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Direct Methods in the Calculus of Variations

Second Edition

Springer

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Preface

The present monograph is a revised and augmented edition to Direct Methods in the Calculus of Variations [179] which is now out of print. The core and the structure of the present book are essentially the one of [179], although it has now almost doubled its size. While writing the present volume, it clearly appeared to me that a new subject has emerged and that it deserves to be called "*quasiconvex analysis*". This name, of course, refers to "convex analysis", although the new subject is still in its infancy when compared with the classical one.

The calculus of variations is an immense and very active field. It is therefore, when writing a book, necessary to make a severe selection. This was already the case for [179] and is even more so for this new edition. Rather than superficially covering a lot of materials, I preferred to privilege only some aspects of the field. Here are some main features of the book. I strongly emphasized the resemblances between convex and quasiconvex analysis as well as the "algebraic" aspect of the field, notably through the determinants and singular values. Besides the classical results on lower semicontinuity and relaxation, an important feature of the monograph is the emphasis on the existence of minimizers for non convex problems.

In doing so I missed several important aspects of the calculus of variations such as regularity theory, study of stationary points, existence and relaxation in BV spaces, minimal surfaces, Young measures and the mathematical study of microstructures, Γ convergence and homogenization. However there are already several excellent books on these subjects, some of them very classical, such as: Almgren [18], Ambrosio-Fusco-Pallara [25], Braides-Defranceschi [101], Buttazzo [112], Buttazzo-Giaquinta-Hildebrandt [117], Dal Maso [217], Dierkes-Hildebrandt-Küster-Wohlrab [248], Dolzmann [249], Ekeland [263], Ekeland-Temam [264], Evans [271], Fonseca-Leoni [284], Giaquinta [307], Giaquinta-Hildebrandt [309], Giaquinta-Modica-Soucek [312], Gilbarg-Trudinger [313], Giusti [315], [316], Ladyzhenskaya-Uraltseva [388], Mawhin-Willem [440], Morrey [455], Müller [462], Nitsche [476], Pedregal [492], Roubicek [517] or Struwe [546], [547]. I have also added in the bibliography several articles which present important developments that I did not discuss in the present monograph, but are still closely related.

For a reader not very familiar with the calculus of variations, it might be advisable to start with an introductory book such as [180], which could be considered as a companion to the present one. Nevertheless, the present monograph, which is essentially a reference book on the subject of quasiconvex analysis, can be used, as was [179], for an advanced course on the calculus of variations.

I would next like to reiterate my thanks to the people who helped me while writing the earlier version [179], namely J.M. Ball, L. Boccardo, P. Ciarlet, I. Ekeland, J.C. Evard, B. Kawohl, P. Marcellini, J. Moser, C.A. Stuart, E. Zehnder and B. Zwahlen.

However, since then I have benefited of many other important discussions. Surely the most influential ones were with P. Marcellini, with whom I have a long standing collaboration. We have written together several articles and a book [202], which helped me in writing Part III of the present monograph. I want also to recall fruitful discussions with E. Acerbi, J.J. Alibert, N. Ansini, G. Aubert, S. Bandyopadhyay, A.C. Barroso, H. Brézis, G. Buttazzo, P. Cardaliaguet, A. Cellina, G. Croce, G. Dal Maso, F. De Blasi, E. De Giorgi, O. Dosly, J. Douchet, A. Ferriero, I. Fonseca, N. Fusco, W. Gangbo, N. Georgy, F. Gianetti, J.P. Haeberly, H. Hartwig, S. Hildebrandt, T. Iwaniec, O. Kneuss, H. Koshigoe, P.L. Lions, J. Maly, P. Maréchal, A. Martinaglia, E. Mascolo, J. Matias, P. Metzener, G. Mingione, G. Modica, S. Müller, F. Murat, G. Pianigiani, G. Pisante, L. Poggiolini, A.M. Ribeiro, N. Rochat, C. Sbordone, K.D. Semmler, V. Sverak, M. Sychev, R. Tahraoui, C. Tanteri, L. Tartar, M. Troyanov and K. Zhang.

My thanks also go to Mme. G. Rime, who typed the manuscript of [179], and to Mme. M.F. De Carmine, who typed an earlier version of the present monograph. Finally, M. Hägler and C. Hebeisen prepared for me all the figures included in the book.

During the past several years, I have benefited from grants from the Fonds National Suisse and the Troisième Cycle Romand. Of course, particular thanks go to the Section de Mathématiques of the Ecole Polytechnique Fédérale de Lausanne.

Chapter 1

Introduction

1.1 The direct methods of the calculus of variations

The main problem that we will be investigating throughout the present monograph is the following. Consider the functional

$$I(u) := \int_{\Omega} f(x, u(x), \nabla u(x))\, dx$$

where

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded open set and a point in Ω is denoted by $x = (x_1, ..., x_n)$;

- $u : \Omega \to \mathbb{R}^N$, $N \geq 1$, $u = \left(u^1, \cdots, u^N\right)$, and hence

$$\nabla u = \left(\frac{\partial u^j}{\partial x_i}\right)_{1 \leq i \leq n}^{1 \leq j \leq N} \in \mathbb{R}^{N \times n};$$

- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, is a given function.

We say that the problem under consideration is *scalar* if either $N = 1$ or $n = 1$; otherwise we speak of the *vectorial* case.

Associated to the functional I is the minimization problem

$$(P) \quad m := \inf \{I(u) : u \in X\},$$

meaning that we wish to find $\overline{u} \in X$ such that

$$m = I(\overline{u}) \leq I(u) \text{ for every } u \in X.$$

Here X is the space of admissible functions (in most parts, it is the Sobolev space $u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$, where u_0 is a given function).

We now give several examples.

(1) The classical calculus of variations dealt essentially with the case $n = N = 1$, where the most celebrated examples are the *Fermat principle* in geometrical optics, where

$$f(x,u,\xi) := g(x,u)\sqrt{1+\xi^2},$$

the *Newton problem*, where

$$f(x,u,\xi) = f(u,\xi) := 2\pi u \frac{\xi^3}{1+\xi^2},$$

or the *brachistochrone* problem, where

$$f(x,u,\xi) = f(u,\xi) := \frac{\sqrt{1+\xi^2}}{\sqrt{2gu}}.$$

(2) When turning our attention to the case $n > N = 1$ (in our terminology, it is still part of the scalar case), the *Dirichlet integral* surely plays a central role; we have there

$$f(x,u,\xi) = f(\xi) := \frac{1}{2}|\xi|^2.$$

A natural generalization is when $1 < p < \infty$ and

$$f(x,u,\xi) = f(\xi) := \frac{1}{p}|\xi|^p.$$

The *minimal surface in non-parametric form* enters also in this framework; we have in this case

$$f(x,u,\xi) = f(\xi) := \sqrt{1+|\xi|^2}.$$

In geometrical terms, the integral represents the area of the surface given by $(x, u(x)) \in \mathbb{R}^{n+1}$ when $x \in \Omega \subset \mathbb{R}^n$.

(3) In the vectorial case $n, N \geq 2$, the first example is the case of *minimal surfaces in parametric form*, a geometrical framework more general than the preceding one. In this case, we have $N = n+1$ and therefore the matrix $\xi \in \mathbb{R}^{(n+1)\times n}$. We denote by $\operatorname{adj}_n \xi \in \mathbb{R}^{n+1}$ the vector formed by all the $n \times n$ minors of the matrix ξ. Finally, we let

$$f(x,u,\xi) = f(\xi) := |\operatorname{adj}_n \xi|,$$

where $|.|$ stands for the Euclidean norm. In geometrical terms, the integral represents the area of the surface given by $u(x) \in \mathbb{R}^{n+1}$ when $x \in \Omega \subset \mathbb{R}^n$; moreover, $\operatorname{adj}_n \nabla u$ represents the normal to the surface.

Other important examples in the vectorial case are motivated by non-linear elasticity. A particularly simple one is when $N = n$ and

$$f(x,u,\xi) = f(\xi) := g(\det \xi),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given function.

We do not discuss the history of the calculus of variations and we refer for this matter to the books of Dierkes-Hildebrandt-Küster-Wohlrab [248], Giaquinta-Hildebrandt [309], Goldstine [319] and Monna [449].

The first question that arises in conjunction with problem (P) is, of course, the existence of minimizers. This strongly depends on the choice of admissible functions, which we denoted by X. A natural choice would be a subspace of $C^1\left(\Omega;\mathbb{R}^N\right)$, or even $C^2\left(\Omega;\mathbb{R}^N\right)$, if we want to be able to write the differential equation naturally associated to the minimization problem and known as the *Euler-Lagrange equation*. This turns out to be a strategy too hard to implement in most problems, particularly those dealing with partial derivatives (i.e. $n > 1$). The essence of the *direct methods of the calculus of variations* is to split the problem into two parts. First to enlarge the space of admissible functions, for example by considering spaces such as the Sobolev spaces $W^{1,p}$ so as to get a general *existence* theorem and then to prove some *regularity* results that should satisfy any minimizer of (P). In the present book, we are essentially concerned only with the first problem. In most cases, the space of admissible functions is

$$X = u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right),$$

where u_0 is a given function and the notation $u \in X$ is a shortcut meaning that $u = u_0$ on $\partial\Omega$ and $u \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$.

The existence of minimizers in the above space relies on the fundamental property of (sequential) *weak lower semicontinuity*, meaning that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p} \;\Rightarrow\; \liminf_{\nu\to\infty} I\left(u_\nu\right) \geq I\left(\overline{u}\right), \tag{1.1}$$

where $\rightharpoonup$ stands for weak convergence. This property is thoroughly investigated, notably in Chapters 3 and 8.

It turns out that the property (1.1) is intimately related to the convexity of the function $\xi \to f\left(x,u,\xi\right)$ in the scalar case where $N = 1$ or $n = 1$ and to the quasiconvexity (in the sense of Morrey) of the same function in the vectorial case.

This leads us to the study of *convex analysis* in Chapter 2 and *quasiconvex analysis* in Chapters 5, 6 and 7.

We now discuss in more details the content of the monograph and outline some of the main results in every chapter. We state them, most of the time, under slightly stronger hypotheses than needed, but we refer to the precise theorems at each step.

1.2 Convex analysis and the scalar case

We start with the scalar case where $n = 1$ or $N = 1$. The first one corresponds to the case of one single independent variable and is much easier to deal with, in particular from the point of view of regularity. It is discussed in the general framework of the scalar case in Chapter 3 but also has a special treatment in Chapter 4. The second case, $n > N = 1$, involves partial derivatives and is considerably harder; it is discussed in Chapter 3. However, since both cases use in a significant way many results of convex analysis, we start with the study of this classical subject.

1.2.1 Convex analysis

In Chapter 2, we present the most important results of convex analysis. Even though many excellent books exist on the subject, we have decided, for the convenience of the reader, to state and to prove all the results that we need. Another motivation in the presentation of this chapter has been to stress both the similarities and the differences with quasiconvex analysis, which is discussed in Part II.

Traditionally, convex analysis starts with the notion of a convex set and then continues with that of convex functions. This is also the path we have followed, in contrast with the quasiconvex case.

We start by recalling the notion of a convex set. A set $E \subset \mathbb{R}^N$ is said to be *convex* if for every $x, y \in E$ and every $t \in [0,1]$

$$tx + (1-t)\, y \in E.$$

We then give several elementary properties concerning the interior, closure and boundary of convex sets. We next turn to two of the most useful results for convex sets, namely the *separation theorems* (see Corollary 2.11) and *Carathéodory theorem* (see Theorem 2.13). A typical separation theorem is, for example, the following.

Theorem 1.1 *Let $E \subset \mathbb{R}^N$ be convex and $\overline{x} \in \partial E$. Then there exists $a \in \mathbb{R}^N$, $a \neq 0$, so that*

$$\langle \overline{x}; a \rangle \leq \langle x; a \rangle \text{ for every } x \in E,$$

where $\langle .;. \rangle$ denotes the scalar product in $\mathbb{R}^N$.

We also recall that the *convex hull* of a set $E \subset \mathbb{R}^N$ is the smallest convex set containing E and is denoted by $\operatorname{co} E$. Carathéodory theorem then states the following.

Theorem 1.2 *Let $E \subset \mathbb{R}^N$. Then*

$$\operatorname{co} E = \left\{ x \in \mathbb{R}^N : x = \textstyle\sum_{i=1}^{N+1} \lambda_i x_i\,,\ x_i \in E,\ \lambda_i \geq 0 \text{ with } \sum_{i=1}^{N+1} \lambda_i = 1 \right\}.$$

We then conclude this brief account on convex sets by recalling the notion of extreme points of a convex set and *Minkowski theorem*, ensuring that if E is compact and E_{ext} denotes the set of extreme points of $\operatorname{co} E$, then

$$\operatorname{co} E = \operatorname{co} E_{ext}\,.$$

We next discuss the concept of a convex function. We recall that a function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for every $x, y \in \mathbb{R}^N$ and every $t \in [0,1]$. An important property of convex functions that take only finite values (i.e. $f : \mathbb{R}^N \to \mathbb{R}$) is that they are everywhere continuous (see Theorem 2.31).

The notions of convex set and function are related through the *indicator function* of a set E defined by

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E. \end{cases}$$

Indeed the function χ_E is convex if and only if the set E is convex.

As we defined the notion of a convex hull for a set, a natural concept is the *convex envelope* of a given function f, which is, by definition, the largest convex function below f and is denoted by Cf. We can therefore write, for every $x \in \mathbb{R}^N$,

$$Cf(x) := \sup\{g(x) : g \leq f \quad \text{and} \quad g \text{ convex}\}.$$

Of central importance in convex analysis is the concept of a conjugate function (or *Legendre transform*). The *conjugate* of a function f is a function $f^* : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) := \sup_{x \in \mathbb{R}^N} \{\langle x; x^* \rangle - f(x)\},$$

which is a convex function, independently of the convexity of f. Iterating the process, we define the *biconjugate* of f as $f^{**} : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$, it is given by

$$f^{**}(x) = \sup_{x^* \in \mathbb{R}^N} \{\langle x; x^* \rangle - f^*(x^*)\}.$$

It turns out that if f takes only finite values then (see Theorem 2.43)

$$Cf = f^{**}.$$

Finally, we also investigate the differentiability of convex functions, discussing, in particular, the notion of a *subgradient*.

1.2.2 Lower semicontinuity and existence results

The main result of Chapter 3 is the following (more general ones are found in Theorem 3.15 and Corollary 3.24).

Theorem 1.3 *Let $n, N \in \mathbb{N}$, $p \geq 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary, $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ be a non-negative continuous function and*

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx.$$

Part 1. If the function $\xi \to f(x, u, \xi)$ *is convex, then* I *is (sequentially) weakly lower semicontinuous in* $W^{1,p}$ *(meaning that (1.1) is satisfied).*

Part 2. Conversely, if either $N = 1$ *or* $n = 1$ *and* I *is (sequentially) weakly lower semicontinuous in* $W^{1,p}$*, then the function* $\xi \to f(x, u, \xi)$ *is convex.*

We should emphasize that in the vectorial case, $n, N \geq 2$, Part 1 of the theorem is valid but the conclusion of Part 2 does not hold.

This theorem, in the scalar case, has as a first direct consequence that the functional is (sequentially) weakly continuous in $W^{1,p}$, meaning that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p} \;\Rightarrow\; \lim_{\nu\to\infty} I(u_\nu) = I(\overline{u})$$

if and only if $\xi \to f(x, u, \xi)$ is affine. This result again strongly contrasts with the vectorial case.

The main implication of the lower semicontinuity theorem is on the existence of minimizers for the problem

$$(P) \quad \inf\left\{I(u) : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)\right\} = m.$$

Indeed we have, as a special case of our general theorem (see Theorem 3.30), the following result.

Theorem 1.4 *Let* Ω *be a bounded open set of* $\mathbb{R}^n$ *with a Lipschitz boundary. Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ *be continuous and satisfying the* coercivity *condition*

$$f(x, u, \xi) \geq \alpha_1 |\xi|^p - \alpha_2\,, \;\; \forall\, (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n},$$

for some $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$ *and* $p > 1$. *Assume that* $\xi \to f(x, u, \xi)$ *is* convex *and that* $I(u_0) < \infty$. *Then* (P) *has at least one minimizer.*

This theorem is also valid in the vectorial case, but can then be improved a great deal.

As is well known, associated with any variational problem is the differential equation known as the *Euler-Lagrange equation.* Under appropriate regularity hypotheses on the function f and on a minimizer $\overline{u}$ of (P), we find that $\overline{u}$ should satisfy, for every $x \in \Omega$,

$$(E) \quad \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha}\Big[\frac{\partial f}{\partial \xi_\alpha^i}(x, \overline{u}, \nabla\overline{u})\Big] = \frac{\partial f}{\partial u^i}(x, \overline{u}, \nabla\overline{u}),\ i = 1, \cdots, N.$$

The differential equation is a second order ordinary differential equation if $n = N = 1$, a system of such equations if $N > n = 1$, a single second order partial differential equation if $n > N = 1$ and a system of such equations when $n, N \geq 2$. In any case, the convexity of the function $\xi \to f(x, u, \xi)$ ensures the *ellipticity* of the Euler-Lagrange equations. The prototype example is the Dirichlet integral where $n > N = 1$,

$$f(x, u, \xi) = f(\xi) := \frac{1}{2}|\xi|^2,$$

and the associated equation is nothing other than the *Laplace equation*

$$\Delta u = 0.$$

1.2.3 The one dimensional case

In Chapter 4, we specialize to the case where $N = n = 1$, although most of the results are also valid if $N > n = 1$. We are therefore considering the problem

$$(P) \quad \inf \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \;:\; u \in X \right\},$$

where $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p \geq 1$ and

$$X = \left\{ u \in W^{1,p}(a,b),\ u(a) = \alpha,\ u(b) = \beta \right\}.$$

The Euler-Lagrange equation that should satisfy any minimizer $\overline{u}$ of (P) is then given by

$$(E) \quad \frac{d}{dx}\left[f_\xi (x, \overline{u}(x), \overline{u}'(x)) \right] = f_u (x, \overline{u}(x), \overline{u}'(x)),\ x \in [a,b],$$

where $f_\xi = \partial f / \partial \xi$ and $f_u = \partial f / \partial u$. When the function f does not depend explicitly on the variable x, one can find a first integral of (E) that is known as the *second form* of the Euler-Lagrange equation and can be written as

$$f(\overline{u}(x), \overline{u}'(x)) - \overline{u}'(x) f_\xi (\overline{u}(x), \overline{u}'(x)) = \text{constant},\ x \in [a,b].$$

At this stage it might be enlightening to see some examples that show that, even when $n = N = 1$, the hypotheses of the existence theorem (see Theorem 1.4) are essentially optimal. Indeed *non-existence* of minimizers in Sobolev spaces occurs in all the following cases.

(1) Let (see Example 4.4) $f(\xi) = e^{-\xi^2}$ and

$$(P) \quad \inf \left\{ I(u) = \int_0^1 f(u'(x))\, dx : u \in X \right\},$$

where $X = W_0^{1,1}(0,1) = \left\{ u \in W^{1,1}(0,1) : u(0) = u(1) = 0 \right\}$. Here both the convexity and coercivity hypotheses of the theorem are violated.

(2) Consider (see Example 4.5) the case $f(x,u,\xi) = f(u,\xi) = \sqrt{u^2 + \xi^2}$ and

$$(P) \quad \inf \left\{ I(u) = \int_0^1 f(u(x), u'(x))\, dx : u \in X \right\},$$

where $X = \left\{ u \in W^{1,1}(0,1) : u(0) = 0,\ u(1) = 1 \right\}$. In this case, the coercivity condition holds with $p = 1$ (and not, as it should, with $p > 1$).

(3) The present example (see Example 4.6) is known as the *Weierstrass example.* Let $f(x,u,\xi) = f(x,\xi) = x\xi^2$ and

$$(P) \quad \inf\left\{ I(u) = \int_0^1 f(x, u'(x))\, dx : u \in X \right\},$$

where $X = \left\{ u \in W^{1,2}(0,1) : u(0) = 1,\ u(1) = 0 \right\}$. The coercivity hypothesis is violated at just one point (namely at $x = 0$).

(4) Let (the example is known as the *Bolza example*, see Example 4.8)

$$f(x,u,\xi) = f(u,\xi) = \left(\xi^2 - 1\right)^2 + u^4$$

$$(P) \quad \inf\left\{ I(u) = \int_0^1 f(u(x), u'(x))\, dx : u \in W_0^{1,4}(0,1) \right\}.$$

Here it is the convexity assumption on the function $\xi \to f(x,u,\xi)$ that is not satisfied.

Another advantage of the case $N = n = 1$ is that, under appropriate conditions on f, notably the convexity of $\xi \to f(x,u,\xi)$, the solutions of (E) are also solutions and conversely (see Theorem 4.29) of the *Hamiltonian system*

$$(H) \begin{cases} u'(x) = H_v(x, u(x), v(x)) \\ v'(x) = -H_u(x, u(x), v(x)), \end{cases}$$

where $v(x) = f_\xi(x, u(x), u'(x))$ and H is the Legendre transform of $\xi \to f(x,u,\xi)$, namely

$$H(x,u,v) = \sup_{\xi \in \mathbb{R}} \{ v\,\xi - f(x,u,\xi) \}.$$

In classical mechanics, f is called the *Lagrangian* and H the *Hamiltonian.*

We conclude the study of Chapter 4 with a brief discussion on *Lavrentiev phenomenon.* We just study the following example (see Theorem 4.41) exhibited by Mania. We let

$$f(x,u,\xi) = \left(x - u^3\right)^2 \xi^6,$$

$$I(u) = \int_0^1 f(x, u(x), u'(x))\, dx.$$

Consider the two different Sobolev spaces

$$\begin{aligned} \mathcal{W}_\infty &= \left\{ u \in W^{1,\infty}(0,1) : u(0) = 0,\ u(1) = 1 \right\}, \\ \mathcal{W}_1 &= \left\{ u \in W^{1,1}(0,1) : u(0) = 0,\ u(1) = 1 \right\}, \end{aligned}$$

and the corresponding minimization problems

$$\inf\{ I(u) : u \in \mathcal{W}_\infty \} = m_\infty \quad \text{and} \quad \inf\{ I(u) : u \in \mathcal{W}_1 \} = m_1 .$$

We prove that

$$m_\infty > m_1 = 0$$

and that $\overline{u}(x) = x^{1/3}$ is a minimizer of I over $\mathcal{W}_1$.

1.3 Quasiconvex analysis and the vectorial case

We next turn to the vectorial case $n, N \geq 2$, which is the heart of our book and deals with what we call *quasiconvex analysis.* The structure is similar to that of Part I; namely, we develop the quasiconvex analysis in Chapters 5, 6 and 7 and then discuss lower semicontinuity and existence results in Chapter 8.

A first striking difference between our presentations of convex and quasiconvex analyses is the order in which we deal with sets and functions. In convex analysis we first defined, as do essentially all other authors, the concept of convex sets and then that of convex functions. In the present context, we do exactly the reverse. This has some historical reasons. The notion of a quasiconvex function was introduced by Morrey in 1952, while the corresponding notion for sets appeared almost fifty years later and is, in some sense, in its infancy.

The main motivation for introducing the notion of quasiconvexity is to generalize Theorem 1.3 to the vectorial case.

1.3.1 Quasiconvex functions

Unfortunately, when generalizing the notion of a convex function to the vectorial case, several different concepts arise naturally. The notion of a *quasiconvex* function arises, as already said, in conjunction with (sequential) weak lower semicontinuity of the corresponding integral. When dealing with the Euler-Lagrange equation, the right concept is the ellipticity and this leads to the definition of a *rank one convex* function. Finally, when one wants to generalize the separation theorems, Carathéodory theorem, or the notion of duality, one is driven to the concept of *polyconvexity.*

We now describe the content of Chapter 5 and we start with the following definitions.

Definition 1.5 *Let* $f : \mathbb{R}^{N\times n} \to \mathbb{R}$.

(i) The function f *is said to be* rank one convex *if*

$$f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda) f(\eta)$$

for every $\lambda \in [0,1]$, $\xi, \eta \in \mathbb{R}^{N\times n}$ *with* $\operatorname{rank}\{\xi - \eta\} \leq 1$.

(ii) If f *is Borel measurable and locally bounded, then it is said to be* quasiconvex *if*

$$f(\xi) \leq \frac{1}{\operatorname{meas} D} \int_D f(\xi + \nabla\varphi(x))\, dx$$

for every bounded open set $D \subset \mathbb{R}^n$, *for every* $\xi \in \mathbb{R}^{N \times n}$ *and for every* $\varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)$.

(iii) The function f *is said to be* polyconvex *if there exists* $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ *convex, such that*

$$f\left(\xi\right) = F\left(T\left(\xi\right)\right),$$

where $T : \mathbb{R}^{N \times n} \to \mathbb{R}^{\tau(n,N)}$ *is such that*

$$T\left(\xi\right) = \left(\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_{n \wedge N} \xi\right).$$

In the previous definition, $\operatorname{adj}_s \xi$ *stands for the matrix of all* $s \times s$ *minors of the matrix* $\xi \in \mathbb{R}^{N \times n}$, $2 \leq s \leq n \wedge N = \min\left\{n, N\right\}$, *and*

$$\tau\left(n, N\right) = \sum_{s=1}^{n \wedge N} \sigma\left(s\right)$$

where

$$\sigma\left(s\right) = \binom{N}{s}\binom{n}{s} = \frac{N!n!}{\left(s!\right)^2\left(N-s\right)!\left(n-s\right)!}.$$

(iv) A function f *is said to be* rank one affine, quasiaffine *or* polyaffine *if* f and $-f$ *are rank one convex, quasiconvex or polyconvex respectively.*

Remark 1.6 (i) Note that in the case $N = n = 2$, the notion of polyconvexity can be read as follows:

$$\begin{cases} \tau\left(n, N\right) = \tau\left(2, 2\right) = 5 \ \ (\text{since } \sigma\left(1\right) = 4,\ \sigma\left(2\right) = 1) \\ T\left(\xi\right) = \left(\xi, \det \xi\right),\ f\left(\xi\right) = F\left(\xi, \det \xi\right). \end{cases}$$

(ii) The first and third definitions extend in a straightforward manner to functions $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \left\{+\infty\right\}$. However this is not the case for quasiconvex functions. At the moment, no good definition of quasiconvexity for such functions is available. This fact is a strong source of difficulty when dealing with the definition of quasiconvex sets. ♢

The main properties of these functions are now given (see Theorems 5.3 and 5.20).

Theorem 1.7 *Let* $f : \mathbb{R}^{N \times n} \to \mathbb{R}$.

(i) The following implications hold

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

(ii) If $N = 1$ *or* $n = 1$, *then all these notions are equivalent.*

(iii) If $f \in C^2\left(\mathbb{R}^{N \times n}\right)$, *then rank one convexity is equivalent to the* Legendre-Hadamard *condition (or* ellipticity *condition)*

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f\left(\xi\right)}{\partial \xi_\alpha^i \partial \xi_\beta^j} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0$$

for every $\lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^n, \xi = \left(\xi_\alpha^i\right)_{1\leq\alpha\leq n}^{1\leq i\leq N} \in \mathbb{R}^{N\times n}$.

(iv) The notions of rank one affine, quasiaffine and polyaffine are equivalent. Moreover, any quasiaffine function is of the form

$$f(\xi) = \alpha + \langle \beta; T(\xi) \rangle ,$$

where $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R}^{\tau(n,N)}$ *and* $\langle .; . \rangle$ *denotes the scalar product in* $\mathbb{R}^{\tau(n,N)}$.

We now give some significant examples. The first one (see Theorem 5.25) concerns quadratic forms and is one of the most important, since then the associated Euler-Lagrange equations are *linear*.

Theorem 1.8 *Let* M *be a symmetric matrix in* $\mathbb{R}^{(N\times n)\times(N\times n)}$. *Let*

$$f(\xi) = \langle M\xi; \xi \rangle ,$$

where $\xi \in \mathbb{R}^{N\times n}$ *and* $\langle \cdot; \cdot \rangle$ *denotes the scalar product in* $\mathbb{R}^{N\times n}$. *The following statements then hold.*

(i) f *is rank one convex if and only if* f *is quasiconvex.*

(ii) If $N = 2$ *or* $n = 2$, *then*

$$f \text{ polyconvex } \Leftrightarrow f \text{ quasiconvex } \Leftrightarrow f \text{ rank one convex.}$$

(iii) If $N, n \geq 3$, *then in general*

$$f \text{ rank one convex } \not\Rightarrow f \text{ polyconvex.}$$

We next turn to some more examples.

1) Let $N = n = 2$. The function

$$f(\xi) = \det \xi$$

is quasiaffine and thus polyconvex, quasiconvex or rank one convex, but not convex.

2) When $n \geq 2$ and $N \geq 3$, Sverak (see Theorem 5.50) produced an example of a function that is rank one convex but not quasiconvex, answering a long standing conjecture of Morrey. It is still not known if there are rank one convex but not quasiconvex functions in the case $N = n = 2$, or more generally $n \geq N = 2$.

3) Let $N = n = 2$. The function studied by Alibert-Dacorogna-Marcellini (see Theorem 5.51) and given by $f_\gamma : \mathbb{R}^{2\times 2} \to \mathbb{R}$, for $\gamma \in \mathbb{R}$, where

$$f_\gamma(\xi) = |\xi|^2 \left(|\xi|^2 - 2\gamma \det \xi \right),$$

is such that

$$\begin{array}{rcl} f_\gamma \text{ is convex} & \Leftrightarrow & |\gamma| \leq \gamma_c = 2\sqrt{2}/3, \\ f_\gamma \text{ is polyconvex} & \Leftrightarrow & |\gamma| \leq \gamma_p = 1, \\ f_\gamma \text{ is quasiconvex} & \Leftrightarrow & |\gamma| \leq \gamma_q \ \text{ and } \gamma_q > 1, \\ f_\gamma \text{ is rank one convex} & \Leftrightarrow & |\gamma| \leq \gamma_r = 2/\sqrt{3}\,. \end{array}$$

It is not presently known if $\gamma_q = 2/\sqrt{3}\,$.

1.3.2 Quasiconvex envelopes

In Chapter 6, we define the convex Cf (already defined in Section 1.2.1) polyconvex Pf, quasiconvex Qf and rank one convex envelope Rf, which are, respectively, defined as the largest convex, polyconvex, quasiconvex and rank one convex functions below f. We therefore have, for every $\xi \in \mathbb{R}^{N\times n}$,

$$\begin{array}{rcl} Cf(\xi) & = & \sup\{g(\xi) : g \leq f \ \text{ and } \ g \text{ convex}\}, \\ Pf(\xi) & = & \sup\{g(\xi) : g \leq f \ \text{ and } \ g \text{ polyconvex}\}, \\ Qf(\xi) & = & \sup\{g(\xi) : g \leq f \ \text{ and } \ g \text{ quasiconvex}\}, \\ Rf(\xi) & = & \sup\{g(\xi) : g \leq f \ \text{ and } \ g \text{ rank one convex}\}. \end{array}$$

Observe that Theorem 1.7 immediately implies

$$Cf \leq Pf \leq Qf \leq Rf \leq f.$$

Several representation formulas exist for computing these envelopes, we just give a formula for the quasiconvex envelope (see Theorem 6.9).

Theorem 1.9 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be locally bounded, non-negative and Borel measurable. Then, for every $\xi \in \mathbb{R}^{N\times n}$,*

$$Qf(\xi) = \inf\left\{\frac{1}{\operatorname{meas} D}\int_D f(\xi + \nabla\varphi(x))\,dx : \varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)\right\},$$

where $D \subset \mathbb{R}^n$ is a bounded open set. In particular, the infimum in the formula is independent of the choice of D.

We now give some examples.

(1) Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N\times n} \to \mathbb{R}$ be quasiaffine not identically constant and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(\xi) = g(\Phi(\xi)).$$

Then (see Theorem 6.24)

$$Pf = Qf = Rf = Cg \circ \Phi$$

and in general $Qf > Cf$.

(2) Recall the area type case, where $N = n + 1$. Let $f : \mathbb{R}^{(n+1)\times n} \to \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that

$$f(\xi) = g(\operatorname{adj}_n \xi).$$

Then (see Theorem 6.26)

$$Pf = Qf = Rf = Cg \circ \operatorname{adj}_n$$

and in general $Qf > Cf$.

(3) An interesting problem in optimal design is the following. Let $N = n = 2$ and, for $\xi \in \mathbb{R}^{2\times 2}$,

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

Then (see Theorem 6.28) $Pf = Qf = Rf$ and

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\ 2(|\xi|^2 + 2|\det \xi|)^{1/2} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1. \end{cases}$$

We also have

$$Cf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi| \geq 1 \\ 2|\xi| & \text{if } |\xi| < 1. \end{cases}$$

1.3.3 Quasiconvex sets

We have seen in Section 1.2.1 that the connection between convex functions and sets is made via the indicator function. We recall that, for a set E, the indicator function is defined by

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E. \end{cases}$$

Moreover, the function χ_E is convex if and only if the set E is convex.

The aim of Chapter 7 is to extend the definition of convexity for sets to polyconvexity, quasiconvexity and rank one convexity. A natural way to define polyconvex, quasiconvex or rank one convex set E would be by requiring that χ_E be polyconvex, quasiconvex or rank one convex. This is indeed so (see Proposition 7.5) for the first and third cases but not for quasiconvex sets, since, as we already said, we lack a good definition of quasiconvexity for functions that are allowed to take the value $+\infty$.

Before giving the definitions, let us introduce some notation. In this section we let $O(n)$ be the set of $n \times n$ orthogonal matrices,

$$D := (0,1)^n \subset \mathbb{R}^n$$

and $W^{1,\infty}_{per}(D;\mathbb{R}^N)$ be the space of periodic functions in $W^{1,\infty}(D;\mathbb{R}^N)$, meaning that

$$u(x) = u(x + e_i),\ \text{for every } x \in D \text{ and } i = 1, \cdots, n,$$

where $\{e_1, \cdots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Finally, $\mathcal{W}_{per}$ denotes the subspace of functions in $W^{1,\infty}_{per}(D;\mathbb{R}^N)$, whose gradients take only a finite number of values.

We are now in a position to give the following definitions (see Definition 7.2).

Definition 1.10 *(i) We say that $E \subset \mathbb{R}^{N\times n}$ is* polyconvex *if there exists a convex set $K \subset \mathbb{R}^{\tau(N,n)}$ such that*

$$\left\{\xi \in \mathbb{R}^{N\times n} : T(\xi) \in K\right\} = E.$$

(ii) We say that $E \subset \mathbb{R}^{N\times n}$ is quasiconvex *if we have*

$$\left.\begin{array}{c} \xi + \nabla\varphi(x)R \in E,\ a.e.\ x \in D, \\ \textit{for some } R \in O(n) \textit{ and some } \varphi \in \mathcal{W}_{per} \end{array}\right\} \Rightarrow \xi \in E.$$

(iii) We say that $E \subset \mathbb{R}^{N\times n}$ is rank one convex *if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$ such that* rank $\{\xi - \eta\} = 1$, *then*

$$\lambda\xi + (1-\lambda)\eta \in E.$$

The best definition for quasiconvex sets is unclear. Several definitions have already been considered by other authors. The one we propose here is consistent with known properties for functions and has most properties that are desirable as witnessed by the following theorem (see Theorem 7.7).

Theorem 1.11 *Let $E \subset \mathbb{R}^{N\times n}$. The following implications then hold:*

$$E \textit{ convex} \Rightarrow E \textit{ polyconvex} \Rightarrow E \textit{ quasiconvex} \Rightarrow E \textit{ rank one convex.}$$

All counter implications are false as soon as $N, n \geq 2$.

We should draw attention to the last statement of the theorem. Surprisingly it is better than the corresponding one for functions, where the example of Sverak provides a rank one convex function that is not quasiconvex only when $n \geq 2$ and $N \geq 3$.

Before continuing, one main difference between convex sets and generalized ones should be emphasized. A set can be polyconvex, and thus quasiconvex and rank one convex, and be *disconnected*. Indeed, if $\xi, \eta \in \mathbb{R}^{N\times n}$ are such that rank $\{\xi - \eta\} \geq 2$, then $E = \{\xi, \eta\}$ is polyconvex.

We next point out a fact (the second one in the next proposition) strikingly different from the equivalent one for convex sets (see Proposition 7.24).

Proposition 1.12 *(i) Let* $E \subset \mathbb{R}^{N\times n}$ *be, respectively, a polyconvex, quasiconvex or rank one convex set. Then* $\operatorname{int} E$ *is also, respectively, polyconvex, quasiconvex or rank one convex.*

(ii) There exists a polyconvex and bounded set $E \subset \mathbb{R}^{2\times 2}$ *such that* $\overline{E}$ *is not rank one convex (and hence neither quasiconvex nor polyconvex).*

We next define the *polyconvex*, *quasiconvex* and *rank one convex hulls* of a set $E \subset \mathbb{R}^{N\times n}$ as the smallest polyconvex, quasiconvex and rank one convex sets containing E; they are respectively denoted by $\operatorname{Pco} E$, $\operatorname{Qco} E$ and $\operatorname{Rco} E$.

We clearly have

$$E \subset \operatorname{Rco} E \subset \operatorname{Qco} E \subset \operatorname{Pco} E \subset \operatorname{co} E.$$

Other hulls are also defined in Chapter 7.

We finally conclude this section by giving an example. We first recall that the singular values of a given matrix $\xi \in \mathbb{R}^{n\times n}$, denoted by

$$0 \le \lambda_1(\xi) \le \cdots \le \lambda_n(\xi),$$

are the eigenvalues of $(\xi\xi^t)^{1/2}$. Let $0 < \gamma_1 \le \cdots \le \gamma_n$ and consider the set

$$E = \left\{\xi \in \mathbb{R}^{n\times n} : \lambda_i(\xi) = \gamma_i,\ i = 1, \cdots, n\right\}.$$

We prove (see Theorem 7.43) that

$$\operatorname{co} E = \left\{\xi \in \mathbb{R}^{n\times n} : \textstyle\sum_{i=\nu}^{n} \lambda_i(\xi) \le \sum_{i=\nu}^{n} \gamma_i\,,\ \nu = 1, \cdots, n\right\}$$

$$\operatorname{Pco} E = \operatorname{Qco} E = \operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n} : \textstyle\prod_{i=\nu}^{n} \lambda_i(\xi) \le \prod_{i=\nu}^{n} \gamma_i\,,\ \nu = 1, \cdots, n\right\}.$$

1.3.4 Lower semicontinuity and existence theorems

In Chapter 8, we extend the lower semicontinuity results (see Theorem 1.3) to the vectorial context. This is a delicate matter and, in Chapter 8, we deal with it in several steps. We now gather Theorems 8.1 and 8.11 to obtain the following result.

Theorem 1.13 *Let* $1 \le p < \infty$, $\Omega \subset \mathbb{R}^n$ *be a bounded open set with a Lipschitz boundary and let*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R},\ f = f(x, u, \xi),$$

be a continuous function satisfying

$$0 \le f(x, u, \xi) \le g(x, u)\left(1 + |\xi|^p\right),$$

where

$$g : \Omega \times \mathbb{R}^N \to \mathbb{R},\ g = g(x,u),$$

is a non-negative continuous function. Let

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx.$$

Then I is (sequentially) weakly lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^N)$ if and only if $\xi \to f(x,u,\xi)$ is quasiconvex, i.e.

$$\frac{1}{\operatorname{meas} D} \int_D f(x_0, u_0, \xi_0 + \nabla\varphi(x))\, dx \geq f(x_0, u_0, \xi_0)$$

for every $(x_0, u_0, \xi_0) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$, for every bounded open set $D \subset \mathbb{R}^n$ and for every $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^n)$.

This result has as an immediate corollary that I is (sequentially) weakly continuous in $W^{1,p}$ if and only if $\xi \to f(x,u,\xi)$ is quasiaffine, i.e. all minors of the matrix $\xi \in \mathbb{R}^{N\times n}$ are weakly continuous. We now restate this result, in a more convenient and more general way, in the case where $N = n = 2$ (see Theorem 8.20, Lemma 8.24 and Corollary 8.26). Let us start with the simple but fundamental observation that Jacobian determinants can be written in divergence form. More precisely if $u \in C^2(\Omega;\mathbb{R}^2)$, then letting

$$\operatorname{Det} \nabla u := \frac{\partial}{\partial x_1}\left(u^1 \frac{\partial u^2}{\partial x_2} \right) - \frac{\partial}{\partial x_2}\left(u^1 \frac{\partial u^2}{\partial x_1} \right),$$

we find that

$$\operatorname{Det} \nabla u(x) = \det \nabla u(x),\ \text{for every } x \in \Omega,$$

since we trivially have

$$\begin{aligned} \det \nabla u &= \frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2} - \frac{\partial u^2}{\partial x_1}\frac{\partial u^1}{\partial x_2} \\ &= \frac{\partial}{\partial x_1}\left(u^1 \frac{\partial u^2}{\partial x_2} \right) - \frac{\partial}{\partial x_2}\left(u^1 \frac{\partial u^2}{\partial x_1} \right) = \operatorname{Det} \nabla u. \end{aligned}$$

The quantity $\operatorname{Det} \nabla u$ is called the *distributional Jacobian* of u. We can now state the theorem (see Theorem 8.20, Lemma 8.24, Corollary 8.26 and Example 8.28).

Theorem 1.14 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $1 < p < \infty$, and let*

$$u_\nu \rightharpoonup u \ \text{in } W^{1,p}(\Omega;\mathbb{R}^2).$$

Part 1. *If $p > 2$, then*

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \ \text{in } L^{p/2}(\Omega).$$

If $p = 2$, *the result is false, but the following convergence holds*

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } \mathcal{D}'(\Omega) .$$

Part 2. *If* $p \geq 4/3$, *then* $\operatorname{Det} \nabla u \in \mathcal{D}'(\Omega)$ *and if* $p \geq 2$, *then*

$$\operatorname{Det} \nabla u = \det \nabla u \text{ in } \mathcal{D}'(\Omega) .$$

Part 3. *If* $p > 4/3$, *then*

$$\operatorname{Det} \nabla u_\nu \rightharpoonup \operatorname{Det} \nabla u \text{ in } \mathcal{D}'(\Omega) .$$

If $p \leq 4/3$, *the result is false.*

Theorem 1.13 also has as a direct consequence the following existence theorem (see Theorem 8.29).

Theorem 1.15 *Let* $p > 1$, $\Omega \subset \mathbb{R}^n$ *be a bounded open set with a Lipschitz boundary. Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, *be a continuous function satisfying*

$$\xi \to f(x, u, \xi) \text{ is quasiconvex,}$$

$$\alpha_1 |\xi|^p + \beta_1 \leq f(x, u, \xi) \leq \alpha_2 (|\xi|^p + 1) ,$$

for every $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n}$, *where* $\alpha_2 \geq \alpha_1 > 0$, $\beta_1 \in \mathbb{R}$. *Let*

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N) \right\} .$$

Then (P) *admits at least one minimizer.*

Using Theorem 1.14, we can also prove some existence theorems for polyconvex functions (see Theorem 8.31).

1.4 Relaxation and non-convex problems

In Part III, we go back to the study of

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N) \right\} ,$$

where

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded open set;
- $u : \Omega \to \mathbb{R}^N$, $N \geq 1$ and $u_0 \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a given function;
- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, is a given non-convex (non-quasiconvex in the vectorial case) function.

The direct methods (see Theorems 1.4 and 1.15) do not apply and the general rule is that (P) has no minimizers, as already pointed out in Section 1.2.3.

However, there is a way of defining generalized solutions of (P) via the so called *relaxed problem*

$$(QP) \quad \inf\left\{\overline{I}\left(u\right)=\int_{\Omega}Qf\left(x,u\left(x\right),\nabla u\left(x\right)\right)dx:u\in u_{0}+W_{0}^{1,p}\left(\Omega;\mathbb{R}^{N}\right)\right\},$$

where Qf is the quasiconvex envelope of f (with respect to the last variable ∇u), defined in Section 1.3.2.

The relaxed problem is useful not only to define generalized solutions of (P), but also to show that in many cases, although the direct methods do not apply, the problem (P) does have minimizers.

1.4.1 Relaxation theorems

In Chapter 9, we prove the relaxation theorem (see Theorems 9.1 and 9.8) and we state it here, as usual under stronger hypotheses, in the case where f does not depend on lower order terms.

Theorem 1.16 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Borel measurable function satisfying, for $1 \le p < \infty$,*

$$0 \le f\left(\xi\right) \le \alpha\left(1+\left|\xi\right|^{p}\right), \text{ for every } \xi \in \mathbb{R}^{N\times n},$$

where $\alpha > 0$ is a constant while for $p = \infty$ it is assumed that f is locally bounded. For every $\xi \in \mathbb{R}^{N\times n}$, let

$$Qf\left(\xi\right) = \sup\left\{g\left(\xi\right) : g \le f \text{ and } g \text{ quasiconvex}\right\}$$

be the quasiconvex envelope of f.

Part 1. *Then*

$$\inf\left(P\right) = \inf(QP).$$

More precisely, for every $u \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$, there exists a sequence $\{u_\nu\}_{\nu=1}^{\infty} \subset u + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)$ such that

$$u_\nu \to u \text{ in } L^p\left(\Omega;\mathbb{R}^N\right) \text{ as } \nu \to \infty,$$

$$\int_\Omega f\left(\nabla u_\nu\left(x\right)\right)dx \to \int_\Omega Qf\left(\nabla u\left(x\right)\right)dx \text{ as } \nu \to \infty.$$

Part 2. *Assume, in addition to the hypotheses of Part 1, that, if $1 < p < \infty$, there exist $\alpha \ge \beta > 0$, $\gamma \in \mathbb{R}$ such that*

$$\gamma + \beta\left|\xi\right|^p \le f\left(\xi\right) \le \alpha\left(1+\left|\xi\right|^p\right) \text{ for every } \xi \in \mathbb{R}^{N\times n}.$$

Then, in addition to the conclusions of Part 1, the following holds:

$$u_\nu \rightharpoonup u \text{ in } W^{1,p}\left(\Omega\right) \text{ as } \nu \to \infty.$$