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Mathematical Analysis in Interdisciplinary Research

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Aims and Scope

Optimization has continued to expand in all directions at an astonishing rate. New algorithmic and theoretical techniques are continually developing and the diffusion into other disciplines is proceeding at a rapid pace, with a spot light on machine learning, artificial intelligence, and quantum computing. Our knowledge of all aspects of the field has grown even more profound. At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the field. Optimization has been a basic tool in areas not limited to applied mathematics, engineering, medicine, economics, computer science, operations research, and other sciences.

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Preface

Mathematical Analysis in Interdisciplinary Research provides an extensive account of research as well as research-expository articles in a broad domain of analysis and its various applications in a plethora of fields.

The book focuses to the study of several essential subjects, including optimal control problems, optimal maintenance of communication networks, optimal emergency evacuation with uncertainty, cooperative and noncooperative partial differential systems, variational inequalities and general equilibrium models, anisotropic elasticity and harmonic functions, nonlinear stochastic differential equations, operator equations, max-product operators of Kantorovich type, perturbations of operators, integral operators, dynamical systems involving maximal monotone operators, the three-body problem, deceptive systems, hyperbolic equations, strongly generalized preinvex functions, Dirichlet characters, probability distribution functions, applied statistics, integral inequalities, generalized convexity, global hyperbolicity of spacetimes, Douglas-Rachford methods, fixed point problems, the general Rodrigues problem, Banach algebras, affine group, Gibbs semigroup, relator spaces, sparse data representation, Meier-Keeler sequential contractions, hybrid contractions, and polynomial equations.

This collective effort, which ranges over the abovementioned broad spectrum of topics, is hoped to be useful to both graduate students and researchers who wish to be informed about the latest developments in the corresponding problems treated. The works published within this book will be of particular value for both theoretical and applicable interdisciplinary research.

We would like to express our gratitude to the authors who contributed their valuable papers in this volume. Last but not least, we wish to extend our sincere thanks to the staff of Springer for their valuable assistance throughout the preparation of this book.

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Contents

Quasilinear Operator Equation at Resonance

A. R. Abdullaev and E. A. Skachkova

Abstract This work considers a quasilinear operator equation at resonance. We obtained solvability theorems and formulated corollaries. The current approach is based on a special generalization of the classical Schauder Fixed Point Theorem.

1 Introduction

Let us consider the following equation:

$$Lx = Fx, \tag{1}$$

with a linear bounded operator $L : X \to Y$ and a continuous (generally speaking, a nonlinear) operator $F : X \to Y$, where X and Y are Banach spaces. If the operator L is non-invertible, Eq. (1) is called a resonance case. In particular, periodic problems for systems of ordinary differential equations can be considered as forms of Eq. (1) with non-invertible operator.

In 1970s, an approach, aimed at studying the Eq. (1) at resonance, was introduced. The approach is based on the Lyapunov-Schmidt method, and it reduces the problem of solvability of Eq. (1) to the problem of the existence of a fixed point of some auxiliary operator. In the literature, statements obtained in a similar manner are called "results of the Landesman-Laser theorem type." However, when applying this methodology to specific classes of resonant boundary value problems, certain

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complications arise. Therefore, the question of the effective solvability conditions for Eq. (1) remains relevant.

By now a large number of papers studying Eq. (1) with a non-invertible operator has been accumulated in the scientific literature. Some intuition about the problems and development trends in this area of research can be obtained from the following sources [1–5].

In this paper we propose an approach to study Eq. (1) at resonance based on a special generalization of the classical Schauder Fixed Point Theorem (Theorem 1). Further, the current paper is structured as follows. In Sect. 2, we explain the notation that we use in our paper. Further in the same section, we provide information related to operators L and F. In Sect. 3, we formulate existence theorems for solutions of Eq. (1). In Sect. 4, we consider applying our approach to the periodic problem for an ordinary second-order differential equation.

2 Preliminaries

The following notations and terminology will be used in the rest of the paper. Let X and Y be Banach spaces. The equality $X = X_1 \bigoplus X_2$ denotes that X is a direct sum of bounded subspaces X_1 and X_2. We denote the kernel and the image of the linear operator $L : X \to Y$ as $\ker L$ and $R(L)$, respectively. Let $P : X \to X$ be a linear bounded projection on $\ker L$, $P^c = I - P$ be an additional projection operator, $Q : Y \to Y$ be a projection on $R(L)$, and Q^c be an additional projection operator. Let $L : X \to Y$ be a Fredholm operator. Then $X = X_0 \bigoplus \ker L$, and $Y = Y_0 \bigoplus R(L)$.

A restriction of operator L is regarded as operator $L_0 : X \to R(L)$, such that $L_0 x = Lx$, for all $x \in X$.

By $U(r)$ we denote a closed ball of radius $r > 0$, centered at the zero element of X or Y. For $R(L) \neq Y$, we assume

$$U_{R(L)}(r)\{y \mid y \in R(L),\ \|y\| \leq r\}.$$

The surjectivity coefficient [6, 7] of an operator $L : X \to Y$ is a non-negative number, defined by

$$q(L) = \inf_{\omega \neq \theta} \frac{\|L^*\omega\|}{\|\omega\|},$$

where $L^* : Y^* \to X^*$ is the adjoint operator of L.

If $q(L) > 0$, then the operator $L : X \to Y$ is surjective, that is $R(L) = Y$. If at the same time $\dim\ \ker L < \infty$, then it holds that $U(q(L)r) \subset L(U(r))$, for all $r > 0$.

If $R(L) \neq Y$, then the following characteristic of the linear operator turns out to be beneficial for studying Eq. (1).

Assume $L_0 : X \to R(L)$ is a contraction of $L : X \to Y$. The relative coefficient of surjectivity of the operator L is the number $q_0(L)$ determined by the equality [8], for all $z \in (R(L))^*$:

$$q_0(L) = \inf_{z \neq \theta} \frac{\left\| L_0^* z \right\|}{\|z\|}.$$

For the Fredholm operator $L : X \to Y$, it holds that $U_{R(L)}(q_0(L)r) \subset L(U(r))$, for all $r > 0$.

If the calculation of the precise meaning of the relative surjectivity coefficient is difficult, then its lower estimation is applied. To do this we consider the following approach.

Operator $K_p : R(L) \to X$ is called generalized inverse [9] to the operator $L : X \to Y$, associated with the projection P, if $K_p L = P^c$ and $LK_p = I_0$ hold, where $I_0 : R(L) \to Y$ is an embedding operator.

The following estimation [8] holds:

$$\left\| K_p \right\|^{-1} \leq q_0(L).$$

For a continuous (generally speaking, a nonlinear) operator, we consider the following functional characteristic:

$$b_F(r) = \sup_{\|x\| \leq r} \|Fx\|,$$

for all $r \geq 0$.

If $b_F(r) < \infty$, then $F(U(r)) \subset U(b_F(r))$. For the linear operator $L : X \to Y$, it holds that $b_L(r) = \|L\| r$. If $\|Fx\| \leq a + b\|x\|$ holds for some non-negative constants a and b, then the estimation $b_F(r) \leq a + br$ holds as well.

3 Existence Theorem

The existence theorems obtained in this section are based on a special modification of the Schauder Fixed Point Theorem [10] for the Eq. (1). It is formulated as follows.

Theorem 1 *Assume the following conditions hold:*

1. *$L : X \to Y$ is a Fredholm operator;*
2. *the operator $F : X \to Y$ is completely continuous;*
3. *there exists a nonempty closed bounded set $M \subset X$, such that $\overline{co} F(M) \subset L(M)$.*

Then there exists at least one solution of the Eq. (1).

Further in the paper, it is assumed that the following conditions hold:

(A) $L : X \to Y$ is a Fredholm an operator with the index $\mathrm{ind} L \geq 0$;
(B) an operator $F : X \to Y$ is completely continuous.

Theorem 2 *Assume the following conditions hold:*

1. *inequality $b_F(r) \leq q_0(L)r$ has a positive solution r_0;*
2. *for all $z \in X_0$ there exists $u \in \ker L$ such that $F(X) \in R(L)$, $x = z + u$, and $\|x\| \leq r_0$.*

Then Eq. (1) *has at least one solution.*

Proof Let $r_0 > 0$ be the number, the existence of which is assumed in condition 1 of Theorem 2. Let M be a set of elements of the form $x = z + u$, $\|x\| \leq r_0$, where $z \in X_0$, and u is an element of $\ker L$, which corresponds to the given z. Since for all $x \in M$ it holds that

$$F(x) \in R(L) \text{ and } \|Fx\| \leq b(r_0),$$

then

$$F(M) \subset U_{R(L)}(b(r_0)).$$

Moreover, it holds that

$$b(r_0) < q_0(L)r_0 \text{ and } U_{R(L)}(q_0(L)r_0) \subseteq L(U(r_0)).$$

Hence,

$$F(M) \subset L(U(r_0)).$$

For a closed set $C = \overline{M}$ we have that

$$F(C) \subset \overline{F}(M) \subset L(U(r_0)).$$

To this embedding, we apply the convex closure operation, taking into account the following equality

$$L(U(r_0)) = L(C),$$

we obtain

$$\overline{\mathrm{co}} F(C) \subset L(C).$$

Now we apply Theorem 1, which guarantees the existence of at least one solution of the Eq. (1) under the conditions of Theorem 2. □

Now we provide a statement that in some situations is more efficient in practice.

Theorem 3 *Assume the following conditions hold:*

1. *For all $x \in X$, there exists $u \in \ker L$ and $u = u(x)$, such that $F(x+u) \in R(L)$, and $\|u\| \leq \eta(\|x\|)$;*
2. *$b_F(r + \eta(r)) \leq q_0(L)r$ has a positive solution.*

Then, Eq. (1) *has at least one solution.*

Proof Let $x \in X$, and $u \in \ker L$, such that condition 1 of the theorem holds. Then

$$\|x+u\| \leq \|x\| + \eta(\|x\|).$$

If $x \in U(r)$, then

$$\|F(x+u)\| \leq b(r_1),$$

where $r_1 = r + \eta(r)$. Let r_0 be a positive solution of inequality from condition 2 of the theorem. Assume

$$M = \{x + u \mid F(x+u) \in R(L),\ \|x\| \leq r_0\}.$$

The following embedding holds

$$F(M) \subset U(r_1),\ r_1 = r_0 + \eta(r_0).$$

Further we follow the lines of the proof of Theorem 2. □

Corollary 1 *Assume the following conditions hold:*

1. *there exist $a, b \geq 0$, such that $\|Fx\| \leq a + b\|x\|$;*
2. *there exist $c, d \geq 0$, such that for all $x \in X$ there exists $u \in \ker L$, such that $u = u(x)$, $F(x+u) \in R(L)$, and $\|u\| \leq c + d\|x\|$;*
3. *it holds that $b(1+d) < q_0(L)$.*

Then Eq. (1) *has at least one solution.*

Proof We have

$$b_F(r + \eta(r)) \leq a + b(r + (c + dr)).$$

If $b(1+d) < q_0(L)$, then

$$a + b(r + (c + dr)) \leq q_0(L)r$$

has a positive solution. Thus, all conditions of 3 hold. □

Remark 1 Under the conditions of Corollary 1, the constant b can be replaced by the following (provided that it exists):

$$b(F) = \limsup \frac{\|Fx\|}{\|x\|}, \text{ where } \|x\| \to \infty.$$

Indeed, for a sufficiently small $\varepsilon > 0$, there exists $a = a(\varepsilon) \geq 0$, such that:

$$\|Fx\| \leq a + (b(F) + \varepsilon) \|x\| .$$

It is easier to check the conditions of Theorem 3 if F has sublinear growth. We consider it in more details below.

Corollary 2 *Assume the following conditions hold:*

1. *there exist $a, b \geq 0$, and $0 \leq \delta < 1$, such that for all $x \in X$ it holds that $\|Fx\| \leq a + b \|x\|^{\delta}$;*
2. *there exist $c, d \geq 0$, such that for all $x \in X$ there exists $u \in \ker L$, such that $u = u(x)$, $F(x + u) \in R(L)$, and $\|u\| \leq c + d \|x\|$.*

Then Eq. (1) *has at least one solution.*

Proof It is enough to observe that for $0 \leq \delta < 1$ inequality

$$a + b(r + (c + dr))^{\delta} \leq q_0(L)r$$

has r_0 as a positive solution. □

Certain difficulties may arise when one applies the above-mentioned statements to specific boundary value problems, which can be expressed in the form of Eq. (1) with a non-invertible operator L. To be more specific, in the said cases it might be problematic to verify the condition 2 of Theorem 2 (or similar conditions of other statements).

For the sake of simplicity, let us consider the case when $\operatorname{ind} L = 0$. Let $n = \dim \ker L$ and $J_1 : R^n \to \ker L$, and $J_2 : Y_0 \to R^n$ be fixed isomorphisms. For an arbitrary fixed $z \in X_0$ we define the mapping $\Phi_z : R^n \to R^n$ by the following equality:

$$\Phi_z(\alpha) = J_2 Q^c F(z + J_1 \alpha),$$

where $Q^c = I - Q$ is an additional projection operator.

Let $B(r) = \{\alpha \mid \alpha \in R^n, |\alpha| < r\}$ be an open ball of radius $r > 0$ in R^n. We use $\deg(\Phi_z, B(r))$ to denote the Brouwer degree [11] of $\Phi_z : R^n \to R^n$ relative to a ball of radius $r > 0$ centered at zero. To verify the mentioned above condition, one may require, for example, that for all $x \in X_0$, there exists $r = r(z) > 0$, such that $\deg(\Phi_z, B(r)) \neq 0$ and $r \leq d \|z\|$.

Taking into account the number of the existing approaches to verify the condition $\deg(\Phi_z, B(r)) \neq 0$, we can talk about efficient sufficient conditions that ensure the existence of a solution to the Eq. (1).

4 Application

As an example of the application of the statements, obtained in Sect. 3, we consider a periodic boundary value problem for an ordinary second-order differential equation:

$$x''(t) = f(t, x(h(t))),\ t \in [0, \omega], \tag{2}$$

$$x(0) = x(\omega),\ x'(0) = x'(\omega), \tag{3}$$

where $f : [0, \omega] \times R^1 \to R^1$ satisfies Carathéodory's criterion, $h : [0, \omega] \to R^1$ is measurable, and $h([0, \omega]) \subset [0, \omega]$.

We consider (2)–(3) on the space $W_2 = W_2[0, \omega]$ of the functions $x : [0, \omega] \to R^1$, which have an absolutely continuous derivative and such that $x'' \in L_2[0, \omega]$. We define the norm in W_2 by

$$\|x\|_W = |x(0)| + |x'(0)| + \left\|x''\right\|_{L_2}.$$

Consider the space $X = \{x \mid x \in W_2,\ x(0) = x(\omega),\ x'(0) = x'(\omega)\}$ and define the operators $L, F : X \to Y$, $Y = L_2$ by

$$(Lx)(t) = x''(t),\ (Fx)(t) = f(t, x(h(t))).$$

Now we consider (2)–(3) as an operator Eq. (1). Obviously:

$$\ker L = \{x \mid x \in X, x(t) \equiv \text{const}\},$$

$$R(L) = \left\{y \mid y \in Y, \int_0^\omega y(s)\,ds = 0\right\}.$$

We define the projection operators $P : X \to X$, $Q : Y \to Y$ by

$$(Px)(t) = x(0),$$

$$(Qy)(t) = y(t) - \frac{1}{\omega}\int_0^\omega y(s)\,ds.$$

Further we will be using the following inequalities. For an arbitrary $x \in W_2$ the following estimation holds:

$$|x(t)| \le \gamma \|x\|_W,\ t \in [0, \omega],$$

where $\gamma = \max\left\{1, \omega, \omega\sqrt{\frac{\omega}{3}}\right\}$.

Lemma 1 *The relative surjectivity coefficient of* $L : X \to Y$, $Lx = x''$ *has the estimation* $\left(1 + \sqrt{\frac{\omega}{3}}^{-1}\right) \le q_0(L)$.

Proof A direct verification shows that the generalized inverse operator $K_p : R(L) \to X$, associated with the projector $Px = x(0)$ has the following form:

$$(K_p y)(t) = \int_0^t (t-s)y(s)\,ds + \frac{t}{\omega}\int_0^\omega sy(s)\,ds.$$

Hence,

$$\left\| K_p y \right\|_W \le \|y\|_{L_2} + \frac{1}{\omega}\left(\frac{\omega}{3}\right)^{\frac{1}{2}} \|y|_{L_2} \le \left(1+\sqrt{\frac{\omega}{3}}\right)\|y\|_{L_2}.$$

Thus, $\left\| K_p \right\| \le 1+\sqrt{\frac{\omega}{3}}$. Since $\left\| K_p \right\|^{-1} \le q_0(L)$, the statement of the lemma holds. □

Theorem 4 *Let the following conditions hold:*

1. *there exist non-negative constants* a, b, *such that* $|f(t,u)| \le a + b|u|$, *and* $(t,u) \in [0,\omega] \times R^1$;
2. *there exists* $u > 0$, *such that for all* $u \in R^1$, $|u| > u^*$, $t \in [0,\omega]$ *it holds that* $sign(u) f(t,u) \ge 0$ $(sign(u) f(t,u) \le 0)$;
3. $b\gamma\sqrt{\omega}(1+\gamma) < \left(1+\sqrt{\frac{\omega}{3}}\right)^{-1}$.

Then, there exists at least one solution for (2)–(3).

Proof To prove the theorem, we use Corollary 1. Note that due to condition 1 of Theorem 4, and since $W_2 \subset L_2$ is completely continuous, we have that the operator $F : W_2 \to L_2$ defined by the equality $F(x)(t) = f(t, x(h(t)))$ is completely continuous.

It is not hard to verify that $|x(h(t))| \le \gamma \|x\|_W$. Due to condition 1 of Theorem 4, we have

$$|f(t, x(h(t)))| \le a + b\gamma \|x\|_W, \ x \in X, \ t \in [0,\omega].$$

Hence, $\|Fx\|_{L_2} \le (a + b\gamma \|x\|_W)\sqrt{\omega}$. Thus, condition 1 of Corollary 1 holds.

For certainty we will check condition 2 of Corollary 1 assuming that:

$$\operatorname{sign}(u) f(t,u) \ge 0,$$

for all $u \in R^1$, $|u| > u^*$, $t \in [0,\omega]$. We fix $x \in X$ and define $\Phi_x : R^1 \to^1$ by

$$\Phi_x(C) = \int_0^\omega f(s, x(h(s)) + C)\,ds.$$

Let $C_1 = u^* + \gamma \|x\|_W$. Then for all $C \ge C_1$ it holds that $x(h(t)) + C > u^*$, hence $\Phi_x(C) \ge 0$. Similarly, $\Phi_x(C) \le 0$ for all $C \le C_2 = -u^* - \gamma \|x\|_W$. Due to the continuity of Φ, there exists a constant $\widetilde{C} = C(x)$, which satisfies $|\widetilde{C}| \le$

$\max\{|C_1|, |C_2|\} \leq \gamma \|x\|_W + u^*$, and such that $\Phi_x(\widetilde{C}) = 0$. Thus, condition 2 of Corollary 1 holds.

For the case $\operatorname{sign}(u) f(t, u) \leq 0$, the proof follows the same lines. □

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A Control Problem for a System of ODE with Nonseparated Multipoint and Integral Conditions

V. M. Abdullayev

Abstract Using gradient-type methods is proposed to solve a control problem with nonseparated multipoint and integral conditions. Therefore, formulas for the gradient of the objective functional are obtained in this study. For the numerical solution of nonlocal direct and conjugate boundary value problems, an approach is proposed that allows folding the integral terms into local ones and then using an analog of the transfer of conditions. As a result, the solving of nonlocal boundary value problems is reduced to the solving of specially constructed Cauchy problems and one system of linear equations. An analysis of the obtained results of computational experiments is carried out.

1 Introduction

Recently, there has been an increase in research on boundary value problems with nonlocal conditions and corresponding control problems.

Note that boundary value problems with nonlocal conditions were started in [1–3] and then continued in the studies of many authors both for equations with ordinary and partial derivatives [4–10]. Control in boundary value problems with nonlocal multipoint integral conditions also aroused great interest [11–17]. Various studies have been carried out in this area, including the necessary conditions for optimality.

For linear nonlocal boundary value problems, numerical methods based on the sweep method are proposed in [18–22]. To take into account the integral conditions, many authors propose to reduce them to problems with multipoint conditions. For this, it was required to introduce new variables and, accordingly, to increase the dimension of the system of differential equations.

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We have studied an approach to the numerical solution of boundary value problems with integral conditions, which does not require an increase in the dimension of the system, and its application to solving the considered control problems. Analytical formulas for the gradient of the objective functional are obtained to use gradient methods for solving optimal control problems. The results of computational experiments and their analysis are given.

2 An Analysis of the Problem under Investigation and Obtaining Basic Formulas

Let the controlled process be described by the following ODE system:

$$\dot{x}(t) = A(t,u)\,x(t) + B(t,u), \quad t \in [t_0, T]. \tag{1}$$

Here $x(t) \in E^n$ is a phase variable; piecewise continuous function $u(t) \in U \subset E^r$ is a control vector function, compact set U is admissible values of control, and n-dimensional square matrix $A(t,u) \neq const$ and the n-dimensional vector function $B(t,u)$ are continuous with respect to t and continuously differentiable with respect to u.

The following conditions are given:

$$\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \overline{D}_i(\tau)\,x(\tau)\,d\tau + \sum_{j=1}^{l_2} \tilde{D}_j x\left(\tilde{t}_j\right) = C_0. \tag{2}$$

Here $\overline{D}_i(\tau)$, $\tilde{D}_j$ are $(n \times n)$-dimensional given matrices and $\overline{D}_i(\tau)$ is continuous; C_0 is the n-dimensional given vector; $\bar{t}_i$, $\tilde{t}_j$ are given time instances from $[t_0, T]$; $\bar{t}_{i+1} > \bar{t}_i, \tilde{t}_{j+1} > \tilde{t}_j, i = 1, \ldots, 2l_1 - 1, \quad j = 1, \ldots, l_2 - 1$; and l_1, l_2 are given.

We assume that, first,

$$\min\left(\bar{t}_1, \quad \tilde{t}_1\right) = t_0, \quad \max\left(\bar{t}_{2l_1}, \quad \tilde{t}_{l_2}\right) = T \tag{3}$$

and, second, condition

$$\tilde{t}_j \overline{\in} \left[\bar{t}_{2i-1}, \bar{t}_{2i}\right] \tag{4}$$

is satisfied for all $i = 1, \ldots, 2l_1, \quad j = 1, \ldots, l_2$.

It is required to determine the admissible control $u(t) \in U$ and the corresponding solution $x(t)$ to the nonlocal boundary value problem (1), (2), such that the pair $(x(t), u(t))$ minimizes the following objective functional:

$$J(u) = \Phi\left(\hat{x}\left(\hat{t}\right)\right) + \int_{t_0}^{T} f^0(x, u, t)\, dt \to \min_{u(t)\in U}, \tag{5}$$

where the given function Φ is continuously differentiable and $f^0(x, u, t)$ is continuously differentiable with respect to (x, u) and continuous with respect to t; $\hat{t} = \left(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2l_1+l_2}\right)$ is the ordered union of the sets $\tilde{t} = \left(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{l_2}\right)$ and $\bar{t} = \left(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{2l_1}\right)$, i.e., $\hat{t}_j < \hat{t}_{j+1}, \quad j = 1, \ldots, 2l_1 + l_2 - 1$ and $\hat{x}\left(\hat{t}\right) = \left(x\left(\hat{t}_1\right), x\left(\hat{t}_2\right), \ldots, x\left(\hat{t}_{2l_1+l_2}\right)\right)$.

The fundamental difference of problem statement (1)–(5) from the optimal control problems considered, for example, in [12–14] lies in nonseparated nonlocal integral and multipoint conditions (2). By introducing some new phase variables, problem (1)–(5) can be reduced to a problem involving multipoint conditions. To demonstrate this, introduce new phase vector $X(t) = \left(x^1(t), \ldots, x^{l_1+1}(t)\right), \quad x^1(t) = x(t)$, which is the solution to the following differential equations system:

$$\begin{aligned} &\dot{x}^1(t) = A(t, u)\, x^1(t) + B(t, u), \\ &\dot{x}^{i+1}(t) = \overline{D}_i(t) x^1(t), \quad t \in \left(\bar{t}_{2i-1}, \bar{t}_{2i}\right], \quad i = 1, \ldots, 2l_1, \end{aligned} \tag{6}$$

involving the following initial conditions:

$$x^{i+1}\left(\bar{t}_{2i-1}\right) = 0, \quad i = 1, \ldots, 2l_1. \tag{7}$$

Then conditions (2) takes the following form:

$$\sum_{i=1}^{l_1} x^{i+1}\left(\bar{t}_{2i}\right) + \sum_{j=1}^{l_2} \tilde{D}_j x^1\left(\tilde{t}_j\right) = C_0. \tag{8}$$

Systems (6)–(8) are obviously equivalent to (1) and (2). In systems (6) and (7), there are $(l_1 + 1)n$ differential equations with respect to the phase vector $X(t)$, and there is the same number of conditions in (7) and (8). Obviously, the drawback of boundary problem (6), (7) is its high dimension. This is an essential point for numerical methods of solution to boundary problems based, as a rule, on the methods of sweep or shift of boundary conditions [18–22]. Also, the increase of the dimension of the phase variable complicates the solution to the optimal control problem itself due to the increase of the dimension of the adjoint problem.

Note that if we use the approach proposed in [22], then at the expense of the additional increase of the dimension of the phase variable vector up to $2(l_1 + l_2 + 1)(l_1 + 1)n$, problem (6)–(8) can be reduced to a two-point problem involving nonseparated boundary conditions.

Using the technique of the works [23–26], we can obtain existence and uniqueness conditions for the solution to problem (1), (2) under every admissible control

$u \in U$, without reducing it to a problem involving multipoint conditions (8). But this kind of investigation is not the objective of the present work.

In the present work, we propose an approach to the numerical solution to both boundary problem (1), (2) and to the corresponding optimal control problem. This approach does not require increase of the order of the differential equations system and of the phase vector.

Assume that under every admissible control $u(t) \in U$, there is a unique solution to problem (1), (2). For this purpose, we assume that the parameters of problem (1), (2), after reducing it to (6)–(8), satisfy the conditions proposed in [2, 23–26] dedicated to differential equations systems involving multipoint and two-point conditions.

To apply gradient methods to solving optimal control problem (1)–(5), we obtain the formulas for the gradient of the functional.

Suppose that $(u(t), x(t;u))$ is an arbitrary admissible process and the control $u(t)$ has received an increment $\Delta u(t)$: $\tilde{u} = u + \Delta u$. Then the phase variable $\tilde{x}(t) = x(t) + \Delta x(t)$ will also receive increments, and the following takes place:

$$\Delta \dot{x}(t) = A(t,u)\,\Delta x(t) + \Delta_u A(t,u)\,x(t) + \Delta_u B(t,u), \quad t \in [t_0, T], \tag{9}$$

$$\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \overline{D}_i(\tau)\,\Delta x(\tau)\,d\tau + \sum_{j=1}^{l_2} \tilde{D}_j \Delta x\left(\tilde{t}_j\right) = 0. \tag{10}$$

Here we use the following designations:

$$\Delta x(t) = x(t,\tilde{u}) - x(t,u), \quad \Delta_u A(t,u) = A(t,\tilde{u}) - A(t,u), \quad \Delta_u B(t,u) = B(t,\tilde{u}) - B(t,u).$$

Consider the as-yet arbitrary almost everywhere continuously differentiable vector function $\psi(t) \in R^n$ and vector $\lambda \in R^n$. To calculate the increment of the functional, taking into account (9)–(10) and (1)–(2), we have

$$J(u) = \Phi\left(\hat{x}\left(\hat{t}\right)\right) + \int_{t_0}^{T} f^0(x,u,t)\,dt + \int_{t_0}^{T} \psi^*(t)\left[\dot{x}(t) - A\Big(t,u\Big)x(t) - B(t,u)\right]dt +$$
$$+ \lambda^*\left[\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \overline{D}_i(\tau)\,x(\tau)\,d\tau + \sum_{j=1}^{l_2} \tilde{D}_j x\left(\tilde{t}_j\right) - C_0\right],$$

$$J(u+\Delta u) = \Phi\left(\hat{x}\left(\hat{t}\right) + \Delta\hat{x}\left(\hat{t}\right)\right) + \int_{t_0}^{T} f^0(x+\Delta x, u+\Delta u, t)\,dt +$$
$$+ \int_{t_0}^{T} \psi^*(t)\left[(\dot{x} + \Delta\dot{x}) - A\Big(t, u+\Delta u\Big)(x+\Delta x) - B(t, u+\Delta u)\right]dt +$$
$$+ \lambda^*\left[\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \overline{D}_i(\tau)\,(x(\tau) + \Delta x(\tau))\,d\tau + \sum_{j=1}^{l_2} \tilde{D}_j\left(x\left(\tilde{t}_j\right) + \Delta x\left(\tilde{t}_j\right)\right) - C_0\right],$$

where * is the transposition operation. Using the formula for integration by parts, we obtain

$$\begin{aligned}\Delta J(u) = \int_{t_0}^{T}\Bigg[&-\dot{\psi}^*(t) - \psi^*(t)A\left(t,u\right) + \lambda^*\sum_{i=1}^{l_1}\left[\chi\left(\bar{t}_{2i}\right) - \chi\left(\bar{t}_{2i-1}\right)\right]\overline{D}_i(t) + f_x^0\left(x,u,t\right)\Bigg]\Delta x(t)dt+\\
&+\int_{t_0}^{T}\left\{f_u^0\left(x,u,t\right) + \psi^*(t)\left[-A_u^*\left(t,u\right)x(t) - B_u\Big(t,u\Big)\right]\right\}\Delta u(t)dt+\\
&+\sum_{k=2}^{2l_1+l_2-1}\left[\psi^{*-}\left(\hat{t}_k\right) - \psi^{*+}\left(\hat{t}_k\right) + \frac{\partial\Phi(\hat{x}(\hat{t}))}{\partial x(\hat{t}_k)}\right]\Delta x\left(\hat{t}_k\right)\\
&+\sum_{j=1}^{l_2}\lambda^*\tilde{D}_j\Delta x\left(\tilde{t}_j\right) + \psi^*(T)\Delta x(T) - \psi^*\left(t_0\right)\Delta x\left(t_0\right)+\\
&+\int_{t_0}^{T}o_1\left(\|\Delta x(t)\|\right)dt + \int_{t_0}^{T}o_2\left(\|\Delta u(t)\|\right)dt + o_3\left(\left\|\Delta\hat{x}\left(\hat{t}_k\right)\right\|\right),\end{aligned} \tag{11}$$

where $\psi^+\left(\hat{t}_k\right) = \psi\left(\hat{t}_k + 0\right), \quad \psi^-\left(\hat{t}_k\right) = \psi\left(\hat{t}_k - 0\right), \quad k = 1,\ldots,(2l_1 + l_2)$, $\chi(t)-$ is the Heaviside function.

$o_1(\|\Delta x(t)\|)$, $o_2(\|\Delta u(t)\|)$, $o_3\left(\left\|\Delta\hat{x}\left(\hat{t}_k\right)\right\|\right)$ are the quantities of less than the first order of smallness.

Henceforth, the norms of the vector functions $\|x(t)\|$ and $\|u(t)\|$ are understood (see [27]) as $\|x(t)\|_{L_2^n[t_0,T]}$ and $\|u(t)\|_{L_2^r[t_0,T]}$, respectively.

$$A_u\left(t,u\right) = \left(\left(\frac{\partial A_{ij}\left(t,u\right)}{\partial u_s}\right)\right) \quad \text{and} \quad B_u\left(t,u\right) = \left(\left(\frac{\partial B_i\left(t,u\right)}{\partial u_s}\right)\right)$$

are considered as matrices of the dimensions $(n \times n \times r)$ and $(n \times r)$, respectively. Applying the transposition operation to these matrices, we obtain the matrices $A_u^*\left(t,u\right)$ and $B_u^*\left(t,u\right)$ of the dimensions $(n \times r \times n)$ and $(r \times n)$.

We require that $\psi(t)$ be a solution to the following nonlocal boundary value problem:

$$\dot{\psi}(t) = -A^*\left(t,u\right)\psi(t) + \sum_{i=1}^{l_1}\left[\chi\left(\bar{t}_{2i}\right) - \chi\left(\bar{t}_{2i-1}\right)\right]\overline{D}^*(t)\lambda + f_x^{0*}\left(x,u,t\right), \tag{12}$$

$$\psi\left(t_0\right) = \begin{cases}\left(\frac{\partial\Phi(\hat{x}(\hat{t}))}{\partial x(\tilde{t}_1)}\right)^* + \tilde{D}_1^*\lambda, & for \quad t_0 = \tilde{t}_1,\\ \left(\frac{\partial\Phi(\hat{x}(\hat{t}))}{\partial x(\bar{t}_1)}\right)^*, & for \quad t_0 = \bar{t}_1,\end{cases} \tag{13}$$

$$\psi(T) = \begin{cases} -\left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\tilde{t}_{l_2})}\right)^* - \tilde{D}^*_{l_2}\lambda, & for \quad \tilde{t}_{l_2} = T, \\ -\left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\bar{t}_{2l_1})}\right)^*, & for \quad \bar{t}_{2l_1} = T, \end{cases} \tag{14}$$

$$\psi^+\left(\tilde{t}_j\right) - \psi^-\left(\tilde{t}_j\right) = \left(\frac{\partial \Phi\left(\hat{x}\left(\hat{t}\right)\right)}{\partial x\left(\tilde{t}_j\right)}\right)^* + \tilde{D}^*_j\lambda, \quad j = 1, 2, \ldots l_2, \tag{15}$$

$$\psi^+\left(\bar{t}_i\right) - \psi^-\left(\bar{t}_i\right) = \left(\frac{\partial \Phi\left(\hat{x}\left(\hat{t}\right)\right)}{\partial x\left(\bar{t}_i\right)}\right)^*, \quad i = 1, 2, \ldots 2l_1. \tag{16}$$

Instead of (12) and (15)–(16), we can use a differential equations system involving impulse actions:

$$\begin{aligned} \dot{\psi}(t) = & -A^*(t, u)\,\psi(t) + \sum_{i=1}^{l_1}\left[\chi\left(\bar{t}_{2i}\right) - \chi\left(\bar{t}_{2i-1}\right)\right]\overline{D}^*(t)\lambda \\ & + \sum_{j=1}^{l_2}\left[\left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\tilde{t}_{v_1})}\right)^* + \tilde{D}^*_j\lambda\right]\delta\left(t - \tilde{t}_j\right) + \\ & + \sum_{i=1}^{l_1}\left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\bar{t}_i)}\right)^*\delta\left(t - \bar{t}_i\right) + f_x^{0*}(x, u, t)\,. \end{aligned} \tag{17}$$

Here $\delta(\cdot)$ is the delta function. Problems (12)–(16) and (17), (13), and (14) are equivalent. Numerical schemes of their approximation and the solution algorithms used are identical.

To obtain estimates of $o_1(\|\Delta x(t)\|)$ and $o_3\left(\left\|\Delta\hat{x}\left(\hat{t}\right)\right\|\right)$ by known methods by increasing the dimension of the system, we can reduce the considered boundary value problem (1), (2) to the Cauchy problem and obtain the estimate of the form:

$$\|\Delta x(t)\| \leq c\,\|\Delta u(t)\|\,, \tag{18}$$

where $c = const > 0$ does not depend on $u(t)$ [15–17].

From formula (11), taking into account that the gradient of the objective functional is determined by the linear part of the functional increment, we have

$$(\nabla J(u))^* = f_u^0(x, u, t) + \psi^*(t)\left[-A_u^*(t, u)\,x(t) - B_u\Big(t, u\Big)\right]. \tag{19}$$

The functions $x(t)$ and $\psi(t)$ here are, for this control, the solutions to nonlocal boundary value problem (1), (2) and the conjugate boundary value problem (12)–(16).

For the numerical solution of the problem, we use methods of minimization of the gradient type, in particular the well-known method of gradient projection [27]:

$$\begin{gathered} u^{k+1}(t) = P_U\left(u^k(t) - \alpha_k \ \nabla \ J\left(u^k(t)\right)\right), \quad k = 0, 1, \ldots, \\ \alpha_k = \arg\min_{\alpha \geq 0} J\left(P_U\left(u^k(t) - \alpha \ \nabla J\left(u^k(t)\right)\right)\right), \end{gathered} \tag{20}$$

where $P_U(\upsilon)$ is the operator of projection of the element $\upsilon \in E^r$ on the set U.

There are two computational difficulties in calculating the gradient of the functional. They are related to the problem of solving direct nonlocal boundary value problem (1), (2) and the conjugate boundary value problem (12)–(16) with an unknown vector λ. It is clear that in system of relations (1), (2), and (12)–(1.16), for a given control $u(t)$, we need to determine $2n$ unknown components of the vector functions $x(t)$, $\psi(t)$, $2n$ their initial values, and n component of the vector λ. For this, there are $2n$ differential Eqs. (1) and (12), n condition (2), and $2n$ conditions (13) and (14).

Proposed below is an algorithm based on the use of the one developed in [13, 18, 21], an operation of shifting conditions for solving systems of ODE with boundary conditions that also include unknown parameters [28, 29]. The proposed operation of shifting intermediate conditions generalizes the well-known operation of transfer of boundary conditions and extends the results of [13, 18, 21] to this class of problems.

3 Numerical Scheme of Solution to the Problem

One of the approaches to the numerical solution to problem (1), (2) could be the reduction of (1) and (2) to a problem involving nonseparated point conditions (6)–(8) by means of introduction of some new variables.

As stated in the first paragraph, the obvious drawback of such an approach is the need for increasing the order of the system, which complicates carrying out the operations of sweep and of shift of the functional matrix of the respective order (see [13, 18, 21]).

Below, we propose and investigate the scheme of the method of reduction of conditions (2) to initial conditions, which do not require increasing the dimension of the differential equations system. For this purpose, we transform first (2) to an integral form.

Introduce the following $(n \times n)$ matrix function:

$$D(t) = \sum_{i=1}^{l_1} \left[\chi\left(\bar{t}_{2i}\right) - \chi\left(\bar{t}_{2i-1}\right)\right] \overline{\overline{D}}_i(t) + \sum_{j=1}^{l_2} \tilde{D}_j \delta\left(t - \tilde{t}_j\right). \tag{21}$$

Function $\overline{\overline{D}}_i(t)$ is as follows:

$$\overline{\overline{D}}_i(t) = \begin{cases} \overline{D}_i(t), & t \in \left[\overline{t}_{2i-1}, \overline{t}_{2i}\right], \\ 0, & t \notin \left[\overline{t}_{2i-1}, \overline{t}_{2i}\right]. \end{cases}$$

From (21), it follows that

$$D(t) \equiv 0 \quad \text{for} \quad t \notin \bigcup_{i=1}^{l_1} \left[\overline{t}_{2i-1}, \overline{t}_{2i}\right] \quad \cup \left(\bigcup_{j=1}^{l_2} \tilde{t}_j \right).$$

In view of (3) and (4), conditions (2) can be written in the equivalent matrix form:

$$\int_{t_0}^{T} D\left(\tau\right) x\left(\tau\right) d\tau = C_0, \tag{22}$$

or each of n conditions in (22) can be written separately:

$$\int_{t_0}^{T} D^{\nu}\left(\tau\right) x\left(\tau\right) d\tau = C_{0\nu}, \quad \nu = 1, \ldots, n, \tag{23}$$

where $D^{\nu}(\tau)$ is the ν th n-dimensional row of the matrix function $D(\tau)$.

Now, in order to replace integral conditions (22) with local (point) initial conditions, we use an operation that is similar to the transfer operation (sweep) of conditions, which we call a convolution operation.

Introduce $n-$ dimensional vector functions:

$$\overline{C}(t) = \int_{t_0}^{t} D\left(\tau\right) x\left(\tau\right) d\tau, \qquad \underline{C}(t) = \int_{t}^{T} D\left(\tau\right) x\left(\tau\right) d\tau, \tag{24}$$

for which, the following relations obviously take place:

$$\overline{C}\left(t_0\right) = \underline{C}(T) = 0, \quad \overline{C}(T) = \underline{C}\left(t_0\right) = C_0. \tag{25}$$

Definition Matrix functions $\overline{\alpha}(t)$, $\underline{\alpha}(t)$ of the dimension $n \times n$ and n-dimensional vector functions $\overline{\beta}(t)$, $\underline{\beta}(t)$ convolve integral conditions (22) into point conditions at the right and left ends, respectively, if for any solution $x(t)$ to system (1), there holds the following conditions:

$$\int_{t_0}^{t} D\left(\tau\right) x\left(\tau\right) d\tau = \overline{\alpha}(t)x(t) + \overline{\beta}(t), \quad t \in [t_0, T], \tag{26}$$

$$\int_{t}^{T} D(\tau)\, x(\tau)\, d\tau = \underline{\alpha}(t)x(t) + \underline{\beta}(t), \quad t \in [t_0, T]. \tag{27}$$

From (26) and (27) in view of (24) and (25), it follows that

$$\overline{\alpha}(T)x(T) + \overline{\beta}(T) = \overline{C}(T) = C_0, \tag{28}$$

$$\underline{\alpha}(t_0)\, x(t_0) + \underline{\beta}(t_0) = \underline{C}(t_0) = C_0. \tag{29}$$

Each of conditions (28) and (29) represents a local boundary condition. Pairs $\overline{\alpha}(t)$, $\overline{\beta}(t)$ and $\underline{\alpha}(t)$, $\underline{\beta}(t)$ are called functions convolving integral conditions (22) into point conditions from left to right and from right to left, respectively.

Denote by $O_{n\times n}$ a matrix of the dimension $(n \times n)$ with null elements, $I_{n\times n}$ is the identity matrix of order n, and by O_n an n-dimensional vector with null elements. The following theorem takes place.

Theorem 1 *If functions* $\overline{\alpha}(t)$, $\overline{\beta}(t)$ *are the solution to the following Cauchy problems:*

$$\dot{\overline{\alpha}}(t) = -\overline{\alpha}(t)A(t) + D(t), \quad \overline{\alpha}(t_0) = O_{n\times n}, \tag{30}$$

$$\dot{\overline{\beta}}(t) = -\overline{\alpha}(t)B(t), \qquad \overline{\beta}(t_0) = O_n, \tag{31}$$

then these functions convolve integral conditions (22) from left to right into point condition (28).

Proof Assume that there exists the dependence:

$$\overline{C}(t) = \overline{\alpha}(t)x(t) + \overline{\beta}(t), \qquad t \in [t_0, T]. \tag{32}$$

Here $\overline{\alpha}(t)$, $\overline{\beta}(t)$ are as-yet arbitrary matrix and vector functions of the dimensions $n \times n$ and n, respectively, which satisfy conditions (31). Then obviously

$$\overline{\alpha}(t_0) = O_{n\times n}, \quad \overline{\beta}(t_0) = O_n.$$

Differentiating (32) and taking (1) and (23) into account, we have

$$[\dot{\alpha}(t) + \alpha(t)A(t) - D(t)]\, x(t) + \left[\dot{\beta}(t) + \alpha(t)B(t)\right] = 0. \tag{33}$$

Taking the arbitrariness of the functions $\overline{\alpha}(t)$, $\overline{\beta}(t)$ into account, as well as the fact that (33) must be satisfied for all the solutions $x(t)$ to system (1), then it is necessary to set each of the expressions in two brackets (33) equal to 0, i.e., to satisfy conditions (30) and (31) of the lemma1.

The following theorem is proven similarly.

Theorem 2 *If functions* $\underline{\alpha}(t)$, $\underline{\beta}(t)$ *are the solution to the following Cauchy problems:*

$$\dot{\underline{\alpha}}(t) = -\underline{\alpha}(t)A(t) - D(t), \quad \underline{\alpha}(T) = O_{n\times n}, \tag{34}$$

$$\dot{\underline{\beta}}(t) = -\underline{\alpha}(t)B(t), \quad \underline{\beta}(T) = O_n, \tag{35}$$

then these functions convolve integral conditions (22) from right to left into point conditions (29).

Thus, to solve problem (1), (2), it is necessary to solve Cauchy problems (30) and (31) or (34) and (35), to obtain the n th order of linear algebraic system (28) or (29), respectively, then $x(T)$ or $x(t_0)$ is determined from (28) or (29). They can be used as initial conditions for solving the Cauchy problem with respect to the main system (1).

The choice of the convolution scheme applied to conditions (2) from right to left or vice versa depends on the properties of the matrix $A(t)$, namely, on its eigenvalues. If they are all positive, then systems (30) and (31) are steady; if they are all negative, then systems (34) and (35) are steady. If some eigenvalues of the matrix $A(t)$ are positive, and the other are negative, and their absolute values are large, then both systems have fast-increasing solutions, and therefore their numerical solution is unsteady, and it can result in low accuracy. In this case, it is recommended to use the convolving functions which are proposed in the following theorem and which have linear growth in time.

Theorem 3 *If n-dimensional vector function* $g_1^\nu(t)$ *and scalar functions* $g_2^\nu(t)$ *and* $m^\nu(t)$ *are the solution to the following nonlinear Cauchy problems:*

$$\dot{g}_1^\nu(t) = S(t)g_1^\nu(t) - A^*(t)g_1^\nu(t) + m^\nu D^{\nu^*}(t), \quad g_1^\nu(t_0) = 0_n, \tag{36}$$

$$\dot{g}_2^\nu(t) = S(t)g_2^\nu(t) - B^*(t)g_1^\nu(t), \; g_2^\nu(t_0) = 0, \tag{37}$$

$$\dot{m}^\nu(t) = S(t)m^\nu(t), \quad m^\nu(t_0) = 1, \tag{38}$$