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Marco A. Arteaga
Alejandro Gutiérrez-Giles
Javier Pliego-Jiménez

Local Stability and Ultimate Boundedness in the Control of Robot Manipulators

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Marco A. Arteaga · Alejandro Gutiérrez-Giles ·
Javier Pliego-Jiménez

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Marco A. Arteaga
Departamento de Control y Robótica, DIE.
Facultad de Ingeniería
UNAM
Coyoacán, Mexico City, Mexico

Alejandro Gutiérrez-Giles
Centro de Estudios en Computación
Avanzada (CECAv)
UNAM
Mexico City, Mexico

Javier Pliego-Jiménez
Departamento de Electrónica y
Telecomunicaciones, División de Física
Aplicada
Centro de Investigación Científica y de
Educación Superior de Ensenada
Ensenada, Baja California, Mexico

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To Heidrun,

with everlasting love.

—Marco A. Arteaga

To my beloved family,

—Alejandro Gutiérrez-Giles

To my beloved family,

—Javier Pliego-Jiménez

Preface

There is a wide variety of applications for industrial robot manipulators, such as welding, painting, coating, gluing, sealing, milling, drilling, grinding, screwing, wiring, fastening or assembling of devices. To carry out any of the different tasks a manipulator is supposed to do, the design of a high-performance control law is advisable. Therefore, the control of robot manipulators has received a lot of attention for the last decades. Although inherently industrial robots are thought for practical applications, many researchers have also considered them as high nonlinear systems whose study from a pure theoretical point of view is quite challenging. Problems like trajectory tracking, state observation, force control, telemanipulation, etc., are just some examples of a wide variety of subjects under study. One of the main challenges always to be considered is global stability. Roughly speaking, it means that no matter how large the initial error state may be, it ultimately will vanish. In fact, theoretically the initial error could be just infinite. Achieving this goal may be at the cost of very complex controllers and/or observers, or at the cost of making many unrealistic assumptions, such as the perfect knowledge of different robot model parameters. But, is it really necessary to invest such a huge effort to prove that the initial error can be infinite for a physical system that is confined in few cubic meters and cannot move with infinite velocity nor apply infinite forces on the environment? Another issue is that even though from a theoretical point of view all errors tend to zero exactly, in practice residual errors arise due to a variety of causes such as sensors resolution, model inaccuracies or the discretization process for computer implementation. Although global asymptotic stability is quite desirable, this book is devoted to the design of control schemes and state observers based on the premise of desirable local stability. Furthermore, ultimate boundedness is considered as an acceptable alternative to asymptotic stability.

Part I contains the most basic and well-known concepts of robot manipulators. On its own, the first part of the book can be used for a postgraduate course on robot control theory. Chapter 1 provides the technical definition of industrial robot manipulators, the only kind of systems under study, and makes a summary of the most employed concepts. The usual categories of robot manipulators are given, and it is explained why it makes just sense to employ the customary kinematic arrangements.

Chapter 2 deals with the robot's kinematics, and it comprises the development of basic rotations among two coordinate frames as well as the definition of homogeneous transformations. Then, the three kinds of kinematics are described in full detail for serial links manipulators, i.e. direct kinematics, inverse kinematics and differential kinematics. By taking advantage of all the previous concepts, Chap. 3 develops the dynamic model of robot manipulators based on the Euler–Lagrange methodology, including the description of external forces by means of the Lagrange multipliers. Furthermore, some of the most useful physical and structural properties of the model are obtained, which prove to be important for control design. Chapter 4 presents some basic concepts and results on control theory. It is not aimed at giving a deep insight, but at making the book self-contained. However, the concepts that are inspected with more detail are precisely those that give name to this book, i.e. local stability in the sense of Lyapunov and ultimate boundedness. A rather colloquial development instead of a strict analytical approach is used for the statement of the main theorems employed for the design of control and observers schemes. Finally, Chap. 5 provides the most elemental control laws for robot manipulators.

Part II is devoted to recapitulate more than two decades of experience by the authors in the development of different control laws and approaches for robot manipulators, as well as pointing out some very well-known schemes that comply with the main goal of this book, i.e. either local stability or ultimate boundedness. Chapter 6 deals with the fact that most industrial robots are not really equipped with velocity sensors, while this quantity is necessary for control laws implementation. Therefore, it is necessary to produce an estimate of joint velocities. Although this can readily be done by numerical differentiation in a digital computer, it is preferable for many reasons the design of an observer. Model-based observers of the Luenberger type are a very common solution even for highly nonlinear systems, but it is explained how much more simpler options can be used to get just as a good performance as for model-based solutions without needing at all the robot dynamic model for implementation. In fact, although theoretically it is possible to obtain a very accurate dynamic model for industrial robots, in practice this is usually not the case because many physical parameters are not known perfectly or they are even unknown. Therefore, Chap. 7 discusses two of the most common solutions to compensate robot models uncertainties, namely adaptive and robust control. The former makes use of the very well-known property of model parameters linearity, while the latter deals with the uncertainties by introducing some extra control terms to the nominal law. Pros and cons are discussed. Chapter 8 introduces some solutions to robot force control. As explained before, these are based on the authors' own experiences, and therefore, the reader will not find common solutions just as impedance or admittance control. However, the proposed schemes take into account a wide variety of situations, just like the possible lack of both velocities and force measurements, accurate robot models or an implementation without computing inverse kinematics. Chapters 9 and 10 study the intercommunication of different robot manipulators. The former deals only with the bilateral teleoperation of two devices, while the latter takes into account three or more, i.e. robot networks. In both cases, it is assumed that time-varying delays are present in the communication channels, which constitutes a more realistic as well as more challenging situation. Also, velocity observers are designed, and for the bilateral teleoperation the concept on delayed kinematic correspondence is introduced in contrast to the usual concept of transparency when no time delays are considered at all. One of the key characteristics of all the chapters of Part II is that experimental results are shown. Much unfortunately, the COVID-19 pandemic that humanity has endured during the writing of this book kept the National Autonomous University of Mexico closed, so that instead of using the industrial robots A255 and A465 for some experiments, the also reliable but simpler *Geomagic Touch* manipulators were employed. While the outcomes are still valid, these smaller robots are easier to control, and it is worthy to point this out.

Part III discusses the modeling of three of the robots available at the Laboratory for Robotics of the School of Engineering of the National Autonomous University of Mexico. Chapters 11 and 12 study the kinematic and dynamic modeling of the industrial robots A465 and A255 by *CRS Robotics*, respectively. While the kinematic models can be gotten with high accuracy, the dynamic ones are computed by making many simplifications. This has to be so because the links geometries are not symmetric, their masses densities are not uniform, and the manufacturer provides little information altogether. One of the main drawbacks of using industrial robots for testing control–observer approaches is that sometimes they do not allow to program them since the manufacturer includes its own control algorithms. Therefore, Chap. 13 makes a proposal about how some hardware devices can be built to replace the original control units, thus allowing the implementation of newly developed control schemes. Finally, Chap. 14 computes the kinematic and dynamic models of the haptic device *Geomagic Touch*, employed as mentioned before in all the experiments of the second part of the book.

Mexico City, Mexico — Marco A. Arteaga
Mexico City, Mexico — Alejandro Gutiérrez-Giles
Ensenada, Mexico — Javier Pliego-Jiménez
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Contents

Part I
Preliminaries

Chapter 1
A General Overview of Robot Manipulators

Abstract Robot manipulators are a very particular kind of robots. They are mainly employed in the industry and therefore they are also known as industrial robots. Although they can be used for very specialized tasks, their main characteristic is the reprogrammability, i.e. the possibility of using the same manipulator for different tasks just by changing the control program code. Great accuracy and repeatability are expected and necessary to accomplish the desired jobs, and these depend on a combination of physical mechanical characteristics as well as on the design of appropriate control schemes. Due to the high nonlinear nature of robot manipulators, control design represents on its own quite a challenge.

1.1 Brief History of Robot Manipulators

Robot manipulators were created in the twentieth century, although some kind of automated mechanical machines existed as early in history as in ancient Greece and Babylon. It is well known that the term *robot* comes from the Czech word *robota* and means subordinated work. It was first used in the science fiction play *Rossum's Universal Robots* by Karel Čapek. Outside science fiction, the first robots were created after the Second World War as a solution to the problem of handling hazardous materials, with the seminal works of Goertz (published several years after their development). Those first robots were of the master-slave type with a mechanical coupling. The objective of these mechanisms was to mimic the arm and hand movements of the operator. Later, the mechanical coupling was substituted by electrical and hydraulic actuators. In the same decade, George C. Devol developed a fully automated mechanism called *programmed articulated transfer device*. The main novelty of this device was its capability to follow a set of instructions that could be reprogrammed. Following this idea, the first industrial robot was presented in 1959 by *Unimation, Inc.* The manipulator called *Unimate*, designed by Joseph F. Engelberger, was successfully installed in a General Motors assembly line in 1961. After these first experimental designs, a boom of commercial robotic arms occurred in the 1960s. The *American Machine and Foundry (AMF)* company introduced the programmable cylindrical manipulator *VERSATRAN* in 1962, designed by Harry

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Johnson and Veljko Milenkovic. In the same decade, the *Freddy* arm, which was pioneer in integrating the manipulator itself with vision and intelligent systems, was developed at the University of Edinburgh. In 1969, Victor Scheinman invented the *Stanford arm*, which was the first six-axes manipulator with a spherical wrist, following the suggestions for easier arm solutions proposed by Donald L. Pieper in his PhD thesis. Years later, Scheinman contributed to the design of the also famous *MIT* and *PUMA* arms. In the next two decades, computer-controller robotic manipulators were widely adopted in the industry, at first mainly in the automotive, but latter became gradually employed also in different areas such as chemical, electronics, and metal industries.

The theory of robotic manipulation was developed at the same time as its physical counterpart. In 1955, Jacques Denavit and Richard S. Hartenberg proposed a systematic method to analyze the kinematics of a general open-chain mechanisms. By employing matrix algebra, they were able to represent the position and orientation of any two coordinate frames as an homogeneous transformation matrix. These coordinate frames can be in turn attached to any link of the manipulator, thus solving the direct kinematics problem. Inverse kinematics turned out to be more involved, even for open-chains, and several methods were developpped to obtain a closed or an approximate solution. As mentioned above, in 1968 Donald L. Pieper proposed a design method for robotic manipulators called *321 kinematic structure*, which always leads to a closed form solution for the inverse kinematics problem. This kind of structure is still used in most of the commercially available robotic arms. The study of the dynamics and its application to the nonlinear control problem was a very intensive research topic in the 1980s and early 1990s, resulting in most of the well known nonlinear control techniques for industrial robots such as sliding modes, adaptive, and passivity-based controllers.

With the millennium change, a new perspective in robotic manipulators arose as well. While in the 20th century the great majority of the installed robots were intended for industrial purposes, in the first two decades of the 21th century an increasing number of applications are focused on human-centered systems and consequently on safe human-robot interaction. One example of such devices is the *da Vinci Surgical System*, developed by *Intuitive Surgery Inc.* Approved for its usage in human interventions in 2000, nowadays around 5000 units are installed worldwide. Some other examples of robots outside industry are the ones employed in hazardous environments (*field robotics*), those used to enhance human capabilities (*human augmentation*) and those designed to improve the quality of life (*service robotics*). All these trends have in common that the manipulators are no longer located in highly structured environments. Such unstructured nature represents a great challenge for the research community in a number of different areas, which in turn makes multidisciplinary and collaborative research mandatory for the upcoming years.

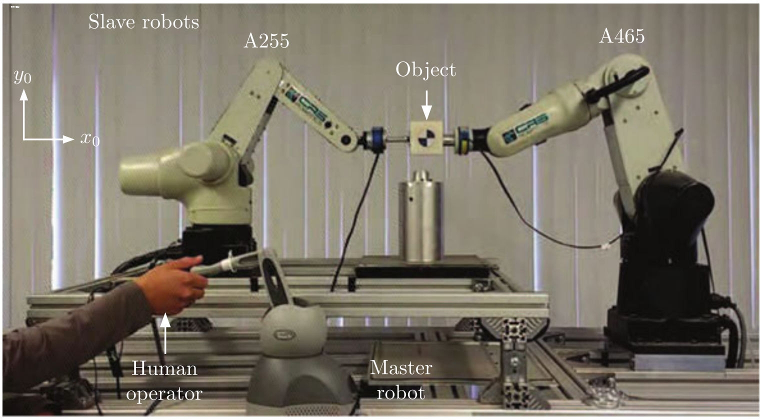


Fig. 1.1 Robots A255 and A465 by *CRS Robotics* and the OMNI Phantom at the Laboratory for Robotics of the National Autonomous University of Mexico

1.2 Industrial Robots

Robot manipulators are also known as industrial robots. Figure 1.1 shows those available at the Laboratory for Robotics of the School of Engineering of the National Autonomous University of Mexico.

First of all, it is necessary to define what a robot manipulator is. The International Organization for Standardization provides Definition 1 of manipulating industrial robots in ISO 8373.

Definition 1 An industrial robot is an automatically controlled, reprogrammable, multipurpose, manipulator programmable in three or more axes, which may be either fixed in place or mobile for use in industrial automation applications. △

One of the key elements of the previous definition is the reprogrammability since it implies that the very same manipulator can be used for many possible applications, and that with good accuracy and flexibility. To understand how this can be done, it is better to begin with the most basic elements of a robot manipulator:

1. Links (connected by joints)
2. Joints:

   a. Revolute (R)
   b. Prismatic (P)

A revolute joint allows to rotate two links around the joint axis, while the prismatic joint allows a linear displacement of the links as depicted in Fig. 1.2.

The joint variables are usually denoted by θ_i for revolute joints and by d_i for the prismatic ones, where $i = 1, \ldots, n$. n is the total number of joints of the manipulator and it is known as the number of **Degrees of Freedom** (DOF) of the manipulator. A

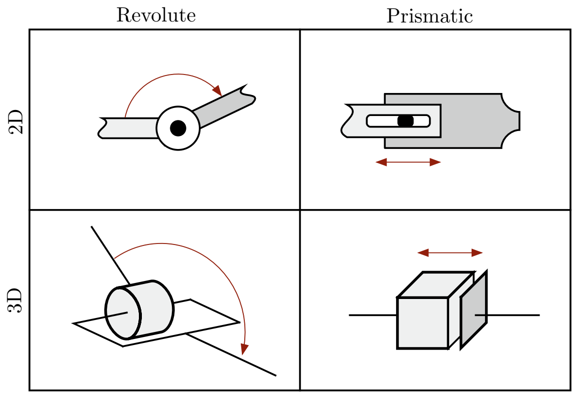


Fig. 1.2 Revolute and prismatic joints connecting two robot links

robot should have 6 DOF because 3 of them are necessary for arbitrary positioning and 3 more for arbitrary orientation. A smaller number of DOF prevents achieving any arbitrary position and orientation of the robot's end-effector, while a bigger number makes the manipulator to be kinematically redundant. Roughly speaking, the manipulator's **reachable workspace** is the total volume spanned by the robot's end-effector when it executes all possible movements. On its own this definition may just be of little utility, except for avoiding collisions. More important is the **dexterous workspace**, within which the manipulador can reach any position with arbitrary orientation. But reaching a position is kind of fuzzy. The **precision** of a manipulator is a measure of how close the robot's end-effector can reach a point within its workspace. The **repeatability** on the other hand is a measure of how close it can reach a point taught with anteriority. Usually, digital encoders are employed to measure the value of a joint variable, while the end-effector position and orientation are computed by taking into consideration the kinematic structure of the manipulator. For that reason, industrial robots are designed to be rigid.

1.3 Common Kinematic Arrangements

As it will be discussed later in full detail in Chap. 2, the inverse kinematics problem can be splitted in two easier challenges, inverse position and inverse orientation. To achieve this, the last three joints of industrial robots are chosen to have the structure of a spherical wrist, while for the first three joints there are usually five categories: articulated, spherical, SCARA, cylindrical and Cartesian manipulators.

1.3.1 Articulated Manipulator

This arrangement is also known as revolute or anthropomorphic. The three robots in Fig. 1.1 are of this kind, while Fig. 1.3 shows that this arrangement is made up of three revolute joints (RRR) and depicts the so-called elbow manipulator, where both axes z_1 and z_2 are parallel and perpendicular to z_0.

The workspace of an articulated manipulator is shown in Fig. 1.4.

1.3.2 Spherical Manipulator

If in the previous configuration the third joint is replaced by a prismatic one (RRP), then the spherical manipulator is gotten as shown in Fig. 1.5.

The name of this configuration comes from the fact that spherical coordinates can straightforwardly be employed to define the position of the end-effector or the wrist center with respect to a frame fixed at the intersection of the z_1 and z_2 axes. The corresponding workspace can be seen in Fig. 1.6.

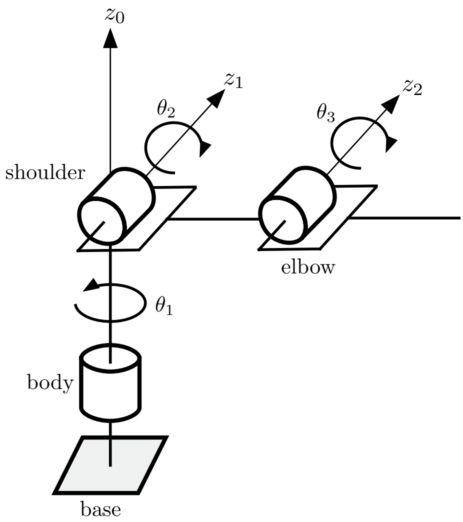


Fig. 1.3 Articulated elbow manipulator

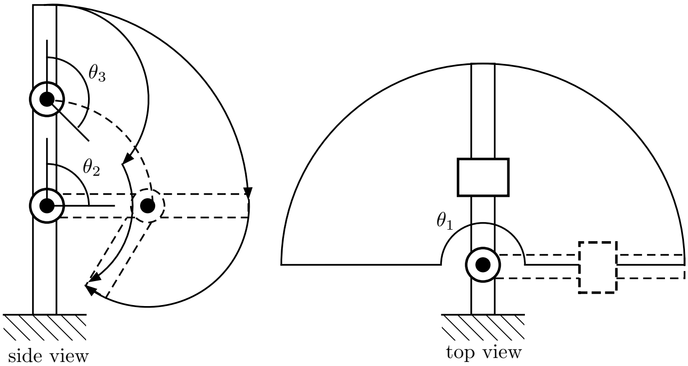
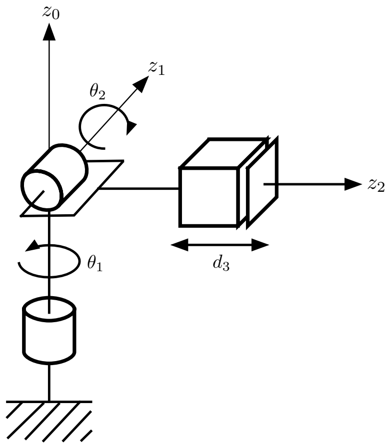


Fig. 1.4 Workspace of an articulated elbow manipulator

Fig. 1.5 Spherical manipulator



1.3.3 SCARA Manipulator

The *Selective Compliant Articulated Robot for Assembly* (SCARA) manipulator does also have a RRP arrangement, but with a different design (the joint axes z_0, z_1 and z_2 are parallel) as shown in Fig. 1.7.

This configuration is widely used in assembly tasks and it is different in appearance to the spherical manipulator despite having both a RRP structure. The corresponding workspace can be seen in Fig. 1.8.

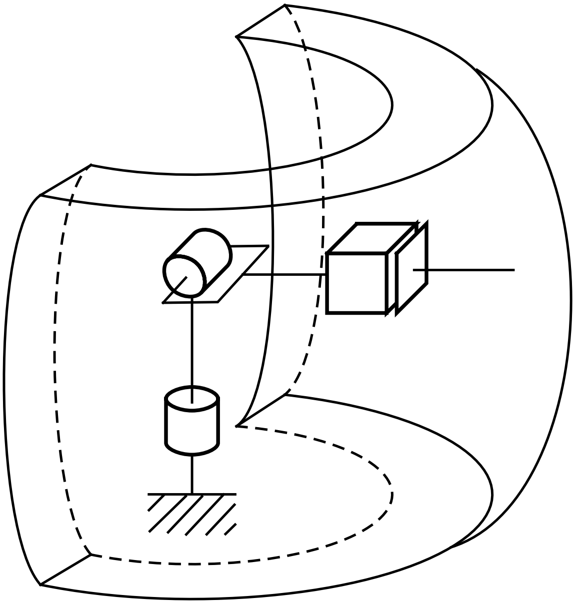


Fig. 1.6 Workspace of a spherical manipulator

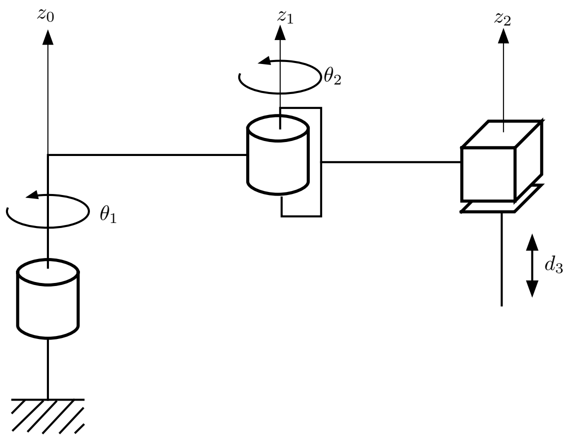


Fig. 1.7 SCARA manipulator

1.3.4 Cylindrical Manipulator

The cylindrical manipulator has a RPP arrangement, as shown in Fig. 1.9. Since the first joint is revolute and the last two are prismatic, this configuration owns its name due to the fact that cylindrical coordinates can be employed to describe the end-effector position with respect to the base. The corresponding workspace is shown in Fig. 1.10.

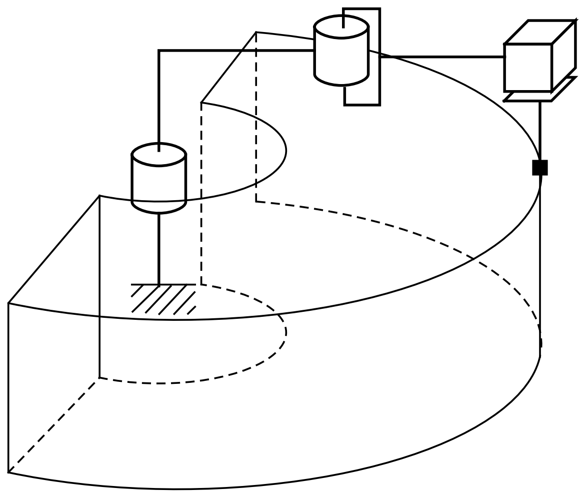


Fig. 1.8 Workspace of a SCARA manipulator

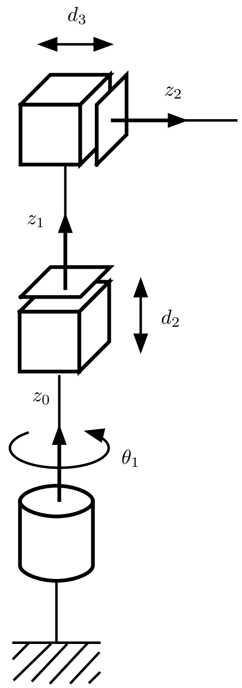


Fig. 1.9 Cylindrical manipulator

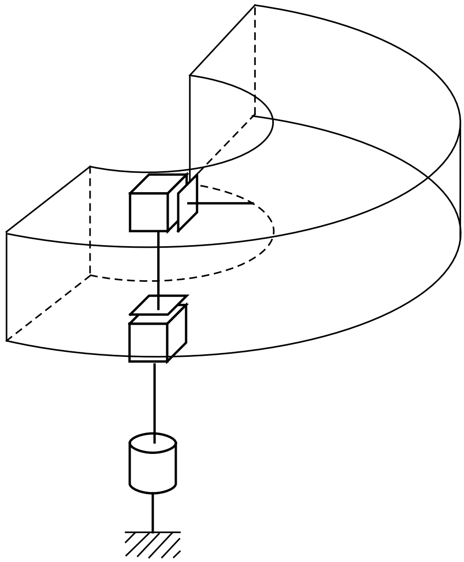


Fig. 1.10 Workspace of the cylindrical manipulator

1.3.5 Cartesian Manipulator

The Cartesian manipulator has all of its joints prismatic (PPP), as it is shown in Fig. 1.11. For this kind of manipulators, the end-effector position, or the center of the wrist, are directly the Cartesian coordinates given by the joint variables leading to the simplest of all configurations. The corresponding workspace is shown in Fig. 1.12.

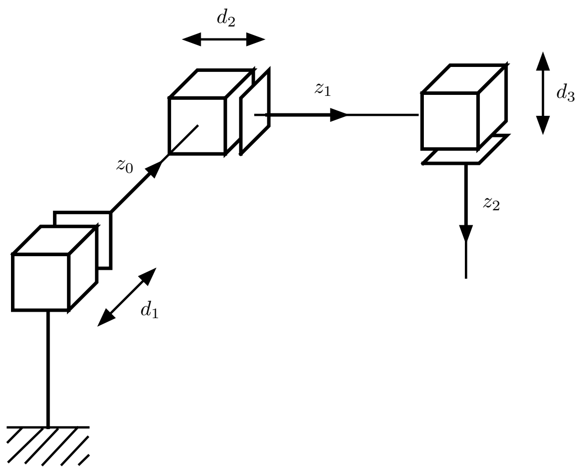


Fig. 1.11 Cartesian manipulator

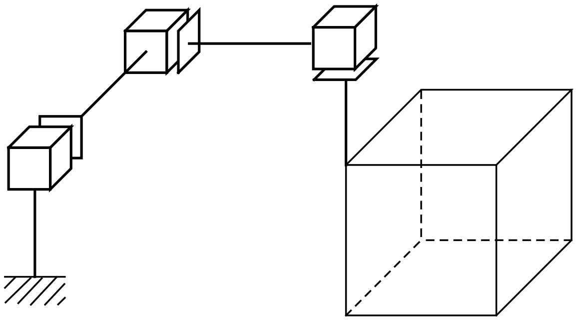


Fig. 1.12 Workspace for a Cartesian manipulator

1.4 Wrists and End-Effectors

1.4.1 Spherical Wrist

The connection between the arm and the end-effector is called the wrist. In the previous section the spherical wrist was mentioned many times because, while the manipulator's main body can take any of the five basic configurations shown before, the last three joints are usually chosen to form precisely a spherical wrist, where its three axes intersect at one single point known as its center. This is shown in Fig. 1.13.

The particular arrangement of the spherical wrist largely simplifies the inverse kinematic analysis as explained in detail in Sect. 2.3. It is worthy pointing out that the wrist may just have less than three joints for some manipulators, which diminishes the capability of reaching any arbitrary orientation.

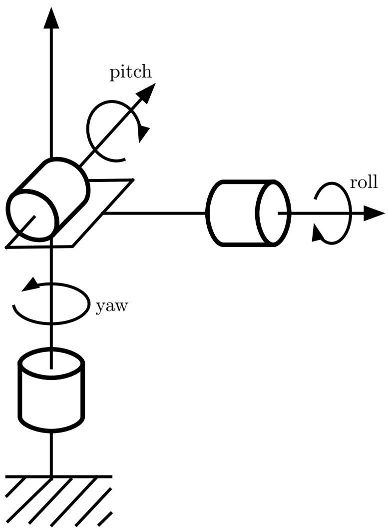


Fig. 1.13 Spherical wrist

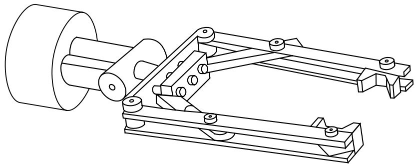


Fig. 1.14 Gripper as end-effector of an industrial robot

1.4.2 Common End-Effectors

The main tool of the manipulator is located at the end of the wrist and it is called end-effector. Perhaps the most usual one is a simple gripper as shown in Fig. 1.14.

Since the gripper can just open and close, a more sophisticated end-effector can be a mechanical hand, for example. Some other important tools are soldering irons, spray guns, cameras and some times even just a single finger (see Fig. 1.1).

1.5 Some Other Important Issues to Take into Account

Other important aspects to consider when working with robot manipulators are the following:

1. To move each of the joints some actuators are needed, being the most commonly used:
   a. Electric actuators, as DC, AC or induction motors can be employed in robot manipulators. They are usually cheap, clean and silence, which explains their popularity. Also, DC motors dynamics can be straightforwardly included in the general robot model, thus allowing an easier design and implementation of control laws.
   b. Hydraulic actuators are fast and they can generate big torques, which allows the manipulator to move heavy loads. However, they require much more peripheral equipment like pumps, there may be some hydraulic fluid leaks and they are noisy.
   c. Pneumatic actuators are cheap, but they are difficult to control, so that its applicability in robotics is not too much.
2. Application area: Depending on the tasks, the robot manipulators can be classified as **assembly** or **non-assembly**. The first class corresponds usually to small electrically driven robots, while to the second one belong those manipulators capable of moving huge heavy loads or those employed for handling different items.
3. Control approach: The precision that a robot manipulator can reach depends both on physical mechanical issues and on the control technique employed for the actuators. Choosing a control scheme depends on many factors, but specially on the task to be accomplished and the information available to implement the

control law. The easiest, yet not meaningless, task is position regulation, while some of the most challenging ones is bilateral teleoperation with time varying delays. Other needs can be the reconstruction of non available measurements or model parameter estimation. This books deals with all these topics.

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Chapter 2
Position, Orientation and Velocity of Rigid Robot Manipulators

Abstract In order to assign a task to a robot manipulator, it is a key matter to describe the position, orientation and velocity of its tool with respect to an inertial frame, usually located at the base of the robot. Although the end result may be involved, the procedure to obtain such a description is rather direct just by defining as many extra coordinate frames as degrees of freedom the manipulator has and by describing the relationship between couples of coordinate systems until arriving to the one fixed at the robot end-effector, where the working tool should be. This allows to define the orientation and position of the end-effector as a function of joint coordinates. Also in a systematic fashion it is possible to compute the linear and angular velocities of the end-effector as a function of both joints positions and velocities.

2.1 Rigid Motions and Homogeneous Transformations

Regulating the position and orientation of a rigid robot manipulator may be quite a big challenge, beginning with how to describe both quantities in the first place. A common solution consists in establishing as many coordinate frames as degrees of freedom the robot has, plus one for the end-effector, where the operating tool is assumed to be fixed. This allows to compute in a rather direct way the relationship between two coordinate systems and then just to accumulate the effect of all frames.

2.1.1 Rotations

Consider Fig. 2.1, where a point $\boldsymbol{p} \in \mathbb{R}^3$ is represented in the coordinate frame o_{x_0,y_0,z_0} as a vector. This is a very common representation in Cartesian coordinates where each axis, x_0, y_0 and z_0, has associated a unit vector in its own direction, $\boldsymbol{i}_0$,$\boldsymbol{j}_0$ and $\boldsymbol{k}_0$ respectively. Therefore, the vector $\boldsymbol{p}$ can be obtained as the following sum:

$$\boldsymbol{p} = p_{x0}\boldsymbol{i}_0 + p_{y0}\boldsymbol{j}_0 + p_{z0}\boldsymbol{k}_0. \tag{2.1}$$

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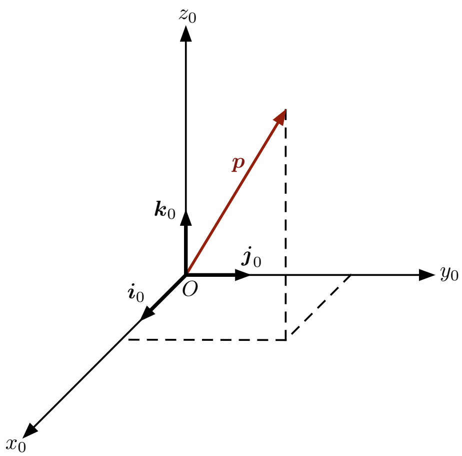

Fig. 2.1 Representation of a point $\boldsymbol{p}$ in the coordinate frame o_{x_0,y_0,z_0}

Since the coordinate system o_{x_0,y_0,z_0} is right handed, the vectors $\boldsymbol{i}_0$, $\boldsymbol{j}_0$ and $\boldsymbol{k}_0$ satisfy

$$\boldsymbol{k}_0 = \boldsymbol{i}_0 \times \boldsymbol{j}_0 \quad \boldsymbol{j}_0 = \boldsymbol{k}_0 \times \boldsymbol{i}_0 \quad \text{and} \quad \boldsymbol{i}_0 = \boldsymbol{j}_0 \times \boldsymbol{k}_0, \tag{2.2}$$

and

$$\boldsymbol{i}_0 \cdot \boldsymbol{i}_0 = 1 \quad \boldsymbol{j}_0 \cdot \boldsymbol{j}_0 = 1 \quad \text{and} \quad \boldsymbol{k}_0 \cdot \boldsymbol{k}_0 = 1. \tag{2.3}$$

The last relationships imply of course that $\boldsymbol{i}_0$,$\boldsymbol{j}_0$ and $\boldsymbol{k}_0$ are unit vectors. Also, the dot product of different vectors is always zero, i.e.

$$\boldsymbol{i}_0 \cdot \boldsymbol{j}_0 = 0 \quad \boldsymbol{j}_0 \cdot \boldsymbol{k}_0 = 0 \quad \text{and} \quad \boldsymbol{k}_0 \cdot \boldsymbol{i}_0 = 0. \tag{2.4}$$

Recall that the dot product is commutative.

Instead of using (2.1), a more practical notation is the following

$$\boldsymbol{p} = \begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix}. \tag{2.5}$$

Consider now Fig. 2.2, where the very same vector is depicted, but this time together with another coordinate frame o_{x_1,y_1,z_1}.

Clearly, $\boldsymbol{p}$ can be expressed as

$$\boldsymbol{p} = p_{x1}\boldsymbol{i}_1 + p_{y1}\boldsymbol{j}_1 + p_{z1}\boldsymbol{k}_1, \tag{2.6}$$

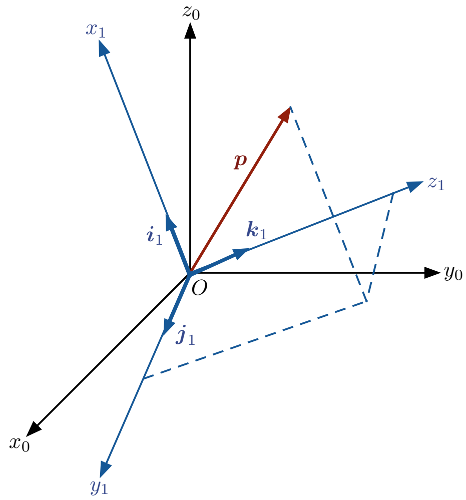

Fig. 2.2 Representation of a point $\boldsymbol{p}$ in the coordinate frame o_{x_1,y_1,z_1}

or

$$\boldsymbol{p} = \begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix}. \tag{2.7}$$

While (2.1) and (2.6) are still perfectly compatible, (2.5) and (2.7) make little sense, and the reason may be rather obvious. The first couple of equations provide a sum of vectors, while the second one provide how those vectors are expressed in different coordinate frames. Therefore, it would be more convenient to rewrite (2.1) and (2.6) as

$$^0\boldsymbol{p} = \begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} \quad \text{and} \quad {}^1\boldsymbol{p} = \begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix}, \tag{2.8}$$

respectively, where in general $^0\boldsymbol{p} \neq {}^1\boldsymbol{p}$ despite they describe the very same vector $\boldsymbol{p}$. The reason for defining two different coordinate frames is that this might constitute an advantage for the representation of a particular point or vector in the space, but now it is necessary to find the relationship between $^0\boldsymbol{p}$ and $^1\boldsymbol{p}$. In fact, from (2.1) and (2.6) one gets

$$\boldsymbol{p} = p_{x0}\boldsymbol{i}_0 + p_{y0}\boldsymbol{j}_0 + p_{z0}\boldsymbol{k}_0 = p_{x1}\boldsymbol{i}_1 + p_{y1}\boldsymbol{j}_1 + p_{z1}\boldsymbol{k}_1. \tag{2.9}$$

This relationship can better be understood when both coordinates systems are removed as shown in Fig. 2.3.

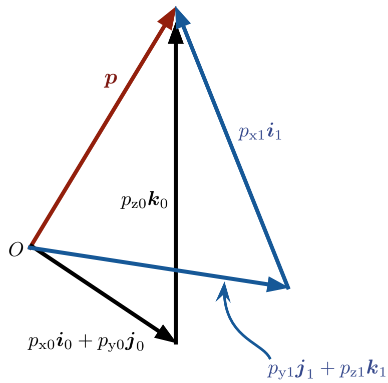

Fig. 2.3 Representing $\boldsymbol{p}$ purely as sum of vectors

By using the properties given in (2.3) and (2.4), one can easily get

$$p_{x0} = \boldsymbol{p} \cdot \boldsymbol{i}_0 = p_{x1}\boldsymbol{i}_1 \cdot \boldsymbol{i}_0 + p_{y1}\boldsymbol{j}_1 \cdot \boldsymbol{i}_0 + p_{z1}\boldsymbol{k}_1 \cdot \boldsymbol{i}_0. \tag{2.10}$$

In the very same fashion it is possible to obtain

$$p_{y0} = p_{x1}\boldsymbol{i}_1 \cdot \boldsymbol{j}_0 + p_{y1}\boldsymbol{j}_1 \cdot \boldsymbol{j}_0 + p_{z1}\boldsymbol{k}_1 \cdot \boldsymbol{j}_0 \tag{2.11}$$

$$p_{z0} = p_{x1}\boldsymbol{i}_1 \cdot \boldsymbol{k}_0 + p_{y1}\boldsymbol{j}_1 \cdot \boldsymbol{k}_0 + p_{z1}\boldsymbol{k}_1 \cdot \boldsymbol{k}_0. \tag{2.12}$$

The last relationships can be written in compact form

$${}^0\boldsymbol{p} = {}^0\boldsymbol{R}_1 {}^1\boldsymbol{p}, \tag{2.13}$$

where

$${}^0\boldsymbol{R}_1 = \begin{bmatrix} \boldsymbol{i}_1 \cdot \boldsymbol{i}_0 & \boldsymbol{j}_1 \cdot \boldsymbol{i}_0 & \boldsymbol{k}_1 \cdot \boldsymbol{i}_0 \\ \boldsymbol{i}_1 \cdot \boldsymbol{j}_0 & \boldsymbol{j}_1 \cdot \boldsymbol{j}_0 & \boldsymbol{k}_1 \cdot \boldsymbol{j}_0 \\ \boldsymbol{i}_1 \cdot \boldsymbol{k}_0 & \boldsymbol{j}_1 \cdot \boldsymbol{k}_0 & \boldsymbol{k}_1 \cdot \boldsymbol{k}_0 \end{bmatrix}. \tag{2.14}$$

${}^0\boldsymbol{R}_1 \in \mathbb{R}^{3\times 3}$ represents a transformation which maps the vector $\boldsymbol{p}$ expressed in o_{x_1,y_1,z_1}, i.e. ${}^1\boldsymbol{p}$, to its representation in the frame o_{x_0,y_0,z_0}, i.e. ${}^0\boldsymbol{p}$. Note that the very same procedure can be employed to find the transformation from o_{x_0,y_0,z_0} to o_{x_1,y_1,z_1}:

$${}^1\boldsymbol{p} = {}^1\boldsymbol{R}_0 {}^0\boldsymbol{p}, \tag{2.15}$$

where

$${}^1\boldsymbol{R}_0 = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 \\ \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 \\ \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix}. \tag{2.16}$$