ be a projector on the null-subspace of a reducibly invertible operator $\widehat{Q}_1$:

$$\widehat{\Pi}\widehat{\varphi} := \sum_{k\in\widehat{E}} \widehat{\pi}_k \widehat{\varphi}_k.$$

Let us denote the potential matrix $\widehat{R}_0 = [\widehat{R}^0_{kl}; 1 \le k, l \le N]$ by the following ratios:

$$\widehat{Q}_1\widehat{R}_0 = \widehat{R}_0\widehat{Q}_1 = \widehat{\Pi} - I.$$

1.1.6. *Processes with independent increments*

The Markov processes with independent increments in the Euclidean space $\mathbf{R}^d$ with the norm $|\cdot|$ will be denoted $\eta(t)$, $t \ge 0$. In the general case, such processes are determined by the corresponding generators (see Bertoin (1996, Chapter I.2) and Korolyuk and Limnios (2005, Chapter 1.2.4))

$$\widetilde{\mathbf{\Gamma}}\varphi(u) = b\varphi'(u) - \frac{\sigma^2}{2}\varphi''(u) + \int_{\mathbf{R}^d} [\varphi(u+v) - \varphi(u) - v\varphi'(u)\mathbf{1}_{(|v|\le 1)}]\widetilde{\Gamma}(dv), \quad [1.6]$$

where $\varphi(u)$ is a real-valued, twice differentiable function in $\mathbf{R}^d$, which is equal to 0 at infinity, and has a sup-norm $||\varphi|| = \sup_{u\in\mathbf{R}^d} |\varphi(u)|, \varphi(u) \in C_0^2(\mathbf{R}^d)$,

$$b = \int_{\mathbf{R}^d} v\Gamma(dv),$$

where $\Gamma(dv)$ is the intensity kernel that satisfies $\Gamma(\{0\}) = 0$.

In the following, we will study the asymptotic behavior of type [1.6] processes, whose generators can be represented as

$$\widetilde{\mathbf{\Gamma}}\varphi(u) = \sum_{k=1}^{d} b_k\varphi'_k(u) + \int_{\mathbf{R}^d} [\varphi(u+v) - \varphi(u) - \sum_{k=1}^{d} v_k\varphi'_k(u)\mathbf{1}_{(|v|\le 1)}]\widehat{\Gamma}(dv),$$

$$\varphi'_k(u) := \partial\varphi(u)/\partial u_k, 1 \le k \le d.$$

Note that despite the absence of a diffusion component at increasing time intervals in Poisson approximation schemes, the diffusion component in the limit process appears as a result of the averaging of small jumps in the initial process.

EXAMPLE 1.1.– *A compound Poisson process is an example of a Markov process with independent increments*

$$\eta(t) = \sum_{k=1}^{\nu(t)} \eta_k,$$

where $\nu(t), t \geq 0$*, is a homogeneous Poisson process with intensity* λ*, and* $\eta_k, k \geq 1$*, are independent and identically distributed real-valued random variables that do not depend on* $\nu(t), t \geq 0$*, and have a distribution function* $F(u)$*. The corresponding generator has the form*

$$\mathbf{\Gamma}\varphi(u) = \lambda \int_{\mathbf{R}} (\varphi(u+v) - \varphi(u)) F(dv).$$

We will also denote by $\eta(t)$, $t \geq 0$, a Markov processes with locally independent increments (also known as PDMP, piecewise deterministic Markov process, Davis (1993)) in the Euclidean space $\mathbf{R}^d$. Such processes are determined by the generator (see Korolyuk and Limnios (2005, Chapter 1.2.5), and Korolyuk (1999))

$$\mathbf{\Gamma}\varphi(u) = b(u)\varphi'(u) + \int_{\mathbf{R}^d} [\varphi(u+v) - \varphi(u) - v\varphi'(u)\mathbf{1}_{(|v|\leq 1)}]\Gamma(u, dv). \quad [1.7]$$

When the intensity kernel is bounded, the form of the generator is simplified, namely:

$$\widetilde{\mathbf{\Gamma}}\varphi(u) = b\varphi'(u) + \int_{\mathbf{R}^d} [\varphi(u+v) - \varphi(u)]\widetilde{\Gamma}(dv) \quad [1.8]$$

for a process with independent increments, and

$$\mathbf{\Gamma}\varphi(u) = b(u)\varphi'(u) + \int_{\mathbf{R}^d} [\varphi(u+v) - \varphi(u)]\Gamma(u, dv) \quad [1.9]$$

for a processes with locally independent increments (see Korolyuk and Limnios (2005, Chapter 1.2.5)).

1.1.7. *Poisson approximation scheme for the processes with independent increments*

The term *Poisson approximation* should be understood in a context similar to the averaging scheme or diffusion approximation, which allows us to study the limit

behavior of random processes at increasing time intervals. Therefore, it is not a classic problem of approximating a random process by Poisson processes.

The main idea of the Poisson approximation (see Korolyuk and Limnios (2005, Chapter 7)) is that the small series parameter normalizes the probabilities (or intensities) of jumps. Thus, the jumps are divided into two types: small jumps with probabilities close to 1 and large jumps that occur with a probability going to 0, together with a small series parameter $\varepsilon \to 0$.

EXAMPLE 1.2.– *Here is a simple example of a random variable with similar properties. For α let:*

$$P\{\alpha = b\} = \varepsilon^2 p,$$

$$P\{\alpha = \varepsilon a_1 + \varepsilon^2 b_1\} = 1 - \varepsilon^2 p.$$

Then, we have:

$$\mathbf{E}\alpha = \varepsilon a_1 + \varepsilon^2(bp + b_1) + o(\varepsilon^2),$$

$$\mathbf{E}\alpha^2 = \varepsilon^2(b^2 p + a_1^2) + o(\varepsilon^2).$$

Such moment conditions characterize the Lévy approximation.

When $a_1 = 0$, we will have

$$\mathbf{E}\alpha = \varepsilon^2(bp + b_1) + o(\varepsilon^2),$$

$$\mathbf{E}\alpha^2 = \varepsilon^2 b^2 p + o(\varepsilon^2),$$

and therefore, by putting $\widetilde{\varepsilon} = \varepsilon^2$, we obtain the moment conditions that characterize the Poisson approximation:

$$\mathbf{E}\alpha = \widetilde{\varepsilon}(bp + b_1) + o(\widetilde{\varepsilon}),$$

$$\mathbf{E}\alpha^2 = \widetilde{\varepsilon} b^2 p + o(\widetilde{\varepsilon}).$$

In the study of processes with independent increments, the moment conditions are imposed on the corresponding intensity kernel, normalized by a small series parameter.

Let $C_3(\mathbf{R}^d)$ be a class of functions that defines a measure and includes real-valued bounded functions, such as $g(u)/|u|^2 \to 0$, as $|u| \to 0$ if $g \in C_3(\mathbf{R}^d)$ (see Jacod and Shiryaev 2003; Korolyuk and Limnios 2005).

Consider a family of normalized Markov processes with trajectories in $D_{\mathbf{R}}[0,\infty)$ and independent increments in the series scheme, with a small series parameter $\varepsilon \to 0, \varepsilon > 0$:

$$\eta_\varepsilon(t) = \eta(t/\varepsilon), t \geq 0,$$

determined by generators (see [1.8])

$$\widetilde{\mathbf{\Gamma}}^\varepsilon \varphi(u) = \varepsilon^{-1} \int_{\mathbf{R}} [\varphi(u+v) - \varphi(u)] \widetilde{\Gamma}^\varepsilon(dv).$$

Let the conditions of the Poisson approximation be satisfied:

C1: *Poisson approximation*.

PA1 Approximation of means:

$$b_\varepsilon = \int_{\mathbf{R}} v \widetilde{\Gamma}^\varepsilon(dv) = \varepsilon[b + \theta_b^\varepsilon],$$

and

$$c_\varepsilon = \int_{\mathbf{R}} v^2 \widetilde{\Gamma}^\varepsilon(dv) = \varepsilon[c + \theta_c^\varepsilon],$$

where

$$b < +\infty, c < +\infty.$$

PA2 For the intensity kernel, there is an asymptotic representation:

$$\widetilde{\Gamma}_g^\varepsilon = \int_{\mathbf{R}} g(v) \widetilde{\Gamma}^\varepsilon(dv) = \varepsilon[\widetilde{\Gamma}_g + \theta_g^\varepsilon]$$

for all $g \in C_3(\mathbf{R})$, which is a class of functions that defines a measure (see Jacod and Shiryaev 2003).

The kernel $\widetilde{\Gamma}^0(dv)$ is set on the class of functions that defines the measure $C_3(\mathbf{R})$ by the ratio

$$\widetilde{\Gamma}_g = \int_{\mathbf{R}} g(v) \widetilde{\Gamma}^0(dv), \quad g \in C_3(\mathbf{R}).$$

The negligibly small terms $\theta_b^\varepsilon, \theta_c^\varepsilon, \theta_g^\varepsilon$ satisfy the condition

$$|\theta_\cdot^\varepsilon| \to 0, \quad \varepsilon \to 0.$$

PA3 There is a relationship:

$$c := \int_{\mathbf{R}} v^2 \widetilde{\Gamma}^0(dv),$$

which causes the absence of a diffusion component in the limit generator.

C2 *Uniform quadratic integrability*:

$$\lim_{c\to\infty}\int_{|v|>c} v^2\widetilde{\Gamma}^0(dv)=0.$$

LEMMA 1.1.– *Generator of the process with independent increments*

$$\widetilde{\mathbf{\Gamma}}^\varepsilon\varphi(u)=\varepsilon^{-1}\int_{\mathbf{R}}[\varphi(u+v)-\varphi(u)]\widetilde{\Gamma}^\varepsilon(dv)$$

in the scheme of Poisson approximation has the following asymptotic representation:

$$\widetilde{\mathbf{\Gamma}}^\varepsilon\varphi(u)=b\varphi'(u)+\int_{\mathbf{R}}[\varphi(u+v)-\varphi(u)-v\varphi'(u)]\widetilde{\Gamma}^0(dv)+\theta^\varepsilon\varphi,$$

where $|\theta^\varepsilon\varphi|\to 0,\varepsilon\to 0,\varepsilon>0.$

REMARK 1.3.– *In the space* $\mathbf{R}^d$*, the asymptotic representation has the form*

$$\widetilde{\mathbf{\Gamma}}^\varepsilon\varphi(u)=\sum_{k=1}^d b_k\varphi'_k(u)+\int_{\mathbf{R}^d}[\varphi(u+v)-\varphi(u)-\sum_{k=1}^d v_k\varphi'_k(u)]\widehat{\Gamma}^0(dv)+\theta^\varepsilon\varphi,$$

$$\varphi'_k(u):=\partial\varphi(u)/\partial u_k, 1\le k\le d.$$

PROOF.– Rewrite the expression for the generator as follows:

$$\widetilde{\mathbf{\Gamma}}^\varepsilon\varphi(u)=\varepsilon^{-1}\int_{\mathbf{R}}[\varphi(u+v)-\varphi(u)-v\varphi'(u)-\frac{v^2}{2}\varphi''(u)]\widetilde{\Gamma}^\varepsilon(dv)$$

$$+\varepsilon^{-1}\int_{\mathbf{R}}v\varphi'(u)\widetilde{\Gamma}^\varepsilon(dv)+\frac{\varepsilon^{-1}}{2}\int_{\mathbf{R}}v^2\varphi'(u)\widetilde{\Gamma}^\varepsilon(dv).$$

It is easy to see that the function $\psi_u(v)=\varphi(u+v)-\varphi(u)-v\varphi'(u)-\frac{v^2}{2}\varphi''(u)$ belongs to the class $C_3(\mathbf{R})$. Really,

$$\psi_u(v)/v^2\to 0, v\to 0$$

uniformly by u, by the condition of derivative boundedness of the function $\varphi(u)$ on a compact. In addition, this function is continuous and bounded for $\varphi(u)\in C_0^2(\mathbf{R})$ by condition **PA1**.

Thus, from conditions **PA1, PA2**, we have:

$$\widetilde{\mathbf{\Gamma}}^\varepsilon\varphi(u)=\int_{\mathbf{R}}[\varphi(u+v)-\varphi(u)-v\varphi'(u)-\frac{v^2}{2}\varphi''(u)]\widetilde{\Gamma}^0(dv)$$

$$+b\varphi'(u)+\frac{c}{2}\varphi''(u)+\theta_b^\varepsilon\varphi+\theta_c^\varepsilon\varphi+\theta_\psi^\varepsilon\varphi.$$