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Time-Synchronized Control: Analysis and Design

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To parents, Huiying and Youning for their companionship in the journey, and to mentors for their guidance without reservation

Dongyu Li

To parents for their love and contentment in life, to Jinlan for her care to the families, and to friends for their supports

Shuzhi Sam Ge

To my family, and to sincere friends

Tong Heng Lee

Preface

In the literature of control science and engineering, we are more concerned with convergence of the states of any dynamic multi-input-multi-output systems, then pay attention to academically challenging and practically relevant finite-time convergence where finite-time control drives the states to converge within a certain time moment, and finally solve the important and critical issue in solving how each state element converges as demanded in many accurate, precise, and delicate operations in many applications including robotic autonomy, manufacturing operations, and exploration in oceans and space.

This monograph develops and introduces various fundamental and basics of time-synchronized control and its stability because of the demand for precise and dedicated operations in practice. Interconnections with finite-time control and sliding mode control are explored. The work forges ahead beyond the well-established notions and outcomes of standard "Finite-Time Stability", and investigates and develops results and outcomes for a newer (and in various important instances, more impactful) notion of "Time-Synchronized Stability". This work is oriented towards researchers, engineers, and students who dedicate themselves for the best control system design in practice yet with sound theoretical support by seamlessly integrating or fusing academic exploration and engineering regulations.

Specifically and importantly, under time-synchronized control, all the state elements of a target system converge to the origin exactly *at the same time*, called *time-synchronized convergence.* The associated time-synchronized stability is built on top of the well-known standard notion of finite-time stability, but adds to it the further notion of time-synchronized convergence. In this monograph, we underscore and discover important outcomes and properties associated with finite-time stability, and then define a notion and concept of a control system—*ratio persistence.* All of these together contribute to time-synchronized stability. Ratio persistence is an essential property that keeps each pair of the closed-loop system state elements being constant. Note that even the classical proportional control (P control) can appropriately enable a control system to be ratio persistent. Thus, P control can be seen as a very special type of time-synchronized control, where its synchronized settling time is $t \to \infty$. Of course herein, our explorations and attained outcomes are much more impactful beyond this in consequence. The results and outcomes here pertain to realizing this ratio persistence, and time-synchronized control with finite-time stability as well.

Time-synchronized control can be viewed as offering coordination of different states of the system in time for different applications in the sense that, as time passes, the system state elements converge to the equilibrium while always preserving a constant ratio among them. These unique nature and property make time-synchronized control highly attractive in providing not only time-synchronized convergence, but also more nice properties for this interesting time-synchronized convergence property attained such as a smooth system output, a short state trajectory, reduced energy consumption, improved convergence accuracy, among others.

The main content of this book includes time-synchronized stability and its relevant Lyapunov theorems, time-synchronized control design for different types of systems with practical considerations, and explorations in time-synchronized consensus.

In Chap. 2, time-synchronized stability, fixed-time-synchronized stability, and predefined-time stability as well as their respective Lyapunov-like theorems are proposed. Fixed- and predefined-time-synchronized stability are regarded as special types of time-synchronized stability.

In Chap. 3, quasi-continuous and continuous time-synchronized sliding modes are introduced to facilitate the achievement of time-synchronized stability. Their convergence rate, singularity issue, types of continuity are properly discussed.

In Chaps. 4–6, time-synchronized controllers are proposed for affine systems, Euler-Lagrange systems, and general MIMO systems under matched and unmatched disturbances, where time-synchronized or fixed-time-synchronized convergence is achieved. An important matter of singularity avoidance is suitably addressed. Further, we introduce the concept of "the least upper bound of synchronized settling time" to accurately estimate the synchronized settling time. An extension to predefined-time-synchronized control is further addressed.

In Chap. 7, the time-synchronized consensus problem is defined and solved. It shows that all the state elements of the multi-agent system are capable of achieving consensus time-synchronously when the graph theory is properly invoked with the proposed time-synchronized consensus controller.

In Chaps. 8 and 9, (practical) time-synchronized control is proposed for disturbed spacecraft under multiple practical constrains with hardware-in-loop experiments.

In Chap. 10, pertinent conclusions are drawn with interesting but challenging possible future directions.

Budding and Genesis

The genesis and mulling of this monography were very much inspired and motivated by many excellent scientific works including *Linear Controller Design: Limits of Performance* by S. Boyd and C. Barratt, *Optimal Control* by F. L. Lewis, D. Vrabie and V. L. Syrmos, *Sliding mode control: theory and applications* by C. Edwards, S. Spurgeon, and many excellent works presented at TCCT Workshop on Cooperative Control and Multi-Agent Systems under the leadership of Jie Chen, Beijing. As we investigated different control performance specifications for multi-agent systems, we further investigated the description and usage of two different versions of the signum function—the usual standard version and a norm-normalized version, i.e. one defined by the signum element and the other by the standard unit vector as illustrated in *Adaptive Neural Network Control of Robotic Manipulators* by S. S. Ge, T. H. Lee and C. J. Harris. We explored that this norm-normalized version had that special property (essentially of ratio persistence) to assure, what we had termed at that initial stage, "Simultaneous Arrival To Origin" (SATO) convergence (despite rather simple underlying mathematical analysis, that is actually, time-synchronized convergence). Motivated by this interesting property, we determined to seek a revisit of existing outcomes in "Finite-Time Stability" and "Sliding Mode Control" to explore SATO convergence in depth. In 2018, we had managed to successfully develop some early promising results along this line, where time-synchronized controls were designed to achieve time-synchronized convergence for simple first-order systems. In 2019, we further developed this work and managed to synthesize methodologies improving the outcomes, where more types of time-synchronized stability and more practical control systems were considered. With the journal manuscripts submitted, review comments came back which additionally benefited our work. It is indeed pleasant on how all the pieces have now fallen nicely in place; here this book/monograph is available to all who had expressed their great interest, and also to many others who, we hope and trust, will appreciate the great potential of the concepts and results here. We welcome interested excellent individuals to further push the boundary well beyond.

Please enjoy reading!

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Contents

Acronyms and Notation

Acronyms

ISS	Input to state stable/stability
MIMO	Multiple-input multiple-output
SISO	Single-input single-output

Notation

$\mathbb{R}$	Field of real numbers
Σ	Summation
max	Maximum
min	Minimum
$\forall$	For all
$\in$	Belongs to
$> (<)$	Greater (less) than
$\geq (\leq)$	Greater (less) than or equal to
$t \rightarrow T$	t approaches T
$t \uparrow T$	t approaches T from below (in an increasing manner)
$t \downarrow T$	t approaches T from above (in an decreasing manner)
$\otimes$	Kronecker product
w.r.t.	With respect to
s.t.	Such That or Subject to
i.e.	The Latin phrase id est, meaning "that is"
e.g.	The Latin phrase exempli gratia, meaning "for example"
$\mathbb{R}^n$	The n-dimensional Euclidean space
I_n	The n-dimensional identity matrix
1_n	The n-dimensional vector with all entries being 1
$\mid \bullet \mid$	The absolute of a scalar $\bullet$
$\\| \bullet \\|$	The norm of a vector $\bullet$

$\dot{f}$	The first derivative of f with respect to time
$\ddot{f}$	The second derivative of f with respect to time
$\mathbf{D}^{+} f$	The right-hand upper Dini derivative of f with respect to time
$\mathrm{diag}\{x_1, \cdots, x_n\}$	A diagonal matrix with diagonal elements x_1 to x_n
$H > 0$	A positive definite matrix H
$H \geq 0$	A positive semi-definite matrix H
X^T	The transpose of matrix/vector X
$\lambda_{\max}(A)/\lambda_{\min}(A)$	The maximum/minimum eigenvalue of matrix A
$A^{\dagger}$	The Moore-Penrose pseudo-inverse of matrix A
$\mathrm{vec}(A)$	The vectorization form of matrix A

Chapter 1
Introduction

Abstract This chapter briefly recalls and re-collects the history of finite-time control, and the budding and genesis of key ideas and developments in the further notion of time-synchronized control. Time-synchronized control is, in fact, a unique type of finite-time control; which already meets all the requirements of finite-time stability, and additionally drives all the system state elements to the equilibrium at the same time. The merit of this noteworthy and so-termed time-synchronized control consists in not only the synchronized property on the convergence time, but also the abilities to smooth the system output, shorten the state travel length, and reduce the energy consumption. Finally in the chapter, the overall book organization is shown in a summary of the content of the remaining chapters.

1.1 Finite-Time Control

With the developments of powerful sensors, actuators, control processors, and computer hardware, sophisticated control algorithms outperform simple ones that have sufficed in the past [9]. Finite-time control, as an important control systems methodology, had emerged and grown in interest in modern control systems design to cater to the growing requirements in a wide range of progressive control tasks; to address the high-precision and fast-response performance. As the notion implies, finite-time stabilization offers a control system convergence in finite-time [21]; under which, more specifically, the situation is such that before a certain time instant, all the state elements are assured to reach the origin (or a bounded neighborhood of the origin). Its wide applications include satellite stabilization [14, 54, 61], manipulator tracking [18, 24, 58], multi-agent consensus [10, 29, 39, 52], state estimation [5, 16, 42, 44], control of network systems [8, 29, 34, 55, 56, 62], control under practical constraints [23, 25, 46, 47, 59], among others.

Dating back to the 1950s in the initial instances of its development, the early idea of finite-time control likely first appeared in [28, 30, 31]. In the 1960s, the concept of finite-time control is further polished in [13, 53]. Again in the 1960s, another instance of optimal control design presented a similar concept of finite-time control

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in [17], which was called the *Fuller effect* therein. We here briefly introduce the main essential elements in [17]. Consider thus a second-order system

$$\ddot{x} = u(x, \dot{x}), \tag{1.1}$$

where $x,\ u \in \mathbb{R}$ denote the system state and the control input under the constraints $|u| \leq 1$. Under the cost function $J = \int_0^\infty x^2 \mathrm{d}t$, the optimal scheme is

$$u = -\operatorname{sign}(s), \tag{1.2}$$

$$s = x + c\dot{x}|\dot{x}|, \tag{1.3}$$

where a positive constant c is chosen such that the system output crosses the surface $s = 0$ at a countably infinite number of points but located within a finite-time interval. From analysis, the time intervals will decrease as a geometric progression, contributing to a finite accumulation point called *Zeno behavior*: regarding a system jumps an infinite number of times in a finite amount of time. This further infers finite-time convergence since it has finite settling time (as also summarized in [50]).

To prove finite-time stability, one can find a Lyapunov function $V(x)$ satisfying

$$\dot{V}(x) + cV(x)^\alpha \leq 0, \tag{1.4}$$

where constants $c > 0$, $\alpha \in [0, 1)$. Then, we have $V(x) = 0$ for $\forall t \geq T(x_0)$, where x_0 is the initial system state, and the settling time is bounded by [6]

$$T\left(x_0\right) \leq \frac{V\left(x_0\right)^{1-\alpha}}{c(1-\alpha)}. \tag{1.5}$$

Another method to prove finite-time stability is given by first proving asymptotic stability and verifying homogeneity of the closed-loop system [7].

Finite-time control drives the system state to the origin within a bounded settling time subject to the initial state [21]. Fixed-time stability is then introduced, where the solution of a system is globally finite-time stable and the bounded settling time is regardless of any initial conditions [43]. To prove fixed-time stability, it requires a Lyapunov function $V(x)$ to satisfy a stronger condition,

$$\dot{V}(x) \leq -\left(\alpha V^p(x) + \beta V^g(x)\right)^\chi, \tag{1.6}$$

where positive constants α, β, p, g and χ satisfy $p\chi < 1$ and $g\chi > 1$. The corresponding settling time $T\left(x_0\right)$ is bounded as

$$T\left(x_0\right) \leq \frac{1}{\alpha^\chi(1-p\chi)} + \frac{1}{\beta^\chi(g\chi-1)}. \tag{1.7}$$

Different from finite-time control and other existing control techniques, the merit of fixed-time control lies in the capability of forcing the system state to the equilibrium in fixed time. This decisive characteristic allows a system engineer to manipulate the set-point value after a priori calculated settling time with disregard of initial states [2].

Recently, predefined-time control is proposed based on time-varying high-gain feedback, which likewise achieves global finite-time convergence with predefined settling time that explicitly selected as a control parameter [40, 60]. For example, given a first-order system $\dot{x} = u \in \mathbb{R}$, the following controller is capable of realizing predefined-time convergence,

$$u = \begin{cases} \frac{-k(e^x-1)}{e^x(t_f-t)}, & \text{if } 0 \leq t < t_f, \\ 0, & \text{otherwise}, \end{cases} \tag{1.8}$$

where k is a positive control gain, and t_f is the predefined settling time. Basically, we have $u \to \infty$ as $t \to t_f$, which ensures convergence at t_f (the detailed proof is already provided in [40]).

1.2 Time-Synchronized Control

Before introducing time-synchronized control, we first briefly recall the history of finite-time control for the single-input single-output (SISO) system and the multi-input multi-output (MIMO) system.

For a SISO system, a simple relay controller is able to drive all the system state to the origin in finite time, i.e., a relative degree one sliding mode [57]. However, the extension of finite-time control from SISO systems to MIMO systems is non-trivial. In the MIMO case, instead of the selection and combination of multiple individual SISO sliding modes [51], elegant controllers are designed to drive all partial state elements to the equilibrium simultaneously by treating the state vector as a whole, the sliding mode control for multivariable uncertain systems [19, 20, 32], the unit vector control for multivariable systems [15] (Sect. 3.6, *The Unit Vector Approach*), the unit controller for MIMO affine systems in [49] (Sect. 3.5, *Unit Control*), the integral sliding mode in [22], the stabilization control of mechanical systems in [45], the quasi-continuous MIMO sliding mode in [33], to name a few.

In contrast with finite-time control, the design of fixed-time control and predefined-time control are more complicated due to the inherent complexity in their stability analysis. For the SISO system (or the MIMO system divided into multiple scalar SISO systems), fixed-time control and predefined-time control are explored and investigated [1, 27, 40, 43, 60]. Similar to the evolution of finite-time control, some recent approaches extend the fixed-time control to MIMO systems via super-twisting-like algorithms [3, 4, 41].

Actually, the performance of practical control systems not only stands on when the system state converges, but also depends largely on when and how each state element converges. Despite the superior convergence property of finite-time control (including fixed-time and predefined-time control), for some applications, standalone finite-time control is not sufficient, when a MIMO system is expected to reach all (or some of) the target values of the state elements *time-synchronously* (at the same time). It happens in practice when one requires a MIMO system (or a system with diversified/networked subsystems) to accomplish multiple *time-synchronized* actions. Considering a robotic hand, to stably grasp an object's surface/contour, its finger joints are to reach the target angles time-synchronously, otherwise the object may escape or slip during the manipulation. By treating a multi-agent system as a whole, its state elements are often required to reach desired locations time-synchronously in cooperative attack, transport, and formation missions [26]. Especially in the field of space offense and defense, it may be even catastrophic when some synchronous criteria are not satisfied during practical applications.

Motivated by the above, the time-synchronized control problem is defined. Naturally, time-synchronized control is usually designed for MIMO systems. It ensures that all the system state elements converge to the equilibrium at the same time, which has unique performance advantages including smoothing the system output trajectory, alleviating the chattering phenomenon, reducing the energy consumption and improving the convergence accuracy. In general, if we use a sliding mode to guarantee theoretically exact compensation of matched disturbances, the chattering phenomenon is impossible to be fully eliminated. However, this chattering could be alleviated when time-synchronized control is applied with the usage of proper sign functions. The concept of time-synchronized control was first put forward in public by the authors of this book together, in a plenary speech at the 2019 International Conference on Automatic Control (CACS 2019, Keelung, Taiwan), called "New Theoretical Developments in Fundamental Signum Functions Applicable in Control Systems", which reveals the time-synchronized control design of an extremely simple first-order system.

In [36], several effective controllers are proposed for different types of systems to achieve simultaneous-arrival-to-origin convergence introduced therein (which is, in fact, time-synchronized convergence). This approach revisits the sign functions well-applied in control design. It shows that when a norm-normalized sign function is properly invoked in the sliding-mode control, simultaneous-arrival-to-origin convergence can be expected. Next, to deeply articulate the time-synchronized convergence property and to formulate the stability associated, time-synchronized stability and its Lyapunov-like theorems are formally defined in [38]. Time-synchronized stability is built on top of finite-time stability. As introduced previously, finite-time stability requires all the system state elements to reach the equilibrium before a certain time instant ignoring how each state element converges. In contrast, time-synchronized stability requires all the system state elements (with non-zero initial values) to reach the equilibrium at exactly the same time. Similar to the various types of finite-time stability, special types of time-synchronized stability include: (1) fixed-synchronized stability where the upper bound of the synchronized settling time is invariant with any initial state; and (2) predefined-time-synchronized stability where the upper bound of the synchronized settling time is selected as a control parameter in advance. In summary, we have the following mapping:

$$\begin{aligned}
\text{Finite}-\text{Time Stability} &\Rightarrow \text{Time}-\text{Synchronized Stability}\\
\text{Fixed}-\text{Time Stability} &\Rightarrow \text{Fixed}-\text{Fime}-\text{Synchronized Stability}\\
\text{Predefined}-\text{Time Stability} &\Rightarrow \text{Predefined}-\text{Time}-\text{Synchronized Stability}\\
\text{Other Types of Finite}-\text{Time Stability} &\Rightarrow \text{Other Types of Time}-\text{Synchronized Stability}
\end{aligned}$$

Time-synchronized and fixed-time-synchronized control schemes are also proposed in [38]. These control methods naturally reduce the energy consumption and provide the shortest output trajectory due to the inherent properties of ratio persistence. Ratio persistence is a critical component of the sufficient conditions of time-synchronized stability. In general, the solution of a system is ratio persistent if the ratio of the closed-loop state elements (non-zero) is time-invariant in forward time. Straightforwardly, since the ratio is constant, the ratio persistent trajectory is the shortest path between any initial state and the equilibrium, which is optimal in terms of its travel length. For example, if the system state is $\in \mathbb{R}^2$ or $\mathbb{R}^3$ Euclidean space, the output trajectory is a straight line. In addition, ratio persistence likewise reduces the energy consumption by aligning the direction of the control signal with the direction of the error vector.

Despite the establishment of time-synchronized stability in [38], the time-synchronized controllers therein are confined to the assumption that the system model is ideally accurate and suffers from no external disturbances. This inevitably degrades the time-synchronized performance in possible practical implementations. In [37], under external disturbances and model uncertainties that can be lumped into disturbances, robust time-synchronized controllers are proposed. Further, based on time-synchronized stability, for a time-synchronized stable system, its analytical solution is derived in [37], providing a quantitative way to explicitly preview and predesign the time-synchronized performance of the closed-loop system.

The discussion in [37, 38] points out that many known control schemes in the existing results (e.g. [15, 19, 22, 33, 45, 48, 49]) already meet or partially meet the Lyapunov conditions of time-synchronized stability. Although many of their numerical results may not directly reflect time-synchronized convergence, they have the potential to achieve such a convergence property after some modifications following the proposed theorems in [37, 38]. It evidently approves the feasibility and generality of the presented conditions of time-synchronized stability, as they widely exist in the state of the art literature although such convergence property and stability results are not studied there.

Despite the time-synchronized convergence attained in [37, 38], they study only one single MIMO system (under some practical considerations). In real-world applications, multi-agent missions depend greatly on how and when the networked system states converge, e.g., for cooperative missile attacks [26], synthetic-aperture radar satellite lineup [11], and spacecraft formation-flying (PROBA-3, EXO-S, SWIFT, etc. [12]), the networked systems are usually required to reach a target or to achieve a spacial configuration synchronously. An extension of time-synchronized control from one single system to the multi-agent system is addressed in [35]. The concept of time-synchronized consensus is formally defined and formulated, where all the elements of all the agent states achieve consensus at exactly the same time. It can be extended to time-synchronized formation control problems by forming a consensus-based spatial configuration synchronously.

1.3 Why Time-Synchronized Control

In this section, we use an extremely simple but representative time-synchronized controller to explain the rationale rather than dig deeply into the mathematical analysis.

For a vector $x = [x_1,\ x_2, \ldots,\ x_n]^T \in \mathbb{R}^n$, we define the classical sign function sign_c and the norm-normalized sign function sign_n

$$\mathrm{sign}_\mathrm{c}\,(x) \triangleq \left[\mathrm{sgn}\,(x_1)\,,\ \mathrm{sgn}\,(x_2)\,, \ldots,\ \mathrm{sgn}\,(x_n)\right]^T, \tag{1.9}$$

$$\mathrm{sign}_\mathrm{n}\,(x) \triangleq \begin{cases} \frac{x}{\|x\|}, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{1.10}$$

where for $i = 1, 2, \ldots, n$,

$$\mathrm{sgn}\,(x_i) \triangleq \begin{cases} +1, & x_i > 0, \\ 0, & x_i = 0, \\ -1, & x_i < 0, \end{cases} \tag{1.11}$$

For a single-integrator system $\dot{x} = u$ in $\mathbb{R}^n$, where $x = [x_1,\ x_2, \ldots,\ x_n]^T \in \mathbb{R}^n$, we propose a time-synchronized controller $u_n = -\,\mathrm{sign}_\mathrm{n}(x)$. The analytical solution of the closed-loop system is

$$x(t) = (\|x_0\| - t)\,\frac{x_0}{\|x_0\|},\ t \leq \|x_0\|\,, \tag{1.12}$$

and $x(t) = 0, t > \|x_0\|$.

In summary, the advantages of time-synchronized control are expected from the following view.

i. *Guarantee Time-Synchronized Convergence*: Naturally and straightforwardly, from (1.12), all the state elements x_i converge to the origin time-synchronously at $t = \|x_0\|$, which guarantees time-synchronized stability.
ii. *Smooth and shorten the output trajectory*: The state trajectory in (1.12) is a straight line (at least in $\mathbb{R}^2$ and $\mathbb{R}^3$ Euclidean space), optimal concerning the travel length. Note that for other time-synchronized controllers, it is not necessary for them to also generate a straight-line trajectory, especially when uncertain and disturbed systems are considered. But in general, compared with other existing control laws, time-synchronized controllers smooth and shorten the system output to a certain degree with a similar rationality as show in (1.12).

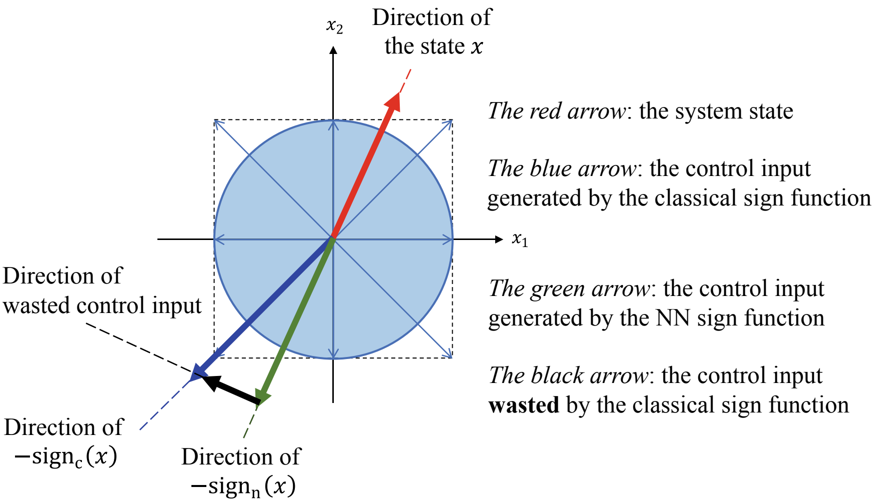

Fig. 1.1 How energy is saved by the norm-normalized (NN) sign function

iii. *Reduce the Energy Consumption*: For a time-synchronized controller designed by the norm-normalized sign function, its energy consumption is in general less than other controllers designed by the classical sign function. The rational could be found in Fig. 1.1. The control object is to drive the system state $x \in \mathbb{R}^2$ in Fig. 1.1 to the origin. A classical sign function-based controller (e.g., $u_c = -\mathrm{sign}_\mathrm{c}(x)$) will produce an input in the direction of the blue arrow; while a norm-normalized sign function-based controller e.g., $u_n = -\mathrm{sign}_\mathrm{n}(x)$) will produce an input in the direction of the green arrow. Clearly, the classical sign function-based controller wastes a part of the energy by additionally producing the control input in the direction of the black arrow (perpendicular to the negative direction of the system state x), while the time-synchronized controller based on the norm-normalized sign function does not.

References

1. Aldana-López, R., Gómez-Gutiérrez, D., Jiménez-Rodríguez, E., Sánchez-Torres, J.D., Defoort, M.: Enhancing the settling time estimation of a class of fixed-time stable systems. International Journal of Robust and Nonlinear Control **29**(12), 4135–4148 (2019)
2. Basin, M.: Finite- and fixed-time convergent algorithms: Design and convergence time estimation. Annual Reviews in Control **48**, 209–221 (2019). https://doi.org/10.1016/j.arcontrol.2019.05.007
3. Basin, M., Panathula, C.B., Shtessel, Y.: Multivariable continuous fixed-time second-order sliding mode control: Design and convergence time estimation. IET Control Theory and Applications **11**(8), 1104–1111 (2017)

4. Basin, M., Rodriguez-Ramirez, P., Garza-Alonso, A.: Continuous Fixed-Time Convergent Super-Twisting Algorithm in Case of Unknown State and Disturbance Initial Conditions. Asian Journal of Control **21**(1), 323–338 (2019)
5. Bejarano, F.J., Fridman, L.: High order sliding mode observer for linear systems with unbounded unknown inputs. International Journal of Control **83**(9), 1920–1929 (2010)
6. Bhat, S.P., Bernstein, D.S.: Finite-time stability of continuous autonomous systems. SIAM Journal on Control and Optimization **38**(3), 751–766 (2000)
7. Bhat, S.P., Bernstein, D.S.: Geometric homogeneity with applications to finite-time stability. Mathematics of Control, Signals and Systems **17**(2), 101–127 (2005)
8. Bidram, A., Davoudi, A., Lewis, F.L.: Finite-time frequency synchronization in microgrids. In: 2014 IEEE Energy Conversion Congress and Exposition (ECCE), pp. 2648–2654. IEEE (2014)
9. Boyd, S.P., Barratt, C.H.: Linear controller design: limits of performance. NJ: Prentice Hall, Englewood Cliffs (1991)
10. Chen, G., Lewis, F.L., Xie, L.: Finite-time distributed consensus via binary control protocols. Automatica **47**(9), 1962–1968 (2011)
11. Colombino, M., Smith, R.S., Summers, T.H.: Mutually quadratically invariant information structures in two-team stochastic dynamic games. IEEE Transactions on Automatic Control **63**(7), 2256–2263 (2018). https://doi.org/10.1109/TAC.2017.2772020
12. Di Mauro, G., Lawn, M., Bevilacqua, R.: Survey on guidance navigation and control requirements for spacecraft formation-flying missions. Journal of Guidance, Control, and Dynamics **41**(3), 1–22 (2017). https://doi.org/10.2514/1.G002868
13. Dorato, P.: Short-time stability in linear time-varying systems. Polytechnic Institute of Brooklyn (1961)
14. Du, H., Li, S., Qian, C.: Finite-time attitude tracking control of spacecraft with application to attitude synchronization. IEEE Transactions on Automatic Control **56**(11), 2711–2717 (2011)
15. Edwards, C., Spurgeon, S.: Sliding mode control: Theory and applications. Crc Press (1998)
16. Engel, R., Kreisselmeier, G.: A continuous-time observer which converges in finite time. IEEE Transactions on Automatic Control **47**(7), 1202–1204 (2002)
17. Fuller, A.T.: Relay control systems optimized for various performance criteria. IFAC Proceedings Volumes **1**(1), 520–529 (1960)
18. Galicki, M.: Finite-time control of robotic manipulators. Automatica **51**, 49–54 (2015)
19. Gutman, S.: Uncertain dynamical systems–A Lyapunov min-max approach. IEEE Transactions on Automatic Control **24**(3), 437–443 (1979)
20. Gutman, S., Leitmann, G.: Stabilizing feedback control for dynamic systems with bounded uncertainties. In: Proceedings of the IEEE Conference on Decision and Control, pp. 94–99 (1976)
21. Haimo, V.T.: Finite time controllers. SIAM Journal on Control and Optimization **24**(4), 760–770 (1986)
22. Hamayun, M.T., Edwards, C., Alwi, H.: Design and analysis of an integral sliding mode fault-tolerant control scheme. IEEE Transactions on Automatic Control **57**(7), 1783–1789 (2012)
23. Hong, Y., Wang, H.O., Bushnell, L.G.: Adaptive finite-time control of nonlinear systems. In: Proceedings of the 2001 American Control Conference.(Cat. No. 01CH37148), vol. 6, pp. 4149–4154. IEEE (2001)
24. Hong, Y., Xu, Y., Huang, J.: Finite-time control for robot manipulators. Systems & control letters **46**(4), 243–253 (2002)
25. Hong, Y., Yang, G., Bushnell, L., Wang, H.O.: Global finite-time stabilization: from state feedback to output feedback. In: Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187), vol. 3, pp. 2908–2913. IEEE (2000)
26. Jeon, I.S., Lee, J.I., Tahk, M.J.: Homing guidance law for cooperative attack of multiple missiles. Journal of guidance, control, and dynamics **33**(1), 275–280 (2010)
27. Jiang, B., Chen, Z., Hu, Q., Qi, J.: Fixed-time attitude control for rigid spacecraft with actuator saturation and faults. IEEE Transactions on Control Systems Technology **24**(5), 4375–4380 (2014)

28. Kamenkov, G.: On stability of motion over a finite interval of time. Journal of Applied Mathematics and Mechanics **17**(2), 529–540 (1953)
29. Khoo, S., Xie, L., Man, Z.: Robust finite-time consensus tracking algorithm for multirobot systems. IEEE/ASME Transactions on Mechatronics **14**(2), 219–228 (2009). https://doi.org/10.1109/TMECH.2009.2014057
30. Lebedev, A.: On stability of motion during a given interval of time. Journal of Applied Mathematics and Mechanics **18**, 139–148 (1954)
31. Lebedev, A.: The problem of stability in a finite interval of time. Journal of Applied Mathematics and Mechanics **18**(1), 75–94 (1954)
32. Leitmann, G.: Guaranteed asymptotic stability for some linear-systems with bounded uncertainties. Journal of Dynamic Systems Measurement and Control-Transactions of the ASME **101**(3), 212–216 (1979)
33. Levant, A., Shustin, B.: Quasi-Continuous MIMO Sliding-Mode Control. IEEE Transactions on Automatic Control **63**(9), 3068–3074 (2018). https://doi.org/10.1109/TAC.2017.2778251
34. Li, D., Ge, S.S., He, W., Ma, G., Xie, L.: Multilayer formation control of multi-agent systems. Automatica **109**, 108558 (2019). https://doi.org/10.1016/j.automatica.2019.108558
35. Li, D., Ge, S.S., Lee, T.H.: Fixed-Time-Synchronized Consensus Control of Multi-Agent Systems. IEEE Transactions on Control of Network Systems **8**(1), 89–98 (2021). https://doi.org/10.1109/TCNS.2020.3034523
36. Li, D., Ge, S.S., Lee, T.H.: Simultaneous-Arrival-to-Origin Convergence: Sliding-Mode Control through the Norm-Normalized Sign Function. IEEE Transactions on Automatic Control, in press. https://doi.org/10.1109/TAC.2021.3069816 (2021)
37. Li, D., Tee, K.P., Xie, L., Yu, H.: Time-Synchronized Control for Disturbed Systems. IEEE Transactions on Cybernetics, early access. https://doi.org/10.1109/TCYB.2021.3054589 (2021)
38. Li, D., Yu, H., Tee, K.P., Wu, Y., Ge, S.S., Lee, T.H.: On Time-Synchronized Stability and Control. IEEE Transactions on Systems, Man, and Cybernetics: Systems, in press. https://doi.org/10.1109/TSMC.2021.3050183 (2021)
39. Liu, X., Lam, J., Yu, W., Chen, G.: Finite-time consensus of multiagent systems with a switching protocol. IEEE transactions on neural networks and learning systems **27**(4), 853–862 (2015)
40. Pal, A.K., Kamal, S., Nagar, S.K., Bandyopadhyay, B., Fridman, L.: Design of controllers with arbitrary convergence time. Automatica **112**, 108710 (2020)
41. Pal, A.K., Kamal, S., Yu, X., Nagar, S.K., Xiong, X.: Free-Will Arbitrary Time Consensus for Multiagent Systems. IEEE Transactions on Cybernetics, early access, https://doi.org/10.1109/TCYB.2020.3032217 (2020)
42. Perruquetti, W., Floquet, T., Moulay, E.: Finite-time observers: application to secure communication. IEEE Transactions on Automatic Control **53**(1), 356–360 (2008)
43. Polyakov, A.: Nonlinear feedback design for fixed-time stabilization of linear control systems. IEEE Transactions on Automatic Control **57**(8), 2106–2110 (2012). https://doi.org/10.1109/TAC.2011.2179869
44. Ríos, H., Efimov, D., Moreno, J.A., Perruquetti, W., Rueda-Escobedo, J.G.: Time-Varying Parameter Identification Algorithms: Finite and Fixed-Time Convergence. IEEE Transactions on Automatic Control **62**(7), 3671–3678 (2017)
45. Sanyal, A.K., Bohn, J.: Finite-time stabilisation of simple mechanical systems using continuous feedback. International Journal of Control **88**(4), 783–791 (2015). https://doi.org/10.1080/00207179.2014.974675
46. Song, Y., Wang, Y., Holloway, J., Krstic, M.: Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. Automatica **83**, 243–251 (2017)
47. Song, Y.D., Lu, Y., Gan, Z.X.: Descriptor sliding mode approach for fault/noise reconstruction and fault-tolerant control of nonlinear uncertain systems. Information Sciences **367**, 194–208 (2016)
48. Sun, C., Hu, G., Xie, L., Egerstedt, M.: Robust finite-time connectivity preserving coordination of second-order multi-agent systems. Automatica **89**, 21–27 (2018)
49. Utkin, V., Guldner, J., Shi, J.: Sliding Mode Control in Electro-Mechanical Systems. CRC Press (2009)

50. Utkin, V., Poznyak, A., Orlov, Y.V., Polyakov, A.: Road map for sliding mode control design. Springer (2020)
51. Utkin, V.I.: Sliding Modes in Control and Optimization. Springer Berlin Heidelberg (1992)
52. Wang, H., Yu, W., Wen, G., Chen, G.: Finite-time bipartite consensus for multi-agent systems on directed signed networks. IEEE Transactions on Circuits and Systems I: Regular Papers **65**(12), 4336–4348 (2018)
53. Weiss, L., Infante, E.: Finite time stability under perturbing forces and on product spaces. IEEE Transactions on Automatic Control **12**(1), 54–59 (1967)
54. Xiao, B., Hu, Q., Zhang, Y.: Adaptive sliding mode fault tolerant attitude tracking control for flexible spacecraft under actuator saturation. IEEE Transactions on Control Systems Technology **20**(6), 1605–1612 (2011)
55. Xiao, F., Wang, L., Chen, J., Gao, Y.: Finite-time formation control for multi-agent systems. Automatica **45**(11), 2605–2611 (2009)
56. Yang, X., Lam, J., Ho, D.W.C., Feng, Z.: Fixed-Time Synchronization of Complex Networks With Impulsive Effects via Nonchattering Control. IEEE Transactions on Automatic Control **62**(11), 5511–5521 (2017)
57. Young, K.D., Utkin, V.I., Özgüner, Ü.: A control engineer's guide to sliding mode control. IEEE Transactions on Control Systems Technology **7**(3), 328–342 (1999). 10.1109/87.761053
58. Yu, S., Yu, X., Shirinzadeh, B., Man, Z.: Continuous finite-time control for robotic manipulators with terminal sliding mode. Automatica **41**(11), 1957–1964 (2005)
59. Zamora-Gomez, G.I., Zavala-Rio, A., Lopez-Araujo, D.J., Cruz-Zavala, E., Nuno, E.: Continuous Control for Fully Damped Mechanical Systems With Input Constraints: Finite-Time and Exponential Tracking. IEEE Transactions on Automatic Control **65**(2), 882–889 (2020)
60. Zhou, B.: Finite-time stability analysis and stabilization by bounded linear time-varying feedback. Automatica **121**, 109191 (2020)
61. Zou, A.M., Kumar, K.D., Hou, Z.G., Liu, X.: Finite-time attitude tracking control for spacecraft using terminal sliding mode and Chebyshev neural network. IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics) **41**(4), 950–963 (2011)
62. Zuo, Z., Tian, B., Defoort, M., Ding, Z.: Fixed-Time Consensus Tracking for Multiagent Systems With High-Order Integrator Dynamics. IEEE Transactions on Automatic Control **63**(2), 563–570 (2018)

Chapter 2
Time-Synchronized Stability

Abstract In this chapter, time-synchronized stability is formally introduced as a unique type of finite-time stability. Its special branches include fixed-time-synchronized stability and predefined-time-synchronized stability, where stronger requirements of the synchronized settling time are imposed. Relevant Lyapunov-like theorems and respective sufficient conditions of these stability formulations are likewise suitably proposed. Among these sufficient conditions, the concept of ratio persistence is one of the key components. A system state is called to be ratio persistent when the ratio of each pair of the state elements is a constant except at the equilibrium. A similar but milder condition is called ratio restriction, requiring only time-varying but bounded ratios, rather than constant ratios. Compared with the classical sign function, we introduced a so-called norm-normalized sign function to facilitate ratio persistence (also ratio restriction) of a system state. Given a state vector x with n dimension, the norm-normalized sign function incorporates the normalization computation of the state x with its norm, while the classical sign function provides only n individual decoupled scalars. Concerning these embedded attributes, it is evidently possible to expect stringent time-synchronized performance from a system design involving the norm-normalized sign function rather than the classical one.

2.1 Properties of Sign Functions

In the literature, the classical sign function is often defined as

$$\operatorname{sign}_{\mathrm{c}}(x_i) \triangleq \begin{cases} +1, & x_i > 0, \\ 0, & x_i = 0, \\ -1, & x_i < 0, \end{cases} \tag{2.1}$$

where its vector form and exponential form are defined as

$$\operatorname{sign}_{\mathrm{c}}(x) = \left[\operatorname{sign}_{\mathrm{c}}(x_1), \ldots, \operatorname{sign}_{\mathrm{c}}(x_n)\right]^T, \tag{2.2}$$

$$\operatorname{sig}_{\mathrm{c}}^{\alpha}(x) = \left[\operatorname{sign}_{\mathrm{c}}(x_1)|x_1|^{\alpha}, \ldots, \operatorname{sign}_{\mathrm{c}}(x_n)|x_n|^{\alpha}\right]^T. \tag{2.3}$$

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