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Manoj Sahni
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Preface

We are delighted to provide the conference proceedings of the 2nd International Conference on Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy (MMCITRE2021), which took place in Pandit Deendayal Energy University (formerly known as Pandit Deendayal Petroleum University), Gandhinagar, Gujarat, on February 6–8, 2021. Due to the COVID-19 pandemic, the conference was streamed live using the Zoom application and the sessions were held in hybrid mode using Microsoft Teams. The primary goal of the conference is to bring together academics, researchers, professionals and educators to interact and share experiences and research results on topics related to science, engineering, computers and mathematics.

This conference is attended by many researchers, scholars and industry persons from nearly all over India as well as many other countries including the USA, UK, Australia, Jordan, Spain, Japan, Chile, Oman, Nepal, etc. Papers in the form of a parallel oral presentation were delivered. This book comprises research papers on numerous topics, primarily focused on the mathematical modeling of many fields, situations based on uncertainty, the modeling of energy systems, statistical analysis, optimization approaches, etc.

Since this conference was organized in a hybrid mode, many researchers from Ahmedabad and Gandhinagar only participated in person and visited Pandit Deendayal Energy University (PDEU), which is the Gujarat's top private university. The 100-acre campus of PDEU is located in Gandhinagar, Gujarat. It provides numerous courses ranging from engineering, arts and management to its students through varied national and international exchange programs with the best universities all around the world. It was set up as a private university by GERMI under the State Act on April 4, 2007. The university has broadened the scope of its programs since its foundation in 2007, delivering a wide variety of courses in technology, management, petroleum, solar and nuclear energy and liberal education through various SOT, SPT, SPM and SLS schools in a relatively short span of time. It aims to extend students and professionals' possibilities to gain key subject knowledge which is appropriately supported by leadership training activities and helps students to create a worldwide imprint. A variety of well-planned courses such as undergraduates, postgraduates and doctorate programs and intensive research initiatives pursue this aim further.

This was the second international conference on the same topic, "Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy (MMCITRE2021)" hosted by the Department of Mathematics, PDEU. The conference's keynote and prominent presenters were requested to participate in the conference as well as in the review process. A significant number of research publications from all around the world were submitted to this international conference. All of these papers are rigorously reviewed by experts in the relevant fields in a double-blind peer review procedure, and only the highest quality research papers are chosen for oral presentation at the conference. Finally, papers that passed the review process were chosen, and a total of 42 research papers are included in this proceedings in the form of chapters based on the quality of work assessed by experts from various fields. These chapters were considered to be relevant not only for researchers but also for postgraduate and undergraduate students in the fields of physics, mathematics and engineering as well as for industrial persons working in many sectors, such as medical, energy and stock market. Both new basic mathematical discoveries and mathematical and computational approaches utilized in interdisciplinary applications are presented in these articles.

We must learn how to deal with new and different challenges at a moment of instability and change on every societal sphere. The growth of future academicians, researchers, programmers, educators and industries is strongly dependent on their capacity to use mathematical tools in diverse applications in real life. In addition, this conference has been organized to appreciate how young researcher and educationalist and prominent scientist should disseminate new information and progress in all aspects of computational and mathematical advancements and their applications. This knowledge must be turned into transformational leadership that motivates and assists practitioners in making meaningful changes in their communities. This conference is a very significant interconnection in the network of change that many believe will lead to a future, more conscious and capable general population.

Basically, the conference is intended to give a place to promote and exchange knowledge of current research and achievements in the field of mathematics and mathematical education to scientific, research and educational staff. It might lead to fresh insights that allow us to form meaningful connections with others and have a good impact on society as a whole, however little. We believe that the papers in this proceedings will aid in broadening scientific understanding and enriching our mathematical abilities for new standard education and will benefit all the academics, researchers and industrialists looking for new mathematical tools.

Gandhinagar, India — Manoj Sahni
Broadway, Australia — José M. Merigó
Gandhinagar, India — Ritu Sahni
Santiago, Chile — Rajkumar Verma

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About the Editors

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Rajkumar Verma is working as Postdoctoral Research Fellow at the Department of Management Control and Information Systems at the University of Chile. Before joining the University of Chile, he was Assistant Professor at the Department of Mathematics, Delhi Technical Campus, India. He holds a M.Sc. degree in mathematics from Chaudhary Charan Singh University, Meerut, and a Ph.D. degree in Mathematics from the Jaypee Institute of Information Technology, Noida, India. He has published more than 45 research papers in journals and conference proceedings, including the *International Journal of Intelligent Systems*, *Journal of Intelligent & Fuzzy Systems*, *Kybernetika*, *Informatica*, *International Journal of Machine Learning and Cybernetics*, *International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems*, *Soft Computing*, *Applied Mathematical Modelling*, and *Neural Computing and Applications*. He has contributed to the scientific committee of several conferences and associations. He also serves as Reviewer for many international journals. He is member of many international professional societies, including the American Mathematical Society, IEEE Computational Intelligence Society, Forum for Interdisciplinary Mathematics, Indian Mathematical Society, and Indian Society of Information Theory and Applications.

Advanced Mathematical Concepts

Strongly Prime Radicals and S-Primary Ideals in Posets

J. Catherine Grace John

Abstract The notion of the strongly prime radical of an ideal in posets is defined in this study. Also, we studied the concepts of S-primary ideals in posets. Characterizations of S-primary ideals with respect to strongly prime radical are discussed. Further, the S-primary decomposition of an ideal is obtained.

Keywords Poset · Ideals · Strongly prime ideal · Strongly m-system · Strongly prime radical · S-primary

1 Introduction

Various radicals play an important role in algebraic structures. All maximal ideals intersection in a commutative ring with unity is named as Jacobson radical, and all prime ideals intersection is called prime radical of ring. The primary ideal that was a development of prime ideal principles was launched using the radical notion [1].

In mathematics, the theory of primary ideals is crucial, particularly in abstract algebra, since a classic pillar of ideal theory is the deconstruction of an ideal into primary ideals. It offers the algebraic basis for decomposing an algebraic variety into its irreducible components. In another sense, primary decomposition is just an extension of the unique-prime-factorization theorem, which states that in number theory, each integer higher than 1 is either a prime number or can be represented as the product of prime integers and that this representation is unique.

Theory of primary ideals played a major position of significance in commutative ring theory, and then, it was taken to commutative semi-groups [2].

Anjaneyulu [3] developed the theory of primary ideals in the arbitrary semigroup. Satyanarayana [11] developed commutative primary semi-groups, in which each ideal in the semi-group is primary. He distinguishes its structure from that of

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commutative primary rings. In addition, he established the necessary and sufficient condition for semi-group to primary semi-group.

Badawi [4] established the notion of a 2-absorbing primary ideal for a commutative ring R with $1 \neq 0$ and established certain 2-absorbing primary ideal features. He also presented a few examples of primary ideals that are 2-absorbing.

Murata [10] applied the idea of a primary ideal into a compactly constructed multiplicative lattice, made use of the m-system. Further, he developed a primary decomposition theorem for ideals in lattice. Joshi [8] later expanded the primary ideal notion to all posets.

Following [8], we have studied the notion of S-primary ideals in posets in this paper.

2 Preliminaries

Throughout the whole paper, a poset with the smallest element 0 is represented by $(\mathbb{X}, \leq)$. We refer [1, 3] to the terminology for fundamental definitions and notations for posets. For $W \subseteq \mathbb{X}$, $W^{\ell} = \{r \in \mathbb{X} : r \leq a \, \forall a \in W\}$ $(W^{u} = \{r \in \mathbb{X} : a \leq r \, \forall a \in W\})$ indicate a lower (upper) cone of W in $\mathbb{X}$. For $H, W \subseteq \mathbb{X}$, it may express $(H, W)^{\ell}$ instead $(H \cup W)^{\ell}$ and $(H, W)^{u}$ instead of $(H \cup W)^{u}$.

If $P = \{r_1, r_2, \ldots, r_n\}$ is a finite set of $\mathbb{X}$, then we are using the notation for the $(r_1, r_2, \ldots, r_n)^{\ell}$ instead of the $(\{r_1, r_2, \ldots, r_n\})^{\ell}$ and dually for the notion of upper cone. This is undeniable that for any subset D of $\mathbb{X}$, we have $D \subseteq D^{u\ell}$ and $D \subseteq D^{\ell u}$. If $D \subseteq E$, then we have $E^{\ell} \subseteq D^{\ell}$ and $E^{u} \subseteq D^{u}$. Also, $D^{\ell u \ell} = D^{\ell}$ and $D^{u \ell u} = D^{u}$.

Following [4], a subset $D(\neq \phi)$ of $\mathbb{X}$ is referred as semi-ideal of $\mathbb{X}$ if $s \in D$ and $r \leq s$, then $r \in D$. Let $D \subseteq \mathbb{X}$. Then, D is referred as ideal if $q, w \in D$ implies $(q, w)^{u\ell} \subseteq D$ [9]. $Id(\mathbb{X})$ denotes set of all ideals in $\mathbb{X}$.

Let D be a proper semi-ideal (ideal) of $\mathbb{X}$. Then D is said to be prime whenever $(s, q)^{\ell} \subseteq D$ implies either $s \in D$ or $q \in D$ for all $s, q \in \mathbb{X}$ [3]. An ideal D of $\mathbb{X}$ is termed as semi-prime whenever $(s, v)^{\ell} \subseteq D$ and $(s, w)^{\ell} \subseteq D$ together imply $(s, (v, w)^{u})^{\ell} \subseteq D$ for all $s, v, w \in \mathbb{X}$ [9].

Given $w \in \mathbb{X}$, a principal ideal of $\mathbb{X}$ generated by an element w is $(w] = (w)^{\ell} = \{k \in \mathbb{X} : k \leq w\}$, and a principle filter of $\mathbb{X}$ constructed by an element w is $[w) = (w)^{u} = \{k \in \mathbb{X} : k \geq w\}$.

Following [6], an ideal D of $\mathbb{X}$ is referred as strongly prime if whenever $(I^*, W^*)^{\ell} \subseteq D$ implies either $I \subseteq D$ or $W \subseteq D$ for all different proper ideals I, W of $\mathbb{X}$, where $I^* = I \backslash \{0\}$. An ideal D of $\mathbb{X}$ has been said that strongly semi-prime if $(I^*, J^*)^{\ell} \subseteq D$ and $(I^*, S^*)^{\ell} \subseteq D$ together imply $(I^*, (J^*, S^*)^{u})^{\ell} \subseteq D$ for different proper ideals I, J and S of $\mathbb{X}$.

Following [5], a subset $S(\neq \phi)$ of $\mathbb{X}$ is referred as m-system if $\forall w_1, w_2 \in S$, there exists $r \in (w_1, w_2)^{\ell}$ such that $r \in S$.

Strongly m-system is defined as an extension of m-system as seen below: A subset $S \neq \phi$ of $\mathbb{X}$ is termed strongly m-system if $I \cap S \neq \phi$, $J \cap S \neq \phi$ imply that $(I^*, J^*)^{\ell} \cap S \neq \phi$ for any proper different ideals I, J of $\mathbb{X}$.

It needs to be noted that for an ideal D of $\mathbb{X}$, D is strongly prime $\Leftrightarrow \mathbb{X}\backslash D$ is a strongly m-system in $\mathbb{X}$. Each strongly m-system is also a m-system. However, in general, reverse part does not need to be true.

3 Main Results

Definition 1 Let D be an ideal of $\mathbb{X}$. We have defined strongly prime radical $sr(D)$ of D to be the set of all $c \in \mathbb{X}$ such that every strongly m-system of $\mathbb{X}$ which contains c has a non-empty intersection with D.

Theorem 1 *Let D_1 and D_2 be ideals of a poset $\mathbb{X}$. Then*

(i) $D_1 \subseteq sr(D_1)$.
(ii) $sr(sr(D_1)) = sr(D_1)$.
(iii) *If $D_2 \subseteq D_1$, then $sr(D_2) \subseteq sr(D_1)$.*
(iv) $sr((D_1^*, D_2^*)^\ell) = sr(D_1 \cap D_2) = sr(D_1) \cap sr(D_2)$.
(v) *In case, D_1 is a strongly prime ideal of $\mathbb{X}$ which implies that $sr(D_1) = D_1$.*
(vi) *If D_1 is strongly prime ideal and $D_2 \subseteq D_1$, then $sr(D_2) \subseteq D_1$.*

Proof (i) Let $c \in D_1$. Then, obviously every strongly m-system containing c has a non-empty intersection with D_1. Therefore, $c \in sr(D_1)$.

(ii) Let $c \in sr(sr(D_1))$ and assume that $c \notin sr(D_1)$. Then, a strongly m-system M_c exists such that $c \in M_c$ and $M_c \cap D_1 = \phi$. Since $c \in sr(sr(D_1))$, we have got $M_c \cap sr(D_1) \neq \phi$. Let $p \in M_c \cap sr(D_1)$. As $p \in sr(D_1)$, then every strongly m-system containing p must intersect D_1. So, in particular $M_c \cap D_1 \neq \phi$, a contradiction.

(iii) It is trivial.

(iv) For any ideals D_1 and D_2 of $\mathbb{X}$, we have $(D_1^*, D_2^*)^\ell \subseteq D_1 \cap D_2 \subseteq D_1$. Then, by (iii), we have $sr((D_1^*, D_2^*)^\ell) \subseteq sr(D_1 \cap D_2) \subseteq sr(D_1)$ which implies $sr((D_1^*, D_2^*)^\ell) \subseteq sr(D_1 \cap D_2) \subseteq sr(D_1) \cap sr(D_2)$. Let $x \in sr(D_1) \cap sr(D_2)$ and K be a strongly m-system of $\mathbb{X}$ containing x. Then, we have $K \cap D_1 \neq \phi$ and $K \cap D_2 \neq \phi$ which implies $(D_1^*, D_2^*)^\ell \cap K \neq \phi$ which implies $x \in sr((D_1^*, D_2^*)^\ell)$.

(v) Let D_1 be strongly prime ideal of $\mathbb{X}$. Suppose $sr(D_1) \nsubseteq D_1$. Then, there exists $c \in sr(D_1)$ such that $c \notin D_1$. As D_1 is strongly prime, we got $\mathbb{X}\backslash D_1$ is a strongly m-system of $\mathbb{X}$ containing c and $(\mathbb{X}\backslash D_1) \cap D_1 = \phi$, a contradiction to the fact that $c \in sr(D_1)$. Hence, $D_1 = sr(D_1)$.

(vi) Let D_1 and D_2 be ideals of $\mathbb{X}$ and D_1 be strongly prime such that $D_2 \subseteq D_1$. Then, by (iii) and (v), we have $sr(D_2) \subseteq sr(D_1) = D_1$. □

Corollary 1 *Let $\mathbb{X}$ be a poset and D be a maximal ideal of $\mathbb{X}$. If D is strongly semi-prime, then $sr(D) = D$.*

Theorem 2 ([6], Theorem 2.1) *Let T be a non-void strongly m-system of $\mathbb{X}$ and K be an ideal of $\mathbb{X}$ with $K \cap T = \phi$. Then, K is contained in a strongly prime ideal D of $\mathbb{X}$ with $D \cap T = \phi$.*

Theorem 3 *For an ideal B of $\mathbb{X}$, we have $\{\bigcap_i Q_i : Q_i$ is strongly prime ideal of $\mathbb{X}$ containing $B\} = \{c \in \mathbb{X}$: every strongly m-system of S which contains c has a non-empty intersection with $B\}$.*

Proof Let $H = \{c \in \mathbb{X}$: Every strongly m-system of $\mathbb{X}$ which contains c has a non-empty intersection with $B\}$ and $c \notin H$. Then, a strongly m-system S of $\mathbb{X}$ which contains c and $S \cap B$ is an empty set. With the support of Theorem 2, $\exists$ a strongly prime ideal Q of $\mathbb{X} \ni B \subseteq Q$ together with $Q \cap S = \phi$ implies that $c \notin \cap Q_i$. So, $\cap Q_i \subseteq H$. Conversely, let $c \notin \cap Q_i$. Then, there will be a strongly prime ideal Q_i of $\mathbb{X}$ with $c \notin Q_i$ which implies $c \in \mathbb{X} \backslash Q_i$ and $\mathbb{X} \backslash Q_i$ is a strongly m-system in $\mathbb{X}$. Since $(\mathbb{X} \backslash Q_i) \cap B = \phi$, we have $c \notin H$. Hence $H \subseteq \cap Q_i$. □

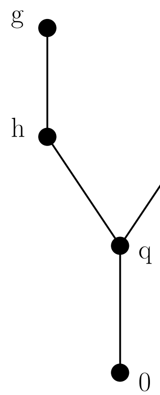
Definition 2 For an ideal D of $\mathbb{X}$ and a strongly prime ideal B of $\mathbb{X}$ with $D \subseteq B$, B is referred as minimal strongly prime ideal of D if there is not any strongly prime ideal K of $\mathbb{X}$ with $D \subset K \subset B$.

$Sspec(\mathbb{X})$ represents all strongly prime ideal collections of $\mathbb{X}$, and $Smin(\mathbb{X})$ indicates all minimal strongly prime ideal collections of $\mathbb{X}$. For any ideal D of $\mathbb{X}$, $SP(D) = \bigcap_{Q_i \supseteq D} Q_i$ and $SP(\mathbb{X}) = \bigcap Q_i$, where Q_is are strongly prime ideal of $\mathbb{X}$. We also have $SP(D) = SP(\mathbb{X})$ if $D = 0$.

Remark 1 For an ideal D of $\mathbb{X}$, we have $SP(D) = sr(D)$. Moreover, $sr(D)$ is an ideal of $\mathbb{X}$ because $\bigcap_{D \in Id(\mathbb{X})} D$ remains an ideal in $\mathbb{X}$. From [7], for an ideal D of $\mathbb{X}$, $\bigcap_{P_i \supseteq D} P_i = D$, where P_is are prime ideals in $\mathbb{X}$

But here is an illustration for $\bigcap_{Q_i \supseteq D} Q_i \neq D$, where Q_is are strongly prime ideals in $\mathbb{X}$.

Example 1 Let $\mathbb{X} = \{0, q, r, h, g\}$ be a poset with the relation $\leq$ on $\mathbb{X}$ as follows:

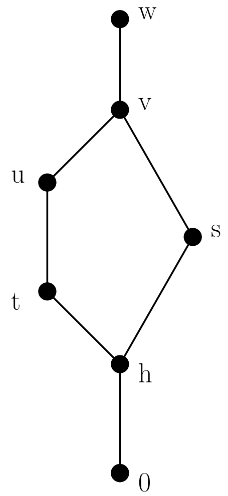


Here, $I_1 = \{0, q, r\}$; $I_2 = \{0, q, h, g\}$ are strongly prime ideals of $\mathbb{X}$, and the ideal $D_1 = \{0, q\}$ gives $SP(D_1) = I_1 \cap I_2 = \{0, q\} = D_1$. But for the ideal $D_2 = \{0, q, h\}$, we have $SP(D_2) = I_2 = \{0, q, h, g\} \neq D_2$. □

Definition 3 An ideal $B(\subsetneq X)$ of $\mathbb{X}$ is referred as S-primary, if $(I^*, D^*)^\ell \subseteq B$ imply that $I \subseteq B$ or $D \subseteq sr(B)$ for any different proper ideals I, D of $\mathbb{X}$ and $I^* = I\backslash\{0\}$.

It is a prompt findings that in a poset $\mathbb{X}$, each strongly prime ideal is also a S-primary ideal. However, reverse does not need to be valid in general using the below example.

Example 2 Let $\mathbb{X} = \{0, h, s, t, u, v, w\}$ be a poset with the relation $\leq$ on $\mathbb{X}$ as follows:



Here, $I = \{0, h, t\}$ is a S-primary ideal of $\mathbb{X}$, but not a strongly prime ideal $(((u)^\ell)^*, ((s)^\ell)^*) \subseteq I$ and $(u)^\ell) \nsubseteq I$ and $(s)^\ell \nsubseteq I$. □

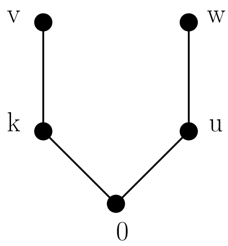
Definition 4 Let D be an ideal of $\mathbb{X}$. Then, D is referred as S_Q-primary if D is S-primary with $sr(D) = Q$, for some strongly prime ideal Q of $\mathbb{X}$.

The S-primary decomposition of D is an expression of the type $D = K_1 \cap K_2 \cap \cdots \cap K_n$, where each K_i is a S_{Q_i}-primary.

Definition 5 Let $\mathbb{X}$ be a poset and D be an ideal of $\mathbb{X}$ with a S-primary decomposition $D = K_1 \cap K_2 \cap \cdots \cap K_m$. A S-primary decomposition of D is called minimal if $K_i \nsupseteq \bigcap_{j \neq i} K_j$ for every $i = 1, 2, \ldots, m$ and all these $K_i's$ are distinct.

Definition 6 For a poset $\mathbb{X}$ and an ideal D of $\mathbb{X}$, $D = \bigcap_{i=1}^{m} K_i$ with $sr(K_i) = Q_i$, $i = 1, 2, \ldots, m$ be a minimal S-primary decomposition of D in $\mathbb{X}$. Then, the strongly prime ideals Q_i, $i = 1, 2, \ldots, m$, are said to be collection of associated strongly prime ideals of the decomposition. Moreover, D is called decomposable if the S-primary decomposition exists.

Example 3 Let $\mathbb{X} = \{0, k, u, v, w\}$ be a poset with the relation $\leq$ on $\mathbb{X}$ as follows:



Here, $D = \{0, k\}$ is a S_Q-primary ideal of $\mathbb{X}$ where $sr(D) = (v] = Q$. □

Remark 2 For a semi-ideal D of $\mathbb{X}$ and $V \subseteq \mathbb{X}$, we have described $\langle V, D\rangle = \{q \in \mathbb{X} : L(v, q) \subseteq D \ \forall v \in V\} = \bigcap_{v\in V} \langle v, D\rangle$ [5]. If $V = \{z\}$,, then we are writing $\langle\{z\}, D\rangle = \langle z, D\rangle$. We say that D satisfies $(*)$ condition if $(S, Q)^\ell \subseteq D$ implies $S \subseteq \langle Q, D\rangle \, \forall S, Q \subseteq \mathbb{X}$ [6].

Theorem 4 *Let D be a S_Q-primary ideal of a poset $\mathbb{X}$ for some strongly prime Q of $\mathbb{X}$ and $v \in \mathbb{X}$. Then,*

(i) *$v \notin D$ and $\langle v, D\rangle$ has $(*)$ condition that implies that $\langle v, D\rangle$ is a S_Q-primary ideal;*
(ii) *$v \notin Q$ gives that $\langle v, D\rangle = D$.*

Proof (i) Let $h \in \langle v, D\rangle$, then $(((v)^\ell)^*, ((h)^\ell)^*)^\ell \subseteq (v, h)^\ell \subseteq D$. As D is S_Q-primary and $(v)^\ell \nsubseteq D$, we have got $h \in (h)^\ell \subseteq sr(D) = Q$ for some strongly primary ideal Q of $\mathbb{X}$. Therefore, $D \subseteq \langle v, D\rangle \subseteq Q$. By Theorem 1, $Q = sr(D) \subseteq sr(\langle v, D\rangle) \subseteq Q$. Thus, $Q = sr(\langle v, D\rangle)$. Now, we have established that $\langle v, D\rangle$ is S-primary. Consider $(((h)^\ell)^*, ((w)^\ell)^*)^\ell \subseteq \langle v, D\rangle$. If $(h)^\ell \nsubseteq \langle v, D\rangle$, then $L(v, h) \nsubseteq D$. So, there is a $t \in L(v, h)$ and $t \notin D$. Consequently, we get $sr(\langle t, D\rangle) = Q$. Since $L(t, w) \subseteq L(h, w) \subseteq \langle v, D\rangle$, we have $L(v, t, w) \subseteq D$. As $t \leq v$, we have $L(t, w) \subseteq D$. Accordingly, $w \in \langle t, D\rangle \subseteq sr(\langle t, D\rangle) = Q = sr(\langle v, D\rangle$. Thus, $w \in sr(\langle v, D\rangle)$ and $(w)^\ell \subseteq sr(\langle v, D\rangle)$. On the other side, if $(h)^\ell \nsubseteq sr(\langle v, D\rangle) = Q$, then we are showing that $(w)^\ell \subseteq \langle v, D\rangle$. Let $t \in (v, w)^\ell$. So, $(t, h)^\ell \subseteq (v, h, w)^\ell \subseteq D$. Since D is S-primary and $h \notin Q = sr(D)$, We have got $t \in D$.

(ii) Assume that $v \notin Q$. Suppose that $D \subset \langle v, D\rangle$. Then there remains that there is $t \in \langle v, D\rangle$ and $t \notin D$. As D is S_Q-primary and $v \notin Q = sr(D)$, we have $t \in D$, which is a contradiction. Hence, $\langle v, D\rangle = D$. □

Theorem 5 *Consider a decomposable ideal W of $\mathbb{X}$, if $W = \bigcap_{i=1}^{m} D_i$ is a minimal S-primary decomposition of W, where $Q_i = sr(D_i)$, $i = 1, 2, 3 \ldots m$ be associated strongly prime ideals of the decomposition, then each strongly prime ideal of the form $sr(\langle h, W\rangle)$ for some $h \in \mathbb{X}$ is one of the associated strongly prime ideals Q_i for some i, and moreover, for each associated strongly prime ideal Q_i, $\exists h_i \in \mathbb{X} \ni sr(\langle h_i, W\rangle) = Q_i$.*

Proof Consider $W = D_1 \cap D_2 \cap \cdots \cap D_m$. For $h \notin W$, $\langle h, W\rangle = \langle h, \bigcap_{i=1}^{m} D_i\rangle = \bigcap_{i=1}^{m}\langle h, D_i\rangle$. Hence $\langle h, W\rangle = \bigcap_{h\notin D_j}(\langle h, D_j\rangle)$, $1 \le j \le m$. By Theorem 1, we get $sr(\langle h, W\rangle) = sr(\bigcap_{h\notin D_j}\langle h, D_j\rangle) = \bigcap_{h\notin D_j} sr\langle h, D_j\rangle$, $1 \le j \le m$. If $sr(\langle h, W\rangle)$ is strongly prime, thereafter it needs to be noted that $sr(\langle h, W\rangle) = Q_j$, for some j, $1 \le j \le m$. Therefore, each strongly prime ideal of the structure $sr(\langle h, W\rangle)$ is one of the Q_js for some j, $1 \le j \le m$. Take the associated strongly prime ideal Q_j, $1 \le j \le m$. We must look out $h \in \mathbb{X} \ni sr(\langle h, W\rangle) = Q_j$. As the decomposition of W is minimal, so $D_j \not\supseteq \bigcap_{i\neq j} D_i$ for each $j \in \{1, 2, 3, \ldots, m\}$. It gives that $\exists\, h_j \in \bigcap_{i\neq j} D_i$ and $h_j \notin D_j$. Now, $\langle h_j, W\rangle = \langle h_j, \bigcap_{i=1}^{n} D_j\rangle = \bigcap_{i=1}^{n}\langle h_j, D_i\rangle$. Since $h_j \in \bigcap_{i\neq j} D_i$ we get that $\langle h_j, W\rangle = \langle h_j, D_j\rangle$. This implies that $sr(\langle h_j, W\rangle) = sr(\langle h_j, D_j\rangle) = Q_j$, by Theorem 4. □

Example 4 In Example 1, consider the ideal $W = (q]$. Observe that $(q] = (r] \cap (h] \cap (g]$ is a S-primary decomposition of $W = (q]$ and a minimal S-primary decomposition of $(q]$ is $(q] = (r] \cap (h]$. Further, $Sr((r]) = (r]$ and $sr((h]) = (g]$. Thus, $(r]$ and $(g]$ are associated strongly prime ideals of the minimal S-primary decomposition of $W = (q]$. For the associated strongly prime ideal $(r]$, there exists h such that $sr(\langle h, W\rangle) = (r]$, and for that associated strongly prime ideal $(g]$, there exists r such that $sr(\langle r, W\rangle) = (g]$.

Following [5], let $\mathbb{X}$ be a poset and B be an ideal of $\mathbb{X}$. For a strongly prime ideal Q of $\mathbb{X}$, we have stated $B_Q = \{w \in \mathbb{X} : (w, s)^{\ell} \subseteq B \text{ for some } s \in \mathbb{X}\backslash Q\} = \bigcup_{s\in\mathbb{X}\backslash Q}\langle s, B\rangle$.

Theorem 6 *Let B be an ideal of $\mathbb{X}$ with $B = \bigcap_{i=1}^{n} D_i$, a minimal S-primary decomposition of B, where $sr(D_i) = Q_i$ be associated strongly prime ideals of the decomposition. Then, the below proclamation holds. If Q is strongly prime and $B \subseteq Q$ which also contains $Q_1, Q_2, \ldots, Q_k$ $(1 \le k \le n)$ but does not having $Q_{k+1}, Q_{k+2}, \ldots, Q_n$, then $B_Q = D_1 \cap D_2 \cap \cdots \cap D_k$ and if Q contains none of the Q_i's then $B_Q = \mathbb{X}$.*

Proof Suppose that Q contains $Q_1, Q_2, \ldots, Q_k$, but does not contain Q_{k+1}, $Q_{k+2}, \ldots, Q_n$. Let $v \in B_Q$, that is, for some $r \notin Q$, $(v, r)^{\ell} \subseteq B$. This implies that $(v, r)^{\ell} \subseteq D_i \subseteq sr(D_i)$, for each $i = 1, 2, \ldots, n$. It can be quickly found that r is precisely not in $Q_1, Q_2, \ldots, Q_k$. For otherwise, if $r \in Q_i$, for some $i \in \{1, 2, \ldots, k\}$, then by assumption, $r \in Q$, an incoherence. So, $r \notin sr(D_1), sr(D_2), \ldots, sr(D_k)$ which gives $(r)^{\ell} \nsubseteq sr(D_1), sr(D_2), \ldots, sr(D_k)$. As $D_1, D_2, \ldots, D_k$ are S-primary, we have got $(v)^{\ell} \in D_1, D_2, \ldots, D_k$. Hence, $(v)^{\ell} \in D_1 \cap D_2 \cap \cdots \cap D_k$. Conversely, let $x \in D_1 \cap D_2 \cap \cap \cap D_k$. As $Q_{k+1}, Q_{k+2}, \ldots, Q_n \nsubseteq Q$, there exist $v_{k+1} \in Q_{k+1}\backslash Q$, $v_{k+2} \in Q_{k+2}\backslash Q, \ldots, v_n \in Q_n\backslash Q$. Since $v_j \in Q_j = sr(D_j)$, $j = k+1, \ldots, n$, hence every strongly m-system including v_j intersects with D_j. In particular, $\mathbb{X}\backslash Q$ is a strongly m-system containing v_j which intersects D_j for every $j = k+1, \ldots, n$.

Choose $h_j \in D_j \cap (\mathbb{X}\backslash Q)$, $j = k+1, \ldots, n$. As Q is strongly prime, we have $(\{h_{k+1}, h_{k+2}, \ldots, h_n\})^\ell \nsubseteq Q$. Therefore, there exists $h \in (\{h_{k+1}, h_{k+2}, \ldots, h_n\})^\ell$ and $h \notin Q$. Thus, $(x, h)^\ell \subseteq \bigcap_{i=1}^{n} D_i = B$ with $h \notin Q$. Hence, $x \in B_Q$. Suppose $Q_i \nsubseteq Q$ for every $i = 1, 2, \ldots, n$. Then, $\exists\, v_i \in Q_i \backslash Q$ for each $i = 1, 2, \ldots, n$. Since $\mathbb{X}\backslash Q$ is a strongly m-system of $\mathbb{X}$, applying similar technical in procedure as above, we get $h_i \in D_i \cap (\mathbb{X}\backslash Q)$, $i = 1, \ldots, n$. As Q is strongly prime ideal, we get $(\{h_1, h_2, \ldots, h_n\})^\ell \nsubseteq Q$. Therefore, there exists $h \in (\{h_1, h_2, \ldots, h_n\})^\ell \ni h \notin Q$. It is obvious that $h \in \bigcap_{i=1}^{n} D_i = B$ and therefore for any $x \in \mathbb{X}$, we have $(x, h)^\ell \subseteq B$, which gives $x \in B_Q$. Hence, $B_Q = \mathbb{X}$. □

Theorem 7 *([6], Theorem 2.4) For an ideal B of $\mathbb{X}$, we have if B has the property below that for $n > 2$, if pairwise distinct ideals $K_1, K_2, \ldots, K_n$ of $\mathbb{X}$ with $(K_1^*, K_2^*, \ldots, K_n^*)^\ell \subseteq B$, then at least $(n-1)$ of n subsets $(K_2^*, K_3^*, \ldots, K_n^*)^\ell$, $(K_1^*, K_3^*, \ldots, K_n^*)^\ell, \ldots, (K_1^*, K_2^*, \ldots, K_{n-1}^*)^\ell$ are subsets of B.*

Theorem 8 *Let $\mathbb{X}$ be a poset and B be an ideal of $\mathbb{X}$. If B has two minimal primary decomposition $I_1 \cap I_2 \cap \cdots \cap I_k = D_1 \cap D_2 \cap \cdots \cap D_s$, where I_i is S_{A_i}-primary and D_j is S_{B_j}-primary and each A_i and B_j are isolated strongly prime, then $k = s$.*

Proof Let $I_1 \cap I_2 \cap \cdots \cap I_k = D_1 \cap D_2 \cap \cdots \cap D_s$ where I_i is S_{A_i}-primary and D_j is S_{B_j}-primary. Then, $A_1 \cap A_2 \cap \cdots \cap A_k = sr(I_1) \cap sr(I_2) \cap \cdots \cap sr(I_k) = sr(I_1 \cap I_2 \cap \cdots \cap I_k) = sr(D_1 \cap D_2 \cap \cdots \cap D_s) = sr(D_1) \cap sr(D_2) \cap \cdots \cap sr(D_s) = D_1 \cap D_2 \cap \cdots \cap D_s$. Now, $L(A_1^*, A_2^*, \ldots, A_k^*) \subseteq A_1 \cap A_2 \cap \cdots \cap A_k \subseteq D_j$ for all j. Since B_j is strongly prime ideal and Theorem 7, we have got $A_i \subseteq B_j$ for some i. Also, $L(B_1^*, B_2^*, \ldots, B_s^*) \subseteq B_1 \cap B_2 \cap \cdots \cap B_s \subseteq A_i$ for all i. Since A_i is strongly prime ideal and Theorem 7, we have got $B_r \subseteq A_i$ for some r. So, $B_r \subseteq A_i \subseteq B_j$. Since B_j is an isolated strongly prime, we have $B_r = B_j$ which implies $B_j = A_i$, so $k = s$. □

4 Conclusion

The definition and its generalization of the prime ideal have a distinguished place in algebraic geometry and commutative algebra. These are useful tools to determine the properties of algebraic structure. In this article, more generalization of primary ideals in posets is given, and some properties of these S-primary ideals are obtained. Also, the S-primary decomposition of an ideal in posets is discussed.

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Association Schemes Over Some Finite Group Rings

Anuradha Sabharwal and Pooja Yadav

Abstract In this paper, we study non-symmetric commutative association schemes for cyclic groups $\underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r \text{ times}}$($p$ is prime), $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_r}$ ($p_i's$ are distinct primes), dihedral group and symmetric group without using conjugacy classes. We also construct commutative association schemes for finite group rings over $\mathbb{Z}_n$, the ring of integers mod n. Moreover, we construct association scheme for $n \times n$ circulant matrices over $\mathbb{Z}_p$, for p prime.

Keywords Group ring · Association scheme · Symmetric group · Dihedral group · Circulant matrices

1 Introduction

In the theory of algebraic combinatorics, association scheme plays a vital role. Association schemes were introduced by Bose and Shimamoto [1]. They are used in coding theory, graph theory, design theory and group theory. Association schemes may also be seen as colorings of the edges of complete graphs which satisfies nice regularity conditions. Jørgensen [5] has listed non-symmetric association schemes with classes less than 96 vertices which stimulate us to study non-symmetric association scheme for various finite groups and group rings. We start with a brief introduction of association scheme. For more basic results on association schemes, we refer to [8]. In this paper, for a finite set X, we denote by $\mathcal{G}$ the partition of $X \times X$.

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2 Preliminaries

Definition 1 Association Scheme (AS): Let X be a finite set, and $\mathcal{G}$ be a partition of $X \times X$ with $R_0, R_1, \ldots, R_n$ binary relations on $\mathcal{G}$. Then, $\chi = (X, \mathcal{G})$ is an association scheme of n-class if the following conditions hold:

1. Existence of identity relation $R_0 = \{(x, x) : x \in X\}$ in $\mathcal{G}$.
2. For any relation $R \in \mathcal{G}$, there exists a relation $R^* \in \mathcal{G}$ such that for every $(x, y) \in R$, $(y, x) \in R^*$.
3. For each i, j, k, if $(x, y) \in R_k$, the cardinality $|xR_i \cap yR_j^*|$ is a constant p_{ij}^k which does not depend on choice of x and y.

The *order* of $\mathcal{G}$ is the number of elements in X. The non-negative integers $\{p_{ij}^k\}_{0 \le i,j,k \le n}$ are the *intersection numbers* or *parameters* of $\mathcal{G}$. The association scheme $\mathcal{G}$ is *commutative* if $p_{ij}^k = p_{ji}^k \; \forall\, 0 \le i, j, k \le n$, and it is *symmetric* if each relation R_i is a symmetric relation, that is, $R_i = R_i^* \; \forall\, i \in \{0, 1, \ldots, n\}$. If $(x, y) \in R_i$ with $x \ne y$, then x and y are called *ith associates*. For $x \in X$ and $R \in \mathcal{G}$, let the set xR be the set of all elements $y \in X$ when $(x, y) \in R$.

Example 1 Let $X = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Let us define the following relations in $\mathcal{G}$:

$$R_0 = \{((0,0),(0,0)), ((0,1),(0,1)), ((1,0),(1,0)), ((1,1),(1,1))\}$$
$$R_1 = \{((0,0),(1,1)), ((0,1),(1,0)), ((1,0),(0,1)), ((1,1),(0,0))\}$$
$$R_2 = \{((0,0),(1,0)), ((0,1),(1,1)), ((1,0),(0,0)), ((1,1),(0,1))\}$$
$$R_3 = \{((0,0),(0,1)), ((0,1),(0,0)), ((1,0),(1,1)), ((1,1),(1,0))\}$$

Then, there exist an identity relation R_0, and if $(x, y) \in R_k$, the cardinality $|xR_i \cap yR_j^*|$ is a constant 0 and 1 depending on i, j, k . Also since, $R_k = R_k^*$ for all $0 \le k \le 3$, $(X, \mathcal{G})$ is a symmetric AS.

Note: Every symmetric AS is commutative.

Definition 2 Group Association Scheme(GAS): A finite group G having conjugacy classes $C_0, C_1, \ldots, C_d$ yields a commutative association scheme with a class of relations R_k on G defined by $R_k = \{(x, y) | yx^{-1} \in C_k\} \; \forall 0 \le k \le d$. This scheme is called *group association scheme* of G.

Any finite group $(X, *)$ yields an association scheme with the following class of relations:

$$R_k = \{(x, y) | x = k * y | x, y \in X\} \quad \text{for all } k \in X.$$

This association scheme is commutative iff X is an abelian group. In this paper, we will study association schemes using these kinds of relations for some finite groups, and further, we will compute its parameters also. We define relations for association schemes in such a way that their intersection numbers are either 0 or 1.

Theorem 1 [4, Theorem 3.3] *An association scheme of a group is commutative, if its order is a prime number.*

Therefore, for $X = \mathbb{Z}_p$ where p is prime, association scheme $(X, \mathcal{G})$ defined in Lemma 1 is commutative.

3 Main Results

We now study association schemes for $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes and for $\underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r \text{ times}}$, where p is odd prime.

Lemma 1 *Let $X = \mathbb{Z}_p$. Let us define relations R_k in $\mathcal{G}$ by $R_k = \{(i, j)|i = k + j|i, j \in \mathbb{Z}_p\}$ $\forall k \in \mathbb{Z}_p$. Then, $(X, \mathcal{G})$ is a non-symmetric commutative AS, and intersection numbers of this AS are as follows:*

$$p_{ij}^k = \begin{cases} 1 & \text{if } k = i + j, \\ 0 & \text{if } k \neq i + j \end{cases}$$

Proof As $\mathbb{Z}_p$ is an abelian group, $(\mathbb{Z}_p, \mathcal{G})$ with the given relations forms a commutative association scheme.

Let R_i, R_j, R_k be arbitrary relations in $\mathcal{G}$. To find cardinality p_{ij}^k such that for all $(x, y) \in R_k$, we have $|xR_i \cap yR_j^*| = p_{ij}^k$, let $(x, y) \in R_k$ and let $xR_i = x'$ and $yR_j^* = y'$. That is, $x = x' + i$, $y' = y + j$ and $x = y + k$. Since every pair of points (x, y) are ith associates for exactly one i, p_{ij}^k can be either 0 or 1. Therefore, $p_{ij}^k = 1$ if $x' = y'$ that is, if $k = i + j$, and $p_{ij}^k = 0$ if $x' \neq y'$ that is, if $k \neq i + j$.

Lemma 2 *Let $X = \mathbb{Z}_p \times \mathbb{Z}_q$, where p and q are distinct primes. Then, the relations R_k in $\mathcal{G}$ defined by*

$$R_k = \{(x, y)|x_1 \equiv (k + y_1) \bmod p, x_2 \equiv (k + y_2) \bmod q \ |x = (x_1, x_2), \\ y = (y_1, y_2) \in \mathbb{Z}_p \times \mathbb{Z}_q\} \quad \forall\, 0 \le k \le pq - 1$$

is a non-symmetric commutative AS with intersection numbers

$$p_{ij}^k = \begin{cases} 1 & \text{if } k \equiv (i + j) \bmod pq, \\ 0 & \text{otherwise} \end{cases}$$

Proof Since $\mathbb{Z}_p \times \mathbb{Z}_q$ is an abelian group, it can be easily verified that $(X, \mathcal{G})$ is a commutative AS with non-symmetric relations as $R_i^* = R_j$ if $i + j = pq$. Let $R_i, R_j, R_k \in \mathcal{G}$. Now, we will find cardinality p_{ij}^k such that for all $(x, y) \in R_k$ we have $|xR_i \cap yR_j^*| = p_{ij}^k$.

Let $(x, y) \in R_k$, where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{Z}_p \times \mathbb{Z}_q$, and let $xR_i = x' = (x_1', x_2')$ and $yR_j^* = y' = (y_1', y_2')$.

That is, $x_1 \equiv (y_1 + k) \bmod p, x_2 \equiv (y_2 + k) \bmod q;\ x_1 \equiv (x_1' + i) \bmod p$, $x_2 \equiv (x_2' + i) \bmod q;\ y_1' \equiv (y_1 + j) \bmod p, y_2' \equiv (y_2 + j) \bmod q$. Using these equations, we have $p_{ij}^k = 1$ iff $x' = y'$ that is, if $k \equiv (i + j) \bmod pq$.

Theorem 2 *Let* $X = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_r}$, *where* $p_1, p_2, \ldots, p_r$ *are distinct primes and* $r \geq 3$. *For each* $t \in \mathbb{Z}_{p_{r-1}p_r}$ *and* $d_s \in \mathbb{Z}_{p_s} \forall\ 1 \leq s \leq r-2$, *the relations* R_k *in* $\mathcal{G}$ *defined by*

$$\begin{aligned} R_k = \{(x, y) |\ & y_r \equiv (k + x_r) \bmod p_r,\ y_{r-1} \equiv (k + x_{r-1}) \bmod p_{r-1}, \\ & y_s \equiv (d_s + x_s) \bmod p_s \ \forall\ 1 \leq s \leq r-2 |\ x = (x_1, x_2, \ldots, x_r), \\ & y = (y_1, y_2, \ldots, y_r) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_r}\} \\ & \forall\ k = p_2 p_3 \cdots p_r d_1 + p_3 p_4 \cdots p_r d_2 + \cdots + p_{r-1} p_r d_{r-2} + t \end{aligned}$$

is a non-symmetric commutative AS with intersection numbers

$$p_{ij}^k = \begin{cases} 1 & \text{if } k \equiv (i + j) \bmod p_{r-1}p_r, \text{ and} \\ & \quad d_s \equiv (d_s^{(1)} + d_s^{(2)}) \bmod p_s \ \forall\ 1 \leq s \leq r-2 \\ 0 & \text{otherwise} \end{cases}$$

where $k = p_2 p_3 \cdots p_r d_1 + p_3 p_4 \cdots p_r d_2 + \cdots + p_{r-1} p_r d_{r-2} + t$; $i = p_2 p_3 \cdots p_r d_1^{(1)} + p_3 p_4 \cdots p_r d_2^{(1)} + \cdots + p_{r-1} p_r d_{r-2}^{(1)} + t^{(1)}$; $j = p_2 p_3 \cdots p_r d_1^{(2)} + p_3 p_4 \cdots p_r d_2^{(2)} + \cdots + p_{r-1} p_r d_{r-2}^{(2)} + t^{(2)}$ *for some* $t, t^{(1)}, t^{(2)} \in \mathbb{Z}_{p_{r-1}p_r}$ *and* $d_s, d_s^{(1)}, d_s^{(2)} \in \mathbb{Z}_{p_s}\ \forall\, 1 \leq s \leq r-2$.

Proof $|X| = |\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_r}| = p_1 p_2 \cdots p_r = |R_k|$ for all $0 \leq k \leq p_1 p_2 \cdots p_r - 1$. All the relations R_k are disjoint, and they form partition of $\mathcal{G}$.

Let R_i, R_j, R_k be arbitrary relations in $\mathcal{G}$. We will show that for each pair x, y with $(x, y) \in R_k$, the cardinality $|\{z \in X | (x, z) \in R_i, (z, y) \in R_j\}|$ is a constant. Let $(x, y) \in R_k$, where $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r) \in X$, and let $xR_i = x' = (x_1', x_2', \ldots, x_r')$, $yR_j^* = y' = (y_1', y_2', \ldots, y_r')$.

Now, $(x, y) \in R_k$ implies $y_r \equiv (k + x_r) \bmod p_r$, $y_{r-1} \equiv (k + x_{r-1}) \bmod p_{r-1}$, $y_s \equiv (d_s + x_s) \bmod p_s \ \forall\, 1 \leq s \leq r-2$ where $k = p_2 p_3 \cdots p_r d_1 + p_3 p_4 \cdots p_r d_2 + \cdots + p_{r-1} p_r d_{r-2} + t$ for some $t \in \mathbb{Z}_{p_{r-1}p_r}$ and $d_s \in \mathbb{Z}_{p_s}\ \forall\, 1 \leq s \leq r-2$.

Similarly, as $(x, x') \in R_i$ and $(y', y) \in R_j$, we have $x_r' \equiv (i + x_r) \bmod p_r, x_{r-1}' \equiv (i + x_{r-1}) \bmod p_{r-1}, x_s' \equiv (d_s^{(1)} + x_s) \bmod p_s \ \forall\, 1 \leq s \leq r-2$ where $i = p_2 p_3 \cdots p_r d_1^{(1)} + p_3 p_4 \cdots p_r d_2^{(1)} + \cdots + p_{r-1} p_r d_{r-2}^{(1)} + t^{(1)}$ for some $t^{(1)} \in \mathbb{Z}_{p_{r-1}p_r}$, $d_s^{(1)} \in \mathbb{Z}_{p_s} \forall\, 1 \leq s \leq r-2$, and $y_r \equiv (j + y_r') \bmod p_r,\ y_{r-1} \equiv (j + y_{r-1}') \bmod p_{r-1}, y_s \equiv (d_s^{(2)} + y_s') \bmod p_s \ \forall\ 1 \leq s \leq r-2$ where $j = p_2 p_3 \cdots p_r d_1^{(2)} + p_3 p_4 \cdots p_r d_2^{(2)} + \cdots + p_{r-1} p_r d_{r-2}^{(2)} + t^{(2)}$ for some $t^{(2)} \in \mathbb{Z}_{p_{r-1}p_r}, d_s^{(2)} \in \mathbb{Z}_{p_s}\ \forall\, 1 \leq s \leq r-2$.

Using above equations, we have $p_{ij}^k = 1$ if and only if $x' = y'$ that is, if $k \equiv (i + j) \bmod p_{r-1}p_r$ and $d_s \equiv (d_s^{(1)} + d_s^{(2)}) \bmod p_s \ \forall\, 1 \leq s \leq r-2$. Hence, $(X, \mathcal{G})$ is a non-symmetric commutative AS.

Lemma 3 *Let $X = \mathbb{Z}_p \times \mathbb{Z}_p$, where p is odd prime. For each $d, t \in \mathbb{Z}_p$, the relations R_k in $\mathcal{G}$ defined by*

$$R_k = \{(x, y) | x_1 \equiv (k + d + y_1) \bmod p, x_2 = (k + t + y_2) \bmod p \,| x = (x_1, x_2), \\ y = (y_1, y_2) \in \mathbb{Z}_p \times \mathbb{Z}_p\} \quad \forall\, k = pd + t$$

yields a non-symmetric commutative association scheme, and its intersection numbers are

$$p_{ij}^k = \begin{cases} 1 & \text{if } 2k + d + t \equiv (2i + 2j + d_1 + d_2 + t_1 + t_2) \bmod p, \\ 0 & \text{otherwise} \end{cases}$$

where $k = pd + t;\ i = pd_1 + t_1;\ j = pd_2 + t_2$ for some $d, d_1, d_2, t, t_1, t_2 \in \mathbb{Z}_p$.

Proof $|X| = |\mathbb{Z}_p \times \mathbb{Z}_p| = p^2$ and $|R_k| = p^2\ \forall 0 \le k \le p^2 - 1$. All the relations R_k are disjoint and $\cup\{R_k : 0 \le k \le p^2 - 1\} = \mathcal{G}$.

Let R_i, R_j, R_k be arbitrary relations in $\mathcal{G}$. We will show that for each $(x, y) \in R_k$, the cardinality $|\{z \in X | (x, z) \in R_i, (z, y) \in R_j\}|$ is a constant. Let $(x, y) \in R_k$ where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{Z}_p \times \mathbb{Z}_p$ and let $xR_i = x' = (x'_1, x'_2),\ yR_j^* = y' = (y'_1, y'_2)$.

Now, $(x, y) \in R_k$ implies $x_1 \equiv (y_1 + k + d) \bmod p,\ x_2 \equiv (y_2 + k + t) \bmod p$, where $k = pd + t$ for some $d, t \in \mathbb{Z}_p$.

Similarly, as $(x, x') \in R_i$ and $(y', y) \in R_j$, $x_1 \equiv (x'_1 + i + d_1) \bmod p, x_2 \equiv (x'_2 + i + t_1) \bmod p$ where $i = pd_1 + t_1$ for some $d_1, t_1 \in \mathbb{Z}_p$ and $y'_1 \equiv (y_1 + j + d_2) \bmod p$, $y'_2 \equiv (y_2 + j + t_2) \bmod p$ where $j = pd_2 + t_2$ for some $d_2, t_2 \in \mathbb{Z}_p$.

Using above equations, we have $p_{ij}^k = 1$ if and only if $x' = y'$ that is, if $2k + d + t \equiv (2i + 2j + d_1 + d_2 + t_1 + t_2) \bmod p$.

Theorem 3 *Let $X = \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r \text{ times}}$, where p is odd prime and $r \ge 3$. For each $t, d_s \in \mathbb{Z}_p\ \forall\ 1 \le s \le r - 1$, the relations R_k in $\mathcal{G}$ defined by*

$$R_k = \{(x, y) |\, y_r \equiv (k + t + x_r) \bmod p,\ y_s \equiv (d_s + x_s) \bmod p\, \forall\ 1 \le s \le r - 1 | \\ x = (x_1, x_2, \ldots, x_r),\ y = (y_1, y_2, \ldots, y_r) \in X\} \\ \forall\, k = p^{r-1}d_1 + p^{r-2}d_2 + \cdots + pd_{r-1} + t$$

is a non-symmetric commutative AS with intersection numbers

$$p_{ij}^k = \begin{cases} 1 & \text{if } \left(\sum_{s=1}^{r-1} d_s^{(k)} + t^{(k)} + k\right) \equiv \left(\sum_{s=1}^{r-1} (d_s^{(i)} + d_s^{(j)}) + t^{(i)} + t^{(j)} + i + j\right) \\ & \bmod p \\ 0 & \text{otherwise} \end{cases}$$

where $k = p^{r-1}d_1^{(k)} + p^{r-2}d_2^{(k)} + \cdots + pd_{r-1}^{(k)} + t^{(k)}; i = p^{r-1}d_1^{(i)} + p^{r-2}d_2^{(i)} + \cdots + pd_{r-1}^{(i)} + t^{(i)}; j = p^{r-1}d_1^{(j)} + p^{r-2}d_2^{(j)} + \cdots + pd_{r-1}^{(j)} + t^{(j)}$ *for some* $t^{(k)}, t^{(i)}, t^{(j)}, d_s^{(k)}, d_s^{(i)}, d_s^{(j)} \in \mathbb{Z}_p \;\forall\, 1 \le s \le r-1$.

Proof $|X| = p^r$ and $|R_k| = p^r$ for all $0 \le k \le p^r - 1$. All the relations R_k are disjoint, and they form partition of $\mathcal{G}$.

Let R_i, R_j, R_k be arbitrary relations in $\mathcal{G}$. We will show that for each $(x, y) \in R_k$, the cardinality $|\{z \in X | (x, z) \in R_i, (z, y) \in R_j\}|$ is a constant. Let $(x, y) \in R_k$ where $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r) \in X$ and let $xR_i = x' = (x_1', x_2', \ldots, x_r')$, $yR_j^* = y' = (y_1', y_2', \ldots, y_r')$.

Now, $(x, y) \in R_k$ implies $y_r \equiv (k + t^{(k)} + x_r) \bmod p$, $y_s \equiv (d_s^{(k)} + x_s) \bmod p$ $\forall\, 1 \le s \le r-1$ where $k = p^{r-1}d_1^{(k)} + p^{r-2}d_2^{(k)} + \cdots + pd_{r-1}^{(k)} + t^{(k)}$ for some $t^{(k)}$, $d_s^{(k)} \in \mathbb{Z}_p \;\forall\, 1 \le s \le r-1$.

Similarly, as $(x, x') \in R_i$ and $(y', y) \in R_j$, we have $x_r' \equiv (i + t^{(i)} + x_r) \bmod p$, $x_s' \equiv (d_s^{(i)} + x_s) \bmod p \;\forall\, 1 \le s \le r-1$ where $i = p^{r-1}d_1^{(i)} + p^{r-2}d_2^{(i)} + \cdots + pd_{r-1}^{(i)} + t^{(i)}$ for some $t^{(i)}, d_s^{(i)} \in \mathbb{Z}_p \;\forall\, 1 \le s \le r-1$, and $y_r \equiv (j + t^{(j)} + y_r') \bmod p$, $y_s \equiv (d_s^{(j)} + y_s') \bmod p \;\forall\, 1 \le s \le r-1$ where $j = p^{r-1}d_1^{(j)} + p^{r-2}d_2^{(j)} + \cdots + pd_{r-1}^{(j)} + t^{(j)}$ for some $t^{(j)}, d_s^{(j)} \in \mathbb{Z}_p \;\forall\, 1 \le s \le r-1$.

We have $p_{ij}^k = 1$ if and only if $x' = y'$ that is, if $k + t^{(k)} \equiv (i + j + t^{(i)} + t^{(j)}) \bmod p$, and $d_s^{(k)} \equiv (d_s^{(i)} + d_s^{(j)}) \bmod p \;\forall\, 1 \le s \le r-1$. Adding these equations, we have the desired result, and hence, $(X, \mathcal{G})$ is a non-symmetric commutative association scheme.

Yue Meng-tian, Li Zeng-ti (see [7]) constructed symmetric association schemes on the dihedral group. In next theorem, we provide a new family of commutative association schemes with non-symmetric relations on the dihedral group.

Theorem 4 *Let* $X = D_{2n} = \langle a, b | a^2, b^n, ab = b^{-1}a\rangle$. *That is, the canonical form of any element of* D_{2n} *is* $a^l b^m$ *where* $0 \le l \le 1$ *and* $0 \le m \le n-1$. *Define relations* R_k *in* $\mathcal{G}$ *by*

$$R_k = \{(a^l b^m, a^{k+l} b^{k+m}) | 0 \le l \le 1, 0 \le m \le n-1\} \quad \text{for all } 0 \le k \le 2n-1.$$

Then, $(X, \mathcal{G})$ *is a non-symmetric AS, and intersection numbers of this scheme are as follows:*

$$p_{ij}^k = \begin{cases} 1 & \text{if } k = i + j, \\ 0 & \text{if } k \neq i + j \end{cases}$$

Proof $R_0 = \{(x, x) | x \in D_{2n}\}$ is an identity relation. It can be verified that $(X, \mathcal{G})$ is an AS and the relations are non-symmetric.

Let $R_i, R_j, R_k \in \mathcal{G}$. To find cardinality p_{ij}^k such that for each $(x, y) \in R_k$, $|xR_i \cap yR_j^*| = p_{ij}^k$. Let $(x, y) \in R_k$ and let $xR_i = x'$ and $yR_j^* = y'$. That is, $x = a^l b^m$, $y = a^{k+l} b^{k+m}$; $x = a^{l_1} b^{m_1}$, $x' = a^{i+l_1} b^{i+m_1}$; $y' = a^{l_2} b^{m_2}$, $y = a^{j+l_2} b^{j+m_2}$ where $0 \le l, l_1$,