Bang-Yen Chen
Mohammad Hasan Shahid
Falleh Al-Solamy *Editors*

Contact Geometry of Slant Submanifolds

Springer

Contact Geometry of Slant Submanifolds

Bang-Yen Chen · Mohammad Hasan Shahid ·
Falleh Al-Solamy
Editors

Contact Geometry of Slant Submanifolds

Springer

Editors
Bang-Yen Chen
Department of Mathematics
Michigan State University
East Lansing, MI, USA

Mohammad Hasan Shahid
Department of Mathematics
Jamia Millia Islamia
New Delhi, India

Falleh Al-Solamy
King Khalid University
Abha, Saudi Arabia

ISBN 978-981-16-0016-6 ISBN 978-981-16-0017-3 (eBook)
https://doi.org/10.1007/978-981-16-0017-3

Mathematics Subject Classification: 53-XX, 53C26, 53C23, 54E35, 53-02, 53A07, 53B25, 53B35

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

An almost Hermitian manifold is an almost complex manifold (M, J) equipped with a Riemannian metric g which satisfies $g(JX, JY) = g(X, Y)$ for vector fields X, Y tangent to M. By a submanifold of an almost Hermitian manifold (M, g_M, J), we mean the image of an isometric immersion

$$\phi : (N, g_N) \to (M, g_M, J)$$

from a Riemannian manifold (N, g_N) into (M, g_M, J).

Dual to the notion of isometric immersions, there exists the notion of Riemannian submersions introduced by B. O'Neill in [11]. By definition, a Riemannian submersion is a surjective map

$$\pi : (M, g_M) \to (B, g_B)$$

from a Riemannian manifold (M, g_M) onto another Riemannian manifold (B, g_B) which preserves the scalar products of vectors normal to fibers.

Based on the action of the almost complex structure J on the tangent bundle of a submanifold, there are three important classes of submanifolds of an almost Hermitian manifold (M, g_M, J), namely the classes of complex, totally real and slant submanifolds.

In terms of the almost complex structure J, a submanifold N of an almost complex manifold (M, g, J) is called a *complex submanifold* (respectively, *totally real submanifold*) if

$$J(T_pN) \subseteq T_pN \quad \text{(respectively, } J(T_pN) \subseteq T_p^{\perp}N) \tag{1}$$

for any point $p \in N$, where $T_p^{\perp}N$ denotes the normal space of N in M at p.

For a unit tangent vector $X \in T_pN$ of a submanifold N in an almost Hermitian manifold (M, g_M, J) at a point $p \in N$, the angle $\theta(X)$ between JX and T_pN is called the Wirtinger angle of X.

In 1990, a more general class of submanifolds than complex and totally real submanifolds was introduced in [5] as follows.

Definition 1 A submanifold N of an almost Hermitian manifold (M, g, J) is called a *slant submanifold* if the Wirtinger angle $\theta(X)$ is independent of the choice of the unit vector $X \in T_pN$ and of $p \in N$. In this case, the constant θ is called the *slant angle*. A slant submanifold with slant angle θ is said to be θ-*slant*.

It follows from the definitions that complex submanifolds and totally real submanifolds are nothing but θ-slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. From J-action points of view, slant submanifolds are the simplest and the most natural submanifolds of an almost Hermitian manifold. In [7, 10], the notion of pointwise slant submanifolds of an almost Hermitian manifold was defined as a generalization of slant submanifolds.

The first results on slant submanifolds were collected in the book [6]. Since then the study of slant submanifolds and of slant submersions has been attracting more and more researchers and a lot of interesting results have been achieved during the past 30 years.

A Riemannian $(2n + 1)$-manifold (M^{2n+1}, g) is called an *almost contact metric manifold* (cf. [1]) if there exist a $(1, 1)$ tensor field φ, a vector field ξ (called the *structure vector field*), and a 1-form η on M^{2n+1} such that

$$\eta(\xi) = 1, \varphi^2(X) = -X + \eta(X)\xi, \varphi\xi = 0, \eta \circ \varphi = 0,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi)$$

for any vector fields X, Y tangent to M^{2n+1}. An almost contact metric structure is called a *contact metric structure* if it satisfies

$$d\eta(X, Y) = g(X, \varphi Y).$$

A contact metric structure is called *normal* if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

where ∇ is the Levi-Civita connection of g. A manifold M endowed with a normal contact metric structure is called a *Sasakian manifold*. A Sasakian manifold with constant φ-sectional curvature is called a *Sasakian space form*.

The study of slant submanifolds was extended by A. Lotta [8] in 1996 to contact slant submanifolds in almost contact geometry as follows. Let N be a submanifold of an almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$. Then N is called *contact slant* if the Wirtinger angle $\theta(X)$ between ϕX and T_pN is a global constant, so that it is independent of the choice of the point $p \in N$ and the vector $X \in T_pN$ such that X and ξ_p are linearly independent. In particular, for $\theta = 0$ and $\theta = \frac{\pi}{2}$, the θ-slant submanifolds of $(M^{2n+1}, \varphi, \xi, \eta, g)$ are called invariant and anti-invariant submanifolds, respectively.

In [8], A. Lotta proved that if M^{2n+1} is a contact metric manifold, then the structure vector field ξ is tangent to every non-anti-invariant slant submanifold. After Lotta's work there are a lot of works done on contact slant submanifolds.

Dual to slant submanifolds, B. Sahin introduced in [12] the notion of slant submersions. Roughly speaking, a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (B, g_B) is called a slant submersion if its vertical distribution is a slant distribution.

Similar to Sahin's work on slant submersions from almost Hermitian manifolds onto Riemannian manifolds, I. K. Erken and C. Murathan defined and studied in [9] slant submersions from Sasakian manifolds onto Riemannian manifolds. Cabrerizo et al. studied in [2, 3] slant, semi-slant, hemi-slant, and bi-slant submanifolds in contact slant geometry. Further, as a generalization of slant submanifolds and semi-slant submanifolds, K. S. Park [4] defined the notion of pointwise slant submanifolds and pointwise semi-slant submanifolds of an almost contact metric manifold.

Given the huge amount of work on contact slant submanifolds and submersions published since the appearance of the last monograph [6], the editors thought it is appropriate to invite a number of specialists to contribute one or more papers to illustrate the state of the art in the theory of contact slant geometry with focuses on contact slant submanifolds and contact slant submersions and many colleagues answered our call. The editors express their gratitude to all the contributors.

The editors hope that the readers will find this book both a good introduction and a useful reference of contact slant geometry to perform their research more successfully and creatively.

East Lansing, Michigan, USA — Bang-Yen Chen
New Delhi, India — Mohammad Hasan Shahid
Abha, Saudi Arabia — Falleh Al-Solamy

References

1. Blair, D. E.: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Vol. 509, Springer-Verlag (1976)
2. Cabrerizo, J. L., Carriazo, A., Fernandez, L. M., Fernandez, M.: Semi-Slant submanifolds of Sasakian manifold, Geom. Dedicata **78**(2), 183–199 (1999)
3. Cabrerizo, J. L., Carriazo, A., Fernandez, L. M., Fernandez, M.: Slant submanifolds in Sasakian manifold, Glasgow Math. J. **42**(1), 125–138 (2000)
4. Park, K. S.: Pointwise slant and pointwise semi-Slant submanifolds of almost contact metric manifolds, Mathematics **8**, Art. 985, pp. 33 (2000)

5. Chen, B.-Y.: Slant immersions, Bull. Austral. Math. Soc. **41**(1), 135–147 (1990)
6. Chen, B.-Y.: Geometry of Slant Submanifolds, Katholieke Universiteit Leuven, Belgium (1990)
7. Chen, B.-Y., Garay, O.-J.: Pointwise Slant submanifolds in almost Hermitian manifolds, Turk. J. Math. **36** (2012), 630--640.
8. Lotta, A.: Slant submanifolds in contact geometry, Bul. Math. Soc. Sci. Math., 39, 183–198 (1996)
9. Erken, I. K., Murathan, C.: Slant Riemannian submersions from Sasakian manifolds, Arab J. Math. Sci. 22(2), 250–264 (2016)
10. Etayo, F.: On quasi-slant submanifolds of an almost Hermitian manifold. Publ. Math. (Debrecen) **53**, 217–223 (1998)
11. O'Neill, B.: The fundamental equations of a submersion, Michigan Math. J. **13**, 459–469 (1966)
12. Sahin, B.: Slant submersions from almost Hermitian manifolds. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **54(102)**(1), 93–105 (2011)

Contents

About the Editors

Bang-Yen Chen, a Taiwanese-American mathematician, is University Distinguished Professor Emeritus at Michigan State University, USA, since 2012. He completed his Ph.D. degree at the University of Notre Dame, USA, in 1970, under the supervision of Prof. Tadashi Nagano. He received his M.Sc. degree from National Tsing Hua University, Hsinchu, Taiwan, in 1967, and B.Sc. degree from Tamkang University, Taipei, Taiwan, in 1965. Earlier at Michigan State University, he served as University Distinguished Professor (1990–2012), Full Professor (1976), Associate Professor (1972), and Research Associate (1970–1972). He taught at Tamkang University, Taiwan, from 1966 to 1968, and at National Tsing Hua University, Taiwan, during the academic year 1967–1968.

He is responsible for the invention of δ-invariants (also known as Chen invariants), Chen inequalities, Chen conjectures, development of the theory of submanifolds of finite type, and co-developed (M+, M–)-theory. An author of 12 books and more than 500 research articles, Prof. Chen has been Visiting Professor at various universities, including the University of Notre Dame, USA; Science University of Tokyo, Japan; the University of Lyon, France; Katholieke Universiteit Leuven, Belgium; the University of Rome, Italy; National Tsing Hua University, Taiwan; and Tokyo Denki University, Japan.

Mohammad Hasan Shahid is Professor at the Department of Mathematics, Jamia Millia Islamia, New Delhi, India. He earned his Ph.D. in Mathematics from Aligarh Muslim University, India, on the topic "On geometry of submanifolds" in 1988 under (Late) Prof. Izhar Husain. Earlier, he served as Associate Professor at King Abdul Aziz University, Jeddah, Saudi Arabia, from 2001 to 2006. He was a recipient of the postdoctoral fellowship from the University of Patras, Greece, from October 1997 to April 1998. He has published more than 100 research articles in various national and international journals of repute. Recently, he was awarded the Sultana Nahar Distinguished Teacher award of the Year 2017–2018 for his outstanding contribution to research. For research works and delivering talks, Prof. Shahid has visited several universities of the world: the University of Leeds, UK; the University of

Montpellier, France; the University of Sevilla, Spain; Hokkaido University, Japan; Chuo University, Japan; and Manisa Celal Bayar University, Turkey.

Falleh Al-Solamy is President at King Khalid University, Abha, Saudi Arabia. Earlier, he was Professor of Differential Geometry at King Abdulaziz University, Jeddah, Saudi Arabia. He studied Mathematics at King Abdulaziz University, Jeddah, Saudi Arabia, and earned his Ph.D. in Mathematics from the University of Wales Swansea, Swansea, UK, in 1998, under Prof. Edwin Beggs. His research interests concern the study of the geometry of submanifolds in Riemannian and semi-Riemannian manifolds, Einstein manifolds, and applications of differential geometry in physics. Professor Al-Solamy's research papers have been published in journals and conference proceedings of repute.

General Properties of Slant Submanifolds in Contact Metric Manifolds

A. Lotta

2000 AMS Mathematics Subject Classification: 53B25 · 53C25 · 53D15

1 Slant Submanifolds of Almost Contact Manifolds

B. Y. Chen's concept of a slant submanifold can be translated into the context of contact metric geometry in a very natural fashion. In this chapter, we shall discuss the basic facts concerning this variant of the theory.

Our standard reference for contact geometry is Blair's book [2], to which we refer the reader for the terminology, the notation and the relevant facts.

Let M be an almost contact metric manifold with structure (φ, ξ, η, g). By a *slant submanifold* of M, we shall mean an immersed submanifold N such that for any $x \in N$ and for any tangent vector $X \in T_x N$, linearly independent on ξ, the angle between φX and $T_x N$ is a constant $\theta \in [0, \frac{\pi}{2}]$, called the *slant angle* of N in M.

Like in complex geometry, when the ambient manifold is a *contact metric* manifold, for $\theta = \frac{\pi}{2}$ one recovers the notion of *anti-invariant* submanifold. For $\theta = 0$, this class coincides with that of *invariant* submanifolds, i.e. those for which each tangent space of the submanifold is invariant under φ. We remark that it is known that such a submanifold must be tangent to the Reeb vector field ξ (see [2, Sect. 8.1], p. 152). We shall see below that this property also holds in the larger class of non-anti-invariant slant submanifolds.

We shall denote by $\bar{\nabla}$ the Levi-Civita connection of the ambient manifold, by ∇ the corresponding connection relative to the metric induced on a submanifold, while the second fundamental form will be denoted by α.

A. Lotta (✉)
Dipartimento di Matematica, Università di Bari Aldo Moro,
Via E. Orabona 4, 70125 BARI, Italy
e-mail: antonio.lotta@uniba.it

B.-Y. Chen et al. (eds.), *Contact Geometry of Slant Submanifolds*,
https://doi.org/10.1007/978-981-16-0017-3_1

We shall also denote the kernel of η by D, which is a distribution on M of rank $\dim(M)-1$. Using the same notation as in the complex case, for every tangent vector $X \in TN$ we write

$$\varphi X = PX + FX$$

where PX is tangent and FX is normal to the submanifold. Then $P : TN \to TN$ is a skew-symmetric (1, 1) tensor field on N with respect to the induced metric. We shall also denote by Q the symmetric operator P^2.

We begin by discussing the following basic result showing that the class of non-anti-invariant slant submanifolds of a given almost contact metric manifold splits into two sub-classes, characterized by the position of the characteristic vector field ξ with respect to the submanifold (cf. [11]).

Theorem 1.1 *Let N be an immersed slant submanifold of the almost contact metric manifold M with structure tensors (φ, ξ, η, g). Let $n = \dim(N)$. Assume that N is not anti-invariant. Then*

$$n \text{ is odd} \iff \xi \text{ is tangent to } N$$

$$n \text{ is even} \iff \xi \text{ is normal to } N.$$

Proof For every point $x \in N$, the orthogonal complement $E \subset T_xN$ of $\mathrm{Ker}(Q_x)$ is even dimensional. Observe that if $X \in \mathrm{Ker}(Q_x)$, then φX is normal to N. By definition of a slant submanifold, this forces that X be a scalar multiple of ξ_x, because we are assuming that the slant angle $\theta \neq \frac{\pi}{2}$. Thus, we have proved that

$$\mathrm{Ker}(Q_x) \subset \mathbb{R}\xi_x.$$

Now, if n is odd, then $\mathrm{Ker}(Q_x) \neq \{0\}$ for every $x \in N$, which yields that ξ is everywhere tangent to N. If n is even, we must have $\mathrm{Ker}(Q_x) = \{0\}$ for all $x \in N$. Fix x and consider an eigenspace H of Q_x relative to the eigenvalue λ. By definition of the slant angle, for every non-null $X \in H$ we have

$$\cos\theta = \frac{||PX||}{||\varphi X||} = \sqrt{-\lambda}\frac{||X||}{||\varphi X||}. \tag{1}$$

On the other hand, since $\dim(H) \geq 2$, H contains some non-null $X \in H$ belonging to D_x, for which $||\varphi X|| = ||X||$. Substituting in (1) yields $\lambda = -\cos^2\theta$. We have thus showed that $Q_x = -\cos^2\theta\, \mathrm{Id}$ and moreover, coming back again to (1) we have that $||\varphi X|| = ||X||$ for every $X \in T_xN$, which implies that $T_xN \subset D_x$. We conclude that ξ is everywhere orthogonal to N. □

We remark that the assumption that N is not anti-invariant in the above result is essential. A significant example is provided by a well-known result of Blair, who classified the contact metric manifolds of dimension at least five, whose curvature tensor annihilates ξ (i.e. $R(X, Y)\xi = 0$ for every vector fields X, Y). The simply connected, complete ones are the Riemannian products

$$M = \mathbb{R}^{n+1} \times \mathbb{S}^n(4), \quad n > 1$$

where $\mathbb{S}^n(4)$ denotes a sphere endowed with a standard metric of constant sectional curvature 4. See [2], Theorem 7.5. Then, it turns out that both the standard immersions of $\mathbb{R}^{n+1}$ and of $\mathbb{S}^n(4)$ are anti-invariant in M. Moreover, for the first one ξ is everywhere tangent, while for the second one ξ is always normal, so that letting $n > 1$ vary, we provide a series of counterexamples to the equivalences in Theorem 1.1.

For a submanifold N tangent to ξ, as a further application of formula (1) the following characterization is readily verified, involving the symmetric tensor Q and the normal bundle valued 1-form $F : TN \to TN^{\perp}$ (see [3]).

Theorem 1.2 *Let N be a submanifold of an almost contact metric manifold M. Assume that ξ is tangent to N. Then the following are equivalent:*

(a) N is slant in M with slant angle θ;
(b) $Q = -\cos^2\theta(I - \eta \otimes \xi)$;
(c) For every unit vector tangent to N and orthogonal to ξ, one has

$$||PX|| = \cos\theta;$$

d) For every unit vector tangent to N and orthogonal to ξ, one has

$$||FX|| = \sin\theta.$$

We remark that, in the general context of almost contact metric manifolds, one can provide simple examples showing that both possibilities regarding the position of ξ with respect to a slant submanifold can occur. Namely, given any almost Hermitian manifold (M, J, g_o), the product $M \times \mathbb{R}$ carries a standard almost contact metric structure (φ, ξ, η, g), where

$$\varphi(X, a\frac{d}{dt}) = (JX, 0),\ \xi = (0, \frac{d}{dt}),\ \eta = dt,$$

g being the product metric of g_o and the standard metric on the real line.

Now, given any θ-slant submanifold N of M, it is not difficult to verify that $N \times \{0\}$ and $N \times \mathbb{R}$ are both θ-slant in $M \times \mathbb{R}$ (cf. [11]). More generally, the same is true if instead of the product metric one considers a warped product metric g on $M \times I$, where $I \subset \mathbb{R}$ is an open interval, namely $g = \lambda^2\pi_1^* g_o + \pi_2^* dt \otimes dt$, where $\lambda : I \to \mathbb{R}$ is a smooth positive function and $\pi_1 : M \times I \to M$ and $\pi_2 : M \times I \to I$ are the canonical projections (see [6]).

Explicit examples of slant submanifolds (most of them in the Sasakian space form $\mathbb{R}^5$) are exhibited in [3]. Other examples and some general results concerning slant submanifolds of some particular classes of almost contact metric manifolds can be found in the recent papers [6, 7] by de Candia and Falcitelli.

We report here the following fact concerning even dimensional submanifolds (it is proved in [7] for the class of $C_5 \oplus C_{12}$-almost contact metric manifolds according to the Chinea-Gonzalez classification scheme [5]).

Theorem 1.3 *Let $(M, \varphi, \xi, \eta, g)$ an almost contact metric manifold and assume that φ is η-parallel, i.e.*

$$g((\bar{\nabla}_X \varphi)Y, Z) = 0$$

for every X, Y, Z vector fields orthogonal to ξ.

Let N be an even dimensional θ-slant submanifold of M, $\theta \neq \frac{\pi}{2}$. Then M induces on N an almost Kähler structure (J, g) where $J = \sec\theta P$.

Proof We know that ξ is normal to N. Moreover, $Q = -\cos^2\theta \mathrm{Id}$. Hence, $J = \sec\theta P$ is an almost complex structure on N, which is Hermitian with respect to the induced metric. Moreover, by the η-parallelism of φ, for every X, Y, Z vector fields tangent to N we get

$$g(\bar{\nabla}_X PY, Z) + g(\bar{\nabla}_X FY, Z) - g(\varphi\nabla_X Y, Z) - g(\varphi\alpha(X, Y), Z) = 0,$$

yielding

$$g((\nabla_X P)Y, Z) = g(A_{FY}X, Z) - g(A_{FZ}X, Y).$$

It follows that

$$\mathfrak{S}_{X,Y,Z}\, g((\nabla_X J)Y, Z) = 0,$$

where $\mathfrak{S}$ is a cyclic sum, and this ensures that the almost Hermitian structure (J, g) is almost Kähler. □

2 Slant Submanifolds of Contact Metric Manifolds

From now on, we shall consider the case when the ambient manifold is a contact metric manifold.

Theorem 2.1 *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. Every non-anti-invariant slant submanifold N of M is tangent to ξ. Moreover, the restriction of η to N is again a contact form and N inherits canonically a contact metric structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$, where*

$$\bar{\varphi} := \sec\theta P, \;\; \bar{\xi} := \sec\theta\xi, \;\; \bar{\eta} := \cos\theta\,\eta, \;\; \bar{g} := \cos^2\theta\, g. \tag{2}$$

Proof If ξ were normal to N, from the formula

$$\bar{\nabla}_X \xi = -\varphi X - \varphi h X, \quad h := \frac{1}{2}\mathcal{L}_\xi \varphi \tag{3}$$

which is valid in the ambient manifold (cf. [2, Lemma 6.2]), for every X and Y vector fields tangent to N we would have

$$g(A_\xi X, Y) = g(\varphi X, Y) + g(\varphi h X, Y),$$

A_ξ being the Weingarten operator in the direction of ξ. But since φh and A_ξ are both symmetric operators, this would imply $g(\varphi X, Y) = 0$ identically, yielding that N is anti-invariant against the assumption. Hence according to Theorem 1.1, ξ must be tangent to N. Concerning the last statement, denoting by the same symbols the restrictions of η, ξ and g to the submanifold, setting $\bar{\varphi} := \sec\theta\, P$, it is easy to check that $(\bar{\varphi}, \xi, \eta, g)$ is an almost contact metric structure satisfying

$$d\eta = \cos\theta \bar{\Phi},$$

where $\bar{\Phi}$ is its fundamental 2-form, i.e. it is a $\cos\theta$-homothetic contact metric structure on N. This implies the last claims. □

The next proposition provides a formula linking the operator h of the ambient manifold and the analogous operator $\bar{h}$ relative to the induced contact metric structure.

Proposition 2.2 *Let N be a θ-slant, non-anti-invariant submanifold of a contact metric manifold $(M, \varphi, \xi, \eta, g)$. Then for every X, Y vectors tangent to N and orthogonal to ξ, we have*

$$g(hX, Y) = \cos^2\theta g(\bar{h}X, Y) - \sin^2\theta g(X, Y) - g(\alpha(X, \xi), FY).$$

In particular,

$$g(\alpha(X, \xi), FY) = g(\alpha(Y, \xi), FX)$$

holds with the same assumption on X, Y.

Proof We shall use (3) and the analogous formula for the induced contact metric structure (2) on N. Observing that the Levi-Civita connections of $g|_N$ and $\bar{g}$ coincide, we have

$$
\begin{aligned}
g(hX,Y) &= g(\varphi\bar{\nabla}_X\xi,Y) - g(X,Y) = \\
&= -g(\alpha(X,\xi),FY) - g(\nabla_X\xi,PY) - g(X,Y) = \\
&= -g(\alpha(X,\xi),FY) - \cos\theta g(\nabla_X\bar{\xi},PY) - g(X,Y) = \\
&= -g(\alpha(X,\xi),FY) + \cos\theta g((\bar{\varphi}+\bar{\varphi}\bar{h})X,PY) - g(X,Y) = \\
&= -g(\alpha(X,\xi),FY) + g(PX,PY) + g(P\bar{h}X,PY) - g(X,Y) = \\
&= -g(\alpha(X,\xi),FY) + \cos^2\theta g(\bar{h}X,Y) - \sin^2\theta g(X,Y),
\end{aligned}
$$

where to deduce the last equality we used (c) in Theorem 1.2. The last claim follows since h and $\bar{h}$ are both symmetric operators. □

As another application of Theorem 1.2, we prove a result concerning contact totally umbilical submanifolds. Recall that a submanifold, tangent to ξ, of a contact metric manifold, is said to be *contact totally umbilical* provided the second fundamental form satisfies (cf. e.g. [15])

$$
\alpha(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}H + \eta(X)\alpha(Y,\xi) + \eta(Y)\alpha(X,\xi). \tag{4}
$$

Here, H is the mean curvature normal vector field. If in addition $H = 0$, one speaks of a *contact totally geodesic* submanifold.

Given a proper slant submanifold, we shall consider the orthogonal splitting

$$
TN^{\perp} = F(TN) \oplus E = F(\bar{D}) \oplus E
$$

of the normal bundle, where $\bar{D}$ denotes the induced contact distribution on the slant submanifold. Of course, this is meaningful because $F_x : \bar{D}_x \to T_xN^{\perp}$ is injective for each point of the submanifold.

Theorem 2.3 *Let (M,φ,ξ,η,g) be a contact metric manifold of dimension $2m+1$, whose almost CR structure $(D,\varphi|_D)$ is integrable, i.e. M is a strongly pseudoconvex CR manifold. Let N be a contact totally umbilical proper slant submanifold of M.*

Then N is contact totally geodesic provided $\dim(N) = m+1$ or $D_XH \in \Gamma(E)$ for every vector field tangent to N and orthogonal to ξ.

Proof First of all, we recall that for contact metric manifolds, the integrability condition for $(D,\varphi|_D)$ is equivalent to φ being η-parallel; indeed, one has the following formula for the covariant derivative of φ:

$$
(\bar{\nabla}_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX);
$$

see [2, Theorem 6.7]. This implies that for every X, Y vector fields tangent to N and orthogonal to ξ (i.e. sections of $\bar{D}$), the vector field $(\bar{\nabla}_X\varphi)Y$ is tangent to N.

Using this, by a standard computation similar to the case of submanifolds of Kähler manifolds, one can derive the following formula for the covariant derivative of F, with the same assumption on the vector fields X, Y:

$$(\nabla_X F)Y = f\alpha(X, Y) - \alpha(X, PY). \tag{5}$$

Here, as usual, for every normal vector field ν to N, we set

$$\varphi\nu = t\nu + f\nu$$

with $t\nu$ tangent resp. $f\nu$ normal to N.

Now, assuming (4), (5) yields, for any local unit vector field X tangent to N and orthogonal to ξ:

$$(\nabla_{PX} F)X = -g(PX, PX)H = -\cos^2\theta\, H. \tag{6}$$

Observe now that the left-hand side of this equation is orthogonal to FX; indeed, by the condition (c) in Theorem 1.2 we have, for every Z, W tangent to N and orthogonal to ξ:

$$g(FZ, FW) = \sin^2\theta g(Z, W);$$

using this, we get, assuming $||X|| = 1$:

$$\begin{aligned} g((\nabla_{PX} F)X, FX) &= g(D_{PX}FX, FX) - g(F\nabla_{PX}X, FX) = \\ &= -\sin^2\theta g(\nabla_{PX}X, X) = 0. \end{aligned}$$

As a consequence, we obtain from (6) that

$$g(H, FX) = 0$$

for every X orthogonal to ξ and tangent to N, showing that $H \in \Gamma E$. If $\dim(N) = m + 1$, this suffices to prove the result, since in this case the subbundle E is trivial, because N has codimension m. Now assume that $D_X H \in \Gamma(E)$ for all sections of $\bar{D}$. Taking the inner product of both sides of (6) with H, one gets

$$-\cos^2\theta g(H, H) = g(D_{PX}FX, H) - g(F(\nabla_{PX}X), H) = -g(FX, D_{PX}H) = 0$$

due to our assumption. □

In [8], a similar result has been proved by Gupta in the case where M is a Kenmotsu manifold.

Corollary 2.4 *Let M be a Sasakian space form with φ-sectional curvature c. Then every totally contact umbilical proper slant submanifold N of M is totally contact geodesic, provided* $\dim(N) > 3$ *or* $c = 1$.

Proof In this case, we shall verify that actually $D_X H = 0$ for every X orthogonal to ξ. This can be proved by using the Codazzi equation. Indeed, observe first that for every X, Y, Z tangent to N and orthogonal to ξ:

$$\alpha(\nabla_X Y, Z) = \{g(\nabla_X Y, Z) - \eta(\nabla_X Y)\eta(Z)\}H = g(\nabla_X Y, Z)H$$

hence the Codazzi equation reads

$$g(Y, Z)D_X H - g(X, Z)D_Y H = (\bar{R}(X, Y)Z)^{\perp}$$

where $\bar{R}$ is the curvature tensor of M. Using the explicit expression of $\bar{R}$ (cf. [2, Theorem 7.19]), we thus obtain

$$g(Y, Z)D_X H - g(X, Z)D_Y H = \frac{c-1}{4}\{g(Z, PY)FX - g(Z, PX)FY + 2g(X, PY)FZ\}.$$

Choosing now $Y = Z$ of length one and orthogonal to X yields

$$D_X H = \frac{3}{4}(c-1)g(X, PY)FY.$$

Hence, the claim follows in the case $c = 1$. If $c \neq 1$, assuming $\dim(N) > 3$ we can choose Y so that Y is also orthogonal to PX, and the same formula yields $D_X H = 0$. □

3 The K-contact Case

In this section, we consider submanifolds of K-contact metric manifolds, i.e. contact metric manifolds whose Reeb vector field ξ is Killing. This is equivalent to requiring that the tensor field h in (3) vanishes. This class contains in particular the class of Sasakian manifolds.

We shall discuss the following characterization of slant submanifolds purely in terms of curvature (cf. [11]).

Theorem 3.1 *Let N be a submanifold of a K-contact metric manifold M with structure (φ, ξ, η, g). Assume that N is tangent to ξ. Fix $\theta \in [0, \frac{\pi}{2}]$; then the following conditions are equivalent:*

(a) N is slant with slant angle θ;
(b) For every $x \in N$ the sectional curvature, with respect to the induced metric, of every 2-plane containing ξ_x equals $\cos^2\theta$.

Moreover, every non-anti-invariant slant submanifold of M is itself a K-contact metric manifold with respect to the induced contact metric structure.

Proof For every $x \in N$, any 2-plane containing ξ_x is spanned by ξ_x and some unit vector X orthogonal to ξ; the corresponding sectional curvature $K(X, \xi)$ is related to the sectional curvature $\bar{K}(X, \xi)$ of the same 2-plane computed in the ambient manifold M by the Gauss equation:

$$K(X, \xi) = \bar{K}(X, \xi) + g(\alpha(X, X), \alpha(\xi_x, \xi_x)) - ||\alpha(X, \xi_x)||^2.$$

Now, M being a K-contact metric manifold, it is known that $\bar{K}(X, \xi) = 1$; moreover, we also have $\alpha(\xi, \xi) = 0$, because $\bar{\nabla}_\xi \xi = 0$. Hence, the above formula can be rewritten as

$$K(X, \xi) = 1 - ||\alpha(X, \xi_x)||^2.$$

On the other hand, remembering (3), in this case we have

$$\alpha(X, \xi_x) = -FX.$$

In conclusion:

$$K(X, \xi) = 1 - ||FX||^2.$$

Now the equivalence of (a) and (b) is clear taking into account the characterization of slant submanifolds provided by Theorem 1.2. Finally, concerning the last claim, observe that (3) also yields

$$\nabla_X \xi = -PX$$

for every vector field tangent to N, which implies that the restriction of ξ to N is again a Killing vector field, since P is skew-symmetric (alternatively, one can infer that the flow of ξ on N consists of local isometries). Hence, assuming that N is slant, the same is true for the Reeb vector field $\bar{\xi}$ of the induced contact metric structure, which is thus K-contact. □

Corollary 3.2 *Any torus admits no slant, non-anti-invariant, isometric immersions into any K-contact metric manifold.*

This is due to the fact that a torus cannot carry a K-contact metric structure [14]. Next, we consider regular K-contact manifolds. Recall that a contact manifold (M, η) is called regular provided the Reeb vector field is, i.e. it determines a regular 1-dimensional foliation on M, so that the space $B = M/\xi$ of maximal integral curves of ξ is a manifold. When M carries a K-contact metric g associated with η, then being $\mathcal{L}_\xi \varphi = \mathcal{L}_\xi g = 0$, g induces in a natural way a metric g' on M/ξ and φ also descends to an almost complex structure J.

Denoting by $\pi : M \to B$ the canonical projection, it turns out by construction that π is a Riemannian submersion with $Ker(d\pi)_x = \mathbb{R}\xi_x$ for every $x \in N$, and

$$d\pi \circ \varphi = J \circ d\pi$$

(see also [13]). Moreover, it is proved by Ogiue in [12] that (B, J, g') is an almost Kähler manifold. If M is Sasakian, then B is Kähler.

A remarkable case is when $M = M(c)$ is a simply connected, complete Sasakian space form; then B is either a flat Euclidean space $\mathbb{C}^m$ (when $c = -3$), a complex hyperbolic space $\mathbb{C}H_m$ with negative constant holomorphic sectional curvature ($c < -3$), or a complex projective space $\mathbb{C}P_m$ with positive constant holomorphic sectional curvature ($c > -3$) (see [2] or [9] for details).

The following result relates slant submanifolds of M with slant submanifolds of B. In particular, it provides a natural way to produce examples of slant submanifolds of the Sasakian space forms, by "lifting up" slant submanifolds of the corresponding complex space form. Another approach for constructing examples in this context has been developed by Cabrerizo, Carriazo, L. M. Fernandez and M. Fernandez in [4], who established a general existence and uniqueness result for slant immersions in Sasakian space forms, along the lines of the corresponding result of Chen-Vrancken for complex space forms.

Theorem 3.3 *Let M be a regular K-contact manifold canonically fibering onto the almost Kähler manifold B, with projection $\pi : M \to B$. Fix $\theta \in [0, \frac{\pi}{2})$. Then*

(a) If S is an embedded θ-slant submanifold of B, then $\pi^{-1}(B)$ is a θ-slant submanifold of M.
(b) If N is a compact θ-slant submanifold of M, then $\pi(N)$ is a θ-slant submanifold of B.

Proof (a) Since π is a surjective submersion, it is known that $N = \pi^{-1}(S)$ is an embedded submanifold of M, having the same codimension as S. Clearly, N is tangent to ξ, because at each point $x \in N$ the tangent space T_xN is $(d\pi)_x^{-1}(T_{\pi(x)}S)$. Moreover, observe that for every normal vector $\nu \in T_xN^{\perp}$, we have that $(d\pi)_x(\nu)$ is normal to S, because for every $X \in T_xN$ orthogonal to ξ, from $g(\nu, X) = 0$ it follows $g'((d\pi)_x\nu, (d\pi)_xX) = 0$, π being a Riemannian submersion.

Now, let X be a unit vector tangent to T_xN and orthogonal to ξ. Then from

$$\varphi X = PX + FX$$

we get

$$J(d\pi)_xX = (d\pi)_x(PX) + (d\pi)_xFX$$

where $(d\pi)_x(PX)$ is tangent and $(d\pi)_xFX$ is orthogonal to S, yielding

$$||PX|| = ||(d\pi)_x(PX)|| = \cos\theta$$

where the last equality holds because S is θ-slant.

(b) Since N is tangent to ξ, and π is a submersion satisfying $Ker(d\pi_x) = \mathbb{R}\xi_x$, for every $x \in M$, we have that the restriction of $d\pi$ to T_xN has rank $\dim(N) - 1$. Hence, $\pi : N \to B$ is a smooth map of constant rank; N being compact, it is known that its image $\pi(N)$ is a submanifold of B (cf. [1, Theorem 3.5.18]). The verification that $S = \pi(N)$ is θ-slant is based on the same argument used in the proof of (a), taking into account that at each point $\pi(x)$ of S we have $T_{\pi(x)}S = (d\pi)_x(T_x N)$. $\square$

Observe that, under the assumption of (b), one deduces that N is also a regular contact manifold. This provides a generalization of a result by Harada [10] concerning invariant submanifolds of regular Sasakian manifolds.

Corollary 3.4 *For every $m \geq 2$, the Sasakian space form $\mathbb{R}^{2m+1}$ of φ-sectional curvature -3 admits no compact proper slant submanifold.*

This holds since $\mathbb{R}^{2m+1}$ fibers onto the flat complex Euclidean space $\mathbb{C}^m$, and Chen-Tazawa's non-compactness result for slant submanifolds of $\mathbb{C}^m$ applies.

References

1. Abraham, R., Marsden, J.E., Ratiu, T.: Manifolds, Tensor Analysis, and Applications. Applied Mathematical Sciences, vol. 75, 3rd edn. Springer, New York (2001)
2. Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds. Progress in Mathematics, vol. 203, 2nd edn. Birkhäuser, Boston (2010)
3. Cabrerizo, J.L., Carriazo, A., Fernández, L.M., Fernández, M.: Slant submanifolds in Sasakian manifolds. Glasgow Math. J. **42**, 125–138 (2000)
4. Cabrerizo, J.L., Carriazo, A., Fernández, L.M., Fernández, M.: Existence and uniqueness theorem for slant immersions in Sasakian space forms. Publ. Math. Debrecen **58**, 559–574 (2001)
5. Chinea, D., Gonzalez, C.: A classification of almost contact metric manifolds. Ann. Mat. Pura Appl. **156**(4), 15–36 (1990)
6. de Candia, S., Falcitelli, M.: Slant immersions in C_5-manifolds. Bull. Math. Soc. Sci. Math. Roum. **60**(108), 239–255 (2017)
7. de Candia, S., Falcitelli, M.: Even-dimensional slant submanifolds of a $C_5 \oplus C_{12}$-manifold. Mediterr. J. Math. **14**, 224 (2017)
8. Gupta, R.S.: Non-existence of contact totally umbilical proper slant submanifolds of a Kenmotsu manifold. Rend. Sem. Mat. Univ. Politec. Torino **69**(1), 51–55 (2011)
9. Jiménez, J.A., Kowalski, O.: The classification of φ-symmetric Sasakian manifolds. Monatsh. Math. **115**, 83–98 (1993)
10. Harada, M.: On Sasakian submanifolds. Tohoku Math. J. **25**, 103–109 (1973)
11. Lotta, A.: Slant submanifolds in contact geometry. Bull. Math. Soc. Sci. Math. Roum. Nouv. Sér. **39**(1–4), 183–198 (1996)
12. Ogiue, K.: On fiberings of almost contact manifolds. Kodai Math. Semin. Rep. **17**(1), 53–62 (1965)
13. Reckziegel, H.: A correspondence between horizontal submanifolds of Sasakian manifolds and totally real submanifolds of Kählerian manifolds. Colloq. Math. Soc. János Bolyai **46**, 1063–1081 (1984). (Topics in Differential Geometry, Debrecen (Hungary))
14. Rukimbira, P.: Some remarks on R-contact flows. Ann. Global Anal. Geom. **11**, 165–171
15. Yano, K., Kon, M.: Structures on Manifolds. Series in Pure Mathematics, vol. 3. World Scientific, Singapore (1984)

Curvature Inequalities for Slant Submanifolds in Pointwise Kenmotsu Space Forms

Gabriel-Eduard Vîlcu

2010 Mathematics Subject Classification: 53C15, 53C25, 53C40, 53C80

1 Introduction

In 1969, Tano [104] proved that the automorphism group of a connected almost contact Riemannian manifold M of dimension $(2n + 1)$ is of maximum dimension $(n + 1)^2$, and the maximum is attained only in the case when M reduces to one of the following spaces: a homogeneous Sasaki manifold (or an ε-deformation of one) with constant ϕ-holomorphic sectional curvature, a global Riemannian product of a line or a circle with a complex space form, and a warped product of the complex space with the real line. In 1972, Kenmotsu [55] investigated the properties of this warped product and characterized it by tensor equations, giving rise to one of the newest chapters of contact geometry, nowadays called Kenmotsu geometry. Although neglected for a long time, these manifolds have attracted the attention of a large number of geometers in the last three decades, proving to be a valuable chapter of the contact geometry (see [89] and the references therein, as well as the recent articles [1, 16, 26, 41, 44, 47, 82, 92, 105, 110, 111, 114]).

The aim of this work is to present a survey on the geometry of Kenmotsu submanifolds, focusing on the curvature properties of slant submanifolds in pointwise Kenmotsu space forms. The present paper is organized as follows.

G.-E. Vîlcu (✉)
University Politehnica of Bucharest, Faculty of Applied Sciences, Department of Mathematics and Informatics, Splaiul Independenţei 313, Bucharest 060042, Romania

Petroleum-Gas University of Ploieşti, Department of Cybernetics, Economic Informatics, Finance and Accountancy, Bd. Bucureşti 39, Ploieşti 100680, Romania
e-mail: gvilcu@upg-ploiesti.ro

B.-Y. Chen et al. (eds.), *Contact Geometry of Slant Submanifolds*,
https://doi.org/10.1007/978-981-16-0017-3_2

In Sect. 2, one recalls some basic facts concerning the geometry of manifolds equipped with almost contact structures and their submanifolds. Section 3 is devoted to the presentation of the definition and some basic properties of Kenmotsu spaces. The aim of Sect. 4 is to overview the main classes of submanifolds investigated in Kenmotsu geometry.

Recall now that one of the basic problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of submanifolds. In a seminal paper published in 1993, Chen [30] proved some sharp inequalities involving such invariants of a Riemannian submanifold. The notions and techniques developed in this article have turned out to be very useful, giving rise to one of the most important research topics in submanifolds geometry: theory of Chen's invariants and inequalities. Later, B. Y. Chen's inequalities have been extensively studied by many authors for different kinds of submanifolds in several ambient spaces. In Sect. 5, we present some Chen-like inequalities for slant submanifolds in a Kenmotsu space form.

Section 6 is devoted to the investigation of the δ-Casorati curvatures of slant submanifolds in Kenmotsu space forms. It is well known that the notion of Casorati curvature has been defined in the geometry of submanifolds as the normalized square of the length of the second fundamental form of the submanifold [39]. Obviously, the Casorati curvature is an extrinsic invariant. This concept, originally introduced by Casorati in 1890 for surfaces in a Euclidean 3-dimensional space [27], was preferred by the author over the classical Gaussian curvature because it seems to correspond better with the common intuition of curvature [106]. Notice that recently, Brubaker and Suceavă [21] obtained sufficient conditions for a smooth hypersurface in Euclidean ambient space to be convex, in terms of Casorati curvatures and mean curvature. On the other hand, Kowalczyk [60] gave a geometrical interpretation of the Casorati curvature of a Riemannian submanifold, as well as a characterization of normally flat submanifolds in Euclidean spaces in terms of a relation between the Casorati curvatures and the normal curvatures of the submanifold. The first basic inequalities involving the Casorati curvatures were proved for submanifolds in real space forms by Decu, Haesen and Verstraelen [38, 39]. Later, these inequalities were generalized to other classes of submanifolds and ambient spaces [2, 6, 8, 10, 15, 40, 46, 53, 54, 62, 65, 69, 70, 76, 96–98, 102, 112, 115, 118–120].

In Sect. 7, we discuss the generalized Wintgen inequality, also referred to in the literature as the DDVV inequality or the DDVV conjecture. This famous inequality has been conjectured in [43] and solved in affirmative in [45, 74]. The aim of this section is to provide the counterpart of this inequality in Kenmotsu geometry. We point out that the study of Wintgen-like inequalities was recently started in a more general setting, namely for submanifolds in statistical manifolds [12, 13, 20, 81]. Notice that Bansal, Uddin and Shahid [16] proved the DDVV inequality for statistical submanifolds of Kenmotsu statistical manifolds of constant ϕ-sectional curvature.

2 Preliminaries

Let $(\bar{M}^m, \bar{g})$ be a Riemannian manifold of dimension m and suppose (M^n, g) is a Riemannian submanifold of $\bar{M}^m$ having dimension n. In the following, we will denote by $K(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_pM,\ p \in M$. If $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_m\}$ are orthonormal bases of T_pM and $T_p^{\perp}M$ (respectively), then it is known that the scalar curvature is defined by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j)$$

and the normalized scalar curvature is given by

$$\rho = \frac{2\tau}{n(n-1)}.$$

Let us denote by $\bar{\nabla}$ the Levi-Civita connection of the metric $\bar{g}$ and suppose ∇ denotes the covariant differentiation induced on M. Then the Gauss-Weingarten formulae are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM)$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^{\perp})$$

where h denotes the second fundamental form of the submanifold M, $\nabla^{\perp}$ is the metric connection in the normal bundle and A_N denotes the shape operator of the submanifold M with respect to the normal vector field N.

Let us recall that a point p of the submanifold M is called totally geodesic if h vanishes at p. Moreover, the submanifold M is said to be totally geodesic if all points of M are totally geodesic points.

On the other hand, we recall Gauss' equation that relates the curvature tensor fields $\bar{R}$ and R of the connections $\bar{\nabla}$ and ∇, respectively:

$$\begin{aligned} \bar{R}(X, Y, Z, W) = {} & R(X, Y, Z, W) + \bar{g}(h(X, W), h(Y, Z)) \\ & -\bar{g}(h(X, Z), h(Y, W)), \end{aligned} \tag{1}$$

for all vector fields X, Y, Z, W on M.

Let H be the mean curvature vector of the submanifold M defined by

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i).$$

Then the squared mean curvature $\|H\|^2$ is given by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2,$$

where

$$h_{ij}^{\alpha} = g(h(e_i, e_j), e_{\alpha}),\ i, j \in \{1, \dots, n\},\ \alpha \in \{n+1, \dots, m\}.$$

On the other hand, the squared norm of the second fundamental form h over the dimension of the submanifold M is an extrinsic geometric invariant known as the Casorati curvature of M. Therefore, this invariant is defined by

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \left(h_{ij}^{\alpha}\right)^2.$$

One can also define the Casorati operator of M as a $(1, 1)$-tensor field given by [51]

$$A^C = \sum_{\alpha=n+1}^{m} A_{e_\alpha}^2,$$

where A_{e_α} denotes the shape operator of the submanifold M with respect to e_α, $\alpha = n+1, \dots, m$. It is easy to see that $\mathcal{C}$ and A^C are linked by

$$\mathcal{C} = \frac{1}{n} \mathrm{Tr} A^C.$$

If $L \subset T_pM$ is a subspace of dimension s, with $s \geq 2$, and $\{e_1, \dots, e_s\}$ is an orthonormal basis of L, then the scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq s} K(e_\alpha \wedge e_\beta).$$

3 Kenmotsu Manifolds

An almost contact metric manifold is a quintuple $(\bar{M}, \phi, \xi, \eta, \bar{g})$ consisting in a Riemannian manifold $(\bar{M}, \bar{g})$ of dimension $(2n+1)$ equipped with a $(1, 1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying the compatibility conditions [19]

$$\begin{aligned} &\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = \eta \circ \phi = 0 \\ &\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y) \\ &\eta(X) = \bar{g}(X, \xi) \end{aligned} \tag{2}$$

for all $X, Y \in \Gamma(T\bar{M})$. Moreover, if the Levi-Civita connection $\bar{\nabla}$ of the metric $\bar{g}$ satisfies

$$\left(\bar{\nabla}_X\phi\right)(Y) = \bar{g}(\phi X, Y)\xi - \eta(Y)\phi X, \quad \bar{\nabla}_X\xi = X - \eta(X)\xi \tag{3}$$

then $(\bar{M}, \phi, \xi, \eta, \bar{g})$ is said to be a Kenmotsu manifold [89].

As remarkable examples of Kenmotsu manifolds, we have the following.

i. The hyperbolic space $\mathbb{H}^{2n+1} = \{(x_1, \ldots, x_{2n+1}) \in \mathbb{R}^{2n+1} | x_1 > 0\}$ equipped with the almost contact structure (ϕ, ξ, η, g) constructed by Chinea and Gonzales in [34] is a Kenmotsu manifold.
ii. The product manifold of a Kähler manifold with a Kenmotsu manifold can be equipped with a Kenmotsu structure [108]. In particular, it follows that the product manifolds $P^n\mathbb{C} \times \mathbb{H}^{2n+1}$, $\mathbb{D}^n \times \mathbb{H}^{2n+1}$ and $\mathbb{C}^n \times \mathbb{H}^{2n+1}$ can be endowed with Kenmotsu structures.
iii. A special class of orientable hypersurfaces of Kähler manifolds admits natural Kenmotsu structures. These submanifolds are called natural Kenmotsu hypersurfaces. For details, see [89, Example 2.1.18].

Recall that a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ with dimension $2n + 1 \geq 5$ is called a pointwise Kenmotsu space form, if the ϕ-sectional curvature of any ϕ-holomorphic plane $\{X, \phi X\}$, where $X \in T_p\bar{M}$, depends only on the point $p \in \bar{M}$, being independently on the ϕ-holomorphic plane at p. A connected Kenmotsu pointwise space form whose ϕ-sectional curvature does not depend on the point is said to be a *Kenmotsu space form*. It is known that a Kenmotsu manifold has constant ϕ-sectional curvature c at a point if and only if the curvature tensor $\bar{R}$ is given by [89]

$$\begin{aligned}\bar{R}(X, Y)Z = \frac{c-3}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + \eta(Y)\bar{g}(X, Z)\xi - \eta(X)\bar{g}(Y, Z)\xi \\ - \bar{g}(\phi X, Z)\phi Y + \bar{g}(\phi Y, Z)\phi X + 2\bar{g}(X, \phi Y)\phi Z\}.\end{aligned} \tag{4}$$

The (pointwise) Kenmotsu space forms are denoted by $\bar{M}(c)$. It is important to point out that any Kenmotsu space form $\bar{M}(c)$ has constant sectional curvature equal to $c = -1$ (see [55, Theorem 13]). As an example of Kenmotsu space form, we have the warped product $\bar{M} = \mathbb{R} \times_f \mathbb{C}^n$, where $f(t) = \exp t$. For details, see [63, Example 1].

Regarding the topology of Kenmotsu manifolds, we recall the following important result, that is, a consequence of the Green theorem and of the fact that $div\xi = 2n$.

Theorem 3.1 ([55]) *Any Kenmotsu manifold is non-compact.*

The local characterization of Kenmotsu manifolds is the following.

Theorem 3.2 ([55]) *Let $(\bar{M}, \phi, \xi, \eta, \bar{g})$ be a Kenmotsu manifold. Then any point of $\bar{M}$ has a neighborhood isometric to the warped product $I \times_f V$, where $I = (-\epsilon, \epsilon)$ is an open interval of $\mathbb{R}$, V is a Kählerian manifold and $f(t) = c \exp t$, $c > 0$.*

For other general properties concerning the geometry and topology of Kenmotsu manifolds, see [18, 42, 57, 88, 103] and [89, Chap. 2].

4 Submanifolds of Kenmotsu Manifolds

4.1 Invariant and Anti-Invariant Submanifolds

If (M, g) is a Riemannian submanifold of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$, then we have the following decomposition for any vector field X tangent to M:

$$\phi X = PX + FX, \tag{5}$$

where PX and FX represent the tangential and the normal components of ϕX, respectively. If the dimension of the submanifold M is $(m + 1)$, then one can define

$$||P||^2 = \sum_{i,j=1}^{m+1} g(e_i, Pe_j)^2, \tag{6}$$

where $\{e_1, e_2, \ldots, e_{m+1}\}$ is a local orthonormal frame of M. Notice that the squared norm of the endomorphism P of TM defined above does not depend on the chosen orthonormal frame. We also point out that F is a normal bundle-valued 1-form on the tangent bundle TM.

On the other hand, for any vector field V normal to M, we have the following decomposition:

$$\phi V = tV + fV, \tag{7}$$

where tV and fV denote the tangential and the normal component of ϕV, respectively.

A Riemannian submanifold M of a Kenmotsu manifold $\bar{M}$ is said to be an invariant submanifold [58] if $F \equiv 0$. On the other hand, if $P \equiv 0$, then the submanifold M is called anti-invariant [93]. These classes of submanifolds were investigated from two perspectives, accordingly as the structure vector field ξ is tangent or normal to the submanifold M. We recall next some results from [58].

Theorem 4.1 ([58]) *Let $(\bar{M}, \phi, \xi, \eta, \bar{g})$ be a Kenmotsu manifold and N be a submanifold of $\bar{M}$ tangent to ξ. Then*

i. N is an invariant submanifold iff t is parallel.
ii. N is an anti-invariant submanifold iff P is parallel.
iii. N is a totally geodesic submanifold iff the second fundamental form h of the submanifold is parallel.

Theorem 4.2 ([58]) *Let $\bar{M}(c)$ be a (pointwise) Kenmotsu space form, and N be a submanifold of $\bar{M}(c)$ tangent to the structure vector field of $\bar{M}(c)$. Suppose that t and f are parallel. Then N is also a (pointwise) Kenmotsu space form.*

Several curvature properties and geometric inequalities involving intrinsic and extrinsic invariants of invariant and anti-invariant submanifolds of (pointwise) Kenmotsu space forms tangent and normal to the structure vector field ξ can be found in [11, 56, 84, 107].

4.2 Contact Semi-Invariant and Normal Semi-Invariant Submanifolds

In 1983, Papaghiuc [86] investigated the concept of semi-invariant submanifold in Kenmotsu geometry, introducing two kinds of such submanifolds: contact semi-invariant submanifolds and normal semi-invariant submanifolds. A submanifold N of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ is called contact semi-invariant if its tangent bundle splits orthogonally into smooth distributions $\mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\xi\rangle$ such that ϕ maps $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) into itself (resp. into the normal bundle of N). Notice that $\mathcal{D}$ is usually called the invariant distribution of the submanifold N, while $\mathcal{D}^{\perp}$ is said to be the anti-invariant distribution of the submanifold N.

In [86], the author investigated the integrability of certain subbundles of the tangent bundle of a contact semi-invariant submanifold of a Kenmotsu manifold, proving the following result.

Theorem 4.3 ([86]) *Let $(\bar{M}, \phi, \xi, \eta, \bar{g})$ be a Kenmotsu manifold and (N, g) be a contact semi-invariant submanifold of $\bar{M}$. Then*

i. *$\mathcal{D}^{\perp}$, $\mathcal{D}^{\perp} \oplus \langle\xi\rangle$ and $\mathcal{D} \oplus \mathcal{D}^{\perp}$ are integrable distributions.*
ii. *$\mathcal{D}$ and $\mathcal{D} \oplus \langle\xi\rangle$ are integrable distributions iff the second fundamental form h satisfies*

$$h(X, \phi Y) = h(\phi X, Y), \ \forall X, Y \in \Gamma(D).$$

Moreover, Sinha and Srivastava [99] derived two criteria for a submanifold of a Kenmotsu space form to be contact semi-invariant, while Papaghiuc [86] obtained some natural conditions that imply the constancy of sectional curvature for a contact semi-invariant submanifold of a pointwise Kenmotsu space form.

The concept of normal semi-invariant submanifold of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ was defined in [86] by considering that the structure vector field ξ is normal to the submanifold and imposing also the condition that the tangent bundle splits orthogonally as $\mathcal{D} \oplus \mathcal{D}^{\perp}$, where $\mathcal{D}$ is an invariant distribution and $\mathcal{D}^{\perp}$ is an anti-invariant distribution. In particular if $\mathcal{D} = 0$ (resp. $\mathcal{D}^{\perp} = 0$), then the submanifold is said to be normal anti-invariant (resp. normal invariant). Papaghiuc [86] investigated the integrability of both distributions involved in the definition of a normal semi-invariant submanifold, proving the following result.