Semiparallel Submanifolds in Space Forms

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Introduction

Among Riemannian manifolds, the most interesting and most important for applications are the symmetric ones. From the local point of view, they were introduced independently by P. A. Shirokov [Shi 25] and H. Levy [Le 25] as Riemannian manifolds with covariantly constant (also called parallel) curvature tensor field R, i.e., with

$$\nabla R = 0, \tag{0.1}$$

where ∇ is the Levi-Civita connection [L-C 17]. An extensive theory of symmetric Riemannian manifolds was worked out by É. Cartan in [Ca 26]. He showed that a Riemannian manifold M has parallel R if and only if every point x has a normal neirhbourhood such that all geodesic symmetries with respect to x are isometries.

If for each point $x \in M$ there exists an involutive isometry s_x of M for which x is an isolated fixed point, then M is called a (globally) symmetric space. The closure of the group of isometries generated by $\{s_x : x \in M\}$ in the compact-open topology is a Lie group G that acts transitively on the symmetric space; hence the typical isotropy subgroup H at a point of M is compact, and $M = G/H$.

The classical examples are connected complete Riemannian manifolds with constant sectional curvature c, called space forms (see [Wo 72], Section 2.4).

Later, a similar development took place in the geometry of submanifolds in space forms, where a fundamental role is played by the first (or metric) form g (as the induced Riemannian metric) and the second fundamental form h. Besides the Levi-Civita connection ∇, with $\nabla g = 0$, a normal connection $\nabla^{\perp}$ is also defined. The submanifolds with parallel fundamental form, i.e., with

$$\bar{\nabla} h = 0, \tag{0.2}$$

where $\bar{\nabla}$ is the pair of ∇ and $\nabla^{\perp}$, deserve special attention. Due to the Gauss identity, each of them is intrinsically a locally symmetric Riemannian manifold.

The first result here was given by V. F. Kagan [Ka 48], who showed that in Euclidean space E^3, the surfaces with parallel h are open subsets of planes, round spheres, and circular cylinders $S^1 \times E^1$. All of these have nonnegative Gaussian

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curvature. The surfaces of negative constant Gaussian curvature in E^3 are therefore examples of submanifolds which are intrinsically locally symmetric, but have nonparallel h.

The hypersurfaces with parallel h in E^n were determined by U. Simon and A. Weinstein [SW 69]. Some new examples of surfaces with parallel h in E^4 were given by C.-S. Houh [Ho 72]: the Clifford tori $S^1 \times S^1$ and the Veronese surfaces. The general theory of submanifolds M^m with parallel h in E^n was initiated by J. Vilms [Vi 72], who showed, in particular, that each of them has totally geodesic Gauss image. Normally flat submanifolds with parallel h in Euclidean spaces and spheres were classified by R. Walden [Wa 73].

A properly developed theory was worked out by D. Ferus [Fe 74, 80]. He proved that a submanifold M^m with parallel h in E^n has the property of local extrinsic symmetry, in the sense that every point has a neighborhood invariant under reflection of E^n with respect to the normal subspace at this point; also conversely, an M^m with this property has parallel h. This was proved in general, for M^m in a Riemannian manifold N^n, by W. Strübing [St 79]. Therefore, the submanifolds with parallel h, especially the complete ones, were called symmetric submanifolds by Ferus (and then by others); here extrinsically was meant, but often not explicitly stated. The other important result of Ferus was that a general symmetric submanifold in E^n reduces to a product of irreducible symmetric submanifolds, each of which (except possibly a Euclidean subspace) lies in a sphere, is minimal in it, and can be obtained as the standard immersion of a Riemannian symmetric R-space. Conversely, each such standard immersion gives a symmetric submanifold; and the products of these immersions (possibly including a Euclidean subspace) exhaust all symmetric submanifolds in E^n. These results gave a classification of such submanifolds in terms of special chapters of the theory of Lie groups and symmetric spaces. All of these submanifolds can be considered as symmetric orbits.

This classification was then extended to submanifolds with parallel h in space forms by M. Takeuchi [Ta 81], who found it more suitable here to use the term parallel submanifolds. This term has become more popular, especially when the local point of view has been considered.

The theory of parallel submanifolds is concisely treated in recent monographic works by B.-Y. Chen [Ch 2000] (Chapter 8), Ü. Lumiste [Lu 2000] (Sections 5–7), and by J. Berndt, S. Console, and C. Olmos [BCO 2003] (Section 3.7: "Symmetric submanifolds").

Already in the first investigations of symmetric Riemannian manifolds [Shi 25] and [Ca 26], it was noted that these manifolds must also satisfy the integrability condition

$$R(X, Y) \cdot R = 0 \tag{0.3}$$

of the differential system $\nabla R = 0$. (Here X and Y are tangent vector fields, and $R(X, Y)$ is considered as a field of linear operators, acting on R.) Riemannian manifolds with this point-wise condition were considered separately by É. Cartan in [Ca 46]. His investigations were continued by A. Lichnerowicz [Li 52, 58] and R. Couty [Co 57]. The term semisymmetric for Riemannian manifolds M satisfying this condition was introduced by N. S. Sinyukov [Si 56, 62], who showed the importance of this condition in the theory of geodesic mappings of Riemannian manifolds (see [Si 79], Chapter 2, Section 3).

A fruitful impulse for investigations of manifolds of this class was given by K. Nomizu in [No 68], who conjectured that all complete irreducible n-dimensional Riemannian manifolds ($n \geq 3$) satisfying $R(X, Y) \cdot R = 0$ are locally symmetric, i.e., that they must also satisfy $\nabla R = 0$. This conjecture was supported by the result that for a Riemannian manifold, $\nabla^k R = 0$ with $k > 1$ implies $\nabla R = 0$, proved for the compact case in [Li 58], and for the complete case in [NO 62]; and this is also valid in general (cf. [KN 63], Vol. 1, Remark 7). However, Nomizu's conjecture was eventually refuted. Namely, in [Ta 72] a hypersurface in E^4 was constructed satisfying $R(X, Y) \cdot R = 0$ but not $\nabla R = 0$. A counterexample of arbitrary dimension was given in [Sek 72].

Semisymmetric Riemannian manifolds were classified by Z. I. Szabó, locally, in [Sza 82]. He showed that for every semisymmetric Riemannian manifold M, there exists an everywhere dense open subset U of M, such that around every point of U, the manifold is locally isometric to a space that is the direct product of an open subset of a Euclidean space and of infinitesimally irreducible simple semisymmetric leaves, each of which is either (i) locally symmetric, or (ii) locally isometric to an elliptic, a hyperbolic, a Euclidean, or a Kählerian cone, or (iii) locally isometric to a space foliated by Euclidean leaves of codimension 2 (or to a two-dimensional manifold, in the case $\dim M = 2$).

These classification results of Szabó were presented briefly in the book [BKV 96], whose main purpose was to summarize recent results on semisymmetric Riemannian manifolds of subclass (iii); these are now called Riemannian manifolds of conullity two, and may be considered the most interesting among semisymmetric Riemannian manifolds.

Parallel submanifolds were likewise later placed in a more general class of submanifolds, generalizing the parallel ones in the same sense as locally symmetric Riemannian manifolds (i.e., with $\nabla R = 0$) were generalized by semisymmetric Riemannian manifolds (i.e., with $R(X.Y) \cdot R = 0$). Namely, the integrability condition of the differential system $\bar{\nabla} h = 0$ is

$$\bar{R}(X, Y) \cdot h = 0, \tag{0.4}$$

where $\bar{R}$ is the curvature operator of the connection $\bar{\nabla} = \nabla \oplus \nabla^{\perp}$, and X, Y are tangent vector fields, as above. This condition in fact already came up in [Fe 74a] and then in [BR 83]. The general concept of submanifolds in E^n satisfying (0.4) was introduced by J. Deprez [De 85], who called them *semiparallel*. He proved that all of them are, intrinsically, semisymmetric Riemannian manifolds and gave a classification of semiparallel surfaces in E^n. In [De 86], he also classified semiparallel hypersurfaces, and in [De 89], summarized these first results.

The investigation of semiparallel submanifolds was continued by the author in [Lu 87a, 88a, b, 89a–c, 90a–e], etc., then by F. Dillen in [Di 90b, 91b], [DN 93], and A. C. Asperti in [As 93], [AM 94]. The first summaries were published in [Lu 91f] and then in the monographic article [Lu 2000a] (whose review in *Mathematical*

Reviews (see [MR 2000j: 53071]) is concluded by A. Bucki as follows: “The author’s contribution to the theory of submanifolds with parallel fundamental form with his more than forty papers on the subject is colossal”). Currently the monograph [Lu 2000a] is no longer completely up to date; several new results have been added to the theory since then.

The present book will give a more complete survey of the theory of semiparallel submanifolds and of some generalizations in space forms. Semiparallel submanifolds are treated here mainly as second-order envelopes of symmetric orbits.

The book consists of twelve chapters. The first three chapters are preparatory in character. In Chapter 1, the necessary background for subsequent chapters is given using frame bundles (i.e., the Cartan moving frame method) and exterior differential calculus, together with vector and tensor bundles. Basic facts from the theories of space forms and of symmetric and semisymmetric Riemannian manifolds are covered.

In Chapter 2, the general theory of smooth submanifolds in space forms is developed. The second fundamental form h is introduced, together with its higher-order generalizations, their fundamental identities, and the corresponding normal and osculating subspaces are covered. This is done by using orthonormal frames suitably adapted to the submanifold.

In Chapter 3, the theory of parallel submanifolds is developed. Here the specifics of their Gauss maps, their local extrinsic symmetry, Ferus’s decomposition theorem and its connection with symmetric R-spaces are presented. The most important examples of complete parallel submanifolds are also given: Segre, Plücker, and Veronese submanifolds.

All of this is in preparation for the main subject, which is the investigation of semiparallel submanifolds. These are introduced in Chapter 4, where some characterizations for their class and several subclasses are given. It is emphasized that (0.4) is a pointwise condition and therefore can be treated purely algebraically. The decomposition theorem for semiparallel submanifolds is also dealt with in the same manner. The analytic fact, that these submanifolds are characterized by the integrability condition of the differential system (0.2) for parallel submanifolds, is interpreted geometrically in the theorem from [Lu 90a], stating that every semiparallel submanifold is a second-order envelope of parallel submanifolds; such envelopes are found for Segre submanifolds, as examples (extending the result of [Lu 91a]).

Chapter 5 is devoted to normally flat semiparallel submanifolds. This class includes all semiparallel submanifolds of principal codimension 1, in particular hypersurfaces, and also semiparallel submanifolds of principal codimension 2 in space forms of nonpositive curvature. A general geometric description is given for normally flat semiparallel submanifolds as immersed warped products of spheres.

Semiparallel submanifolds of low dimensions are considered in Chapters 6 (surfaces) and 7 (three-dimensional submanifolds). They are all classified; the submanifolds of the most general class are described as second-order envelopes of Veronese submanifolds. It is shown that each two-dimensional holomorphic Riemannian manifold can be immersed isometrically into (pseudo-)Euclidean space of dimension ≥ 7, as a surface of this most general class of semiparallel surfaces; but this does not generalize to three dimensions. Some general classes of semiparallel three-dimensional submanifolds are investigated, consisting of second-order envelopes of three-dimensional Segre submanifolds (logarithmic spiral tubes) and of products of Veronese surfaces and plane curves of constant curvature.

In Chapter 8, the decomposition theorems are given: for general parallel and semiparallel submanifolds, for normally flat 2-parallel submanifolds, and for submanifolds with flat van der Waerden–Bortolotti connection. Here the concept of main symmetric orbit is introduced; this is a standardly imbedded symmetric R-space and is minimal in some sphere. The most general semisymmetric submanifold in Euclidean space is locally the second-order envelope of products of main symmetric orbits, some circles and a plane. This is a consequence of the result of [Lu 90a] and of Ferus's famous results [Fe 80].

In Chapter 9, the concept of umbilic-like main symmetric orbit is introduced and studied. A main symmetric orbit is said to be umbilic-like if every second-order envelope of submanifolds congruent or similar to this orbit is a single such orbit; a sphere is an elementary example. Here all known results about umbilic-like main symmetric orbits are presented; the Segre orbits were already investigated from this point of view in Section 4.6 (see Theorem 4.6.1). For the second-order envelope of the family of these main orbits, a differential system is formulated, and then investigated by Cartan's method of differential prolongation. This investigation for Plücker orbits, showing their umbilic-likeness, is carried out in detail. For the other symmetric orbits of the Plücker action, the unitary orbits, this investigation is technically very complicated; only the general scheme is given here and illustrated completely for a model case. For the m-dimensional Veronese orbit, it is shown that in Euclidean space $E^{\frac{1}{2}m(m+3)+1}$, this orbit is not umbilic-like. For the other symmetric orbit of the Veronese action, the Veronese–Grassmann orbit, its umbilic-likeness is asserted, but space and technical complications preclude giving all the details. The general scheme of proof is given, some essential intermediate results are obtained, and the complete proof is illustrated for a model case.

In Chapter 10, it is proved first that a product of umbilic-like main symmetric orbits in Euclidean space is also umbilic-like. This result gives the possibility of extending the description of normally flat semiparallel submanifolds as warped products of spheres to general semiparallel submanifolds, i.e., considering them also as warped products.

Chapter 11 is devoted to semiparallel immersions of semisymmetric Riemannian manifolds, and seeks answers to the problem: can such a manifold be immersed isometrically as a semiparallel submanifold? The answer is positive for dimension $m = 2$, as already shown in Chapter 6. The problem is investigated for dimensions $m > 2$. First, it is proved that if an m-dimensional semiparallel submanifold in E^n is generated by $(m-2)$-dimensional planes, then it is intrinsically a Riemannian manifold of conullity two of the planar type; the other types (i.e., hyperbolic, parabolic, or elliptic type) are not possible. Also, for normally flat semiparallel submanifolds M^m in E^n, it is shown that if such a submanifold is intrinsically of conullity two, then it is of planar type. The same holds for all semiparallel three-dimensional submanifolds. Therefore, it can be conjectured that perhaps this assertion is true in general. The chapter concludes with a theorem that makes this conjecture very plausible. At least, it is certain that there exist semisymmetric Riemannian manifolds, namely of conullity two, that cannot be immersed into Euclidean space as semiparallel submanifolds.

In Chapter 12, some generalizations are considered. First, the k-semiparallel submanifolds for $k > 1$ are introduced and studied. Their relation to envelopes of order k of some family of k-parallel submanifolds is investigated. It is proved that there exist 2-semiparallel submanifolds that are nontrivial, i.e., not parallel and not locally Euclidean; namely, every normally flat semiparallel submanifold (see Chapter 5) turns out to be 2-semiparallel. However, the study of k-semiparallel (in particular 2-semiparallel) submanifolds is still in its initial phase; a complete classification is given only for 2-semiparallel surfaces in space forms (see Section 12.3).

Two other generalizations, namely, the recurrent and the (recently introduced) pseudoparallel submanifolds are discussed briefly in Section 12.4.

Some generalizations have been made by extending the semiparallel condition from the second fundamental form h to some tensor fields (including mixed fields) that are derived from h and the metric form g by some tensor calculus operations. Results of V. Mirzoyan are presented, where the idea of enveloping by corresponding parallel submanifolds is used. These results are illustrated with examples involving surfaces with parallel or semiparallel mean curvature vector field, or normal curvature tensor field, or Ricci tensor field, etc. Hypersurfaces with semiparallel Ricci tensor field are studied in particular, mainly in connection with the famous Ryan's problem: do there exist any hypersurfaces M^m in E^{m+1} with semiparallel Ricci tensor field, that are not intrinsically semisymmetric Riemannian manifolds? It is shown that Mirzoyan's classification result in [Mi 99] covers all known results about this problem, including an affirmative answer for dimension $m \geq 5$ (see [Lu 2002b]). Some special results about the extended Ryan's problem for normally flat submanifolds are also given. The book concludes with a proof that, among the submanifolds of codimension 2 in E^n, there exist normally flat submanifolds that are intrinsically semisymmetric but not semiparallel. This gives additional support to the conjecture stated above (cf. Chapter 11).

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Preliminaries

1.1 Real Spaces with Bilinear Metric

Let A be a point set and G a group with identity element e. A map $A \times G \to A$, $(x, g) \mapsto x \circ g$ with $x \circ e = x$, $(x \circ g_1) \circ g_2 = x \circ (g_1 g_2)$ defines a (right) *action* of the group G on A, also called a (right) *G-action* on A. One also says that G acts on A as a *transformation group*. The action is *effective* if $e \in G$ is the only element of G with the property: $x \circ g = x$ for arbitrary $x \in A$; *transitive*, if for every two $x_1, x_2 \in A$ there exists an element $g \in G$, so that $x_2 = x_1 \circ g$; *simply transitive*, if this g is unique for every $(x_1, x_2) \in A \times A$.

The set $H_x = \{h \in G \mid x \circ h = x\}$ is a subgroup of G, called the *isotropy subgroup* of $x \in A$. Obviously $H_{x \circ g} = g^{-1} H_x g$.

The set $G_x = \{y \in A \mid \exists g, y = x \circ g\}$ is called the *orbit* of x under the G-action on A. The orbits are equivalence classes: $x_1 \sim x_2 \Leftrightarrow \exists g \in G, x_2 = x_1 \circ g$. They form the factor set of this equivalence, called the *orbit set*. A G-action on A induces a transitive G-action on every orbit. Obviously, transitivity of the G-action means that there is only one orbit.

Let G be the additive group of vectors of an n-dimensional real vector space V^n and let there be given an effective transitive and simply transitive action of this G on A. This means that

1. if $x \in A$, $v \in V^n$, there exists $y = x \circ v \in A^n$ (one also denotes this by $v = \vec{xy}$),
2. $(x \circ v) \circ w = x \circ (v + w) = x \circ (w + v) = (x \circ w) \circ v$ (i.e., if $\vec{xy} = \vec{wz}$, then $\vec{xw} = \vec{yz}$),
3. for $x, y \in A$ there exists a unique $v \in V^n$ so that $v = \vec{xy}$.

Then A is called a real n-dimensional *affine space*, denoted by A^n, and V^n is said to be *the vector space of A^n*.

Let T be an m-dimensional vector subspace of V^n. The above action of V^n on A^n induces an action of T on A^n. Every orbit of the latter action is called an m-dimensional *affine subspace* of A^n, or briefly, an *m-plane* in A^n. Intrinsically it is an m-dimensional affine space, and T is called the *direction vector subspace* of this m-plane.

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Let $V^n \times V^n \to \mathbb{R}$, $(v_1, v_2) \mapsto \langle v_1, v_2\rangle$ be a nondegenerate bilinear form, called a *bilinear metric* or, equivalently, a *scalar product*. Two vectors v_1 and v_2 are said to be *orthogonal*, if $\langle v_1, v_2\rangle = 0$. This is denoted by $v_1 \perp v_2$.

If T is an m-dimensional vector subspace of V^n and the scalar product induces a nondegenerate bilinear form on it, then T is called a *regular* subspace, otherwise a *singular* subspace. For a regular subspace T the set $T^\perp = \{v \mid v \perp w$ for every $w \in T\}$ is also a regular subspace, called the *orthogonal complement* of T; here $V^n = T \oplus T^\perp$, thus $T^\perp$ is $(n - m)$-dimensional.

A real n-dimensional affine space A^n whose vector space V^n is equipped with a scalar product as above is called a *space with bilinear metric*. If the scalar product is symmetric, i.e., $\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$ for arbitrary $(v_1, v_2) \in V^n \times V^n$, the space is called *(pseudo-)Euclidean space* ${}_sE^n$; here s is the number of negative coefficients in the canonical representation of the quadratic form $\langle v, v\rangle$. In particular, if this form is positive definite, then the space is *Euclidean space* $E^n (= {}_0E^n)$, otherwise *pseudo-Euclidean space* ${}_sE^n$, $s > 0$ (as is seen, in the latter case without the round brackets around "pseudo-").[1] In particular, ${}_1E^n$ is *Lorentz space*, for $n = 4$ also called *Minkowski space*, the spacetime of the special relativity theory.

An m-plane in ${}_sE^n$ is said to be *regular* if its direction vector subspace is regular, otherwise it is said to be *singular*, in particular *isotropic*, if the scalar product vanishes identically.

In pseudo-Euclidean space ${}_sE^n$ a regular m-plane can be Euclidean or pseudo-Euclidean. In relativity theory, especially the case $s = 1$, such m-planes are also called, correspondingly, *spacelike* or *timelike*, and a singular m-plane is called *lightlike*.

In general, if the bilinear form is not nondegenerate but is symmetric, then A^n is called a *semi-Euclidean space*; for instance, every singular m-plane in pseudo-Euclidean space ${}_sE^n$ is an example of such a semi-Euclidean space.

1.2 Moving Frames

Let A^n be a real affine space with vector space V^n. Let $\varepsilon^0 = (e_1^0, \ldots, e_n^0)$ be a basis of V^n and o a point of A^n. The pair (o, ε^0) is called a *frame* of A^n with *origin* o and *basis vectors* e_I^0, $I \in \{1, \ldots, n\}$. Every basis ε^0 determines an isomorphism $V^n \to \mathbb{R}^n$, $v \mapsto (v^1, \ldots, v^n)$ with $v = v^I e_I^0$. (Henceforth the Einstein summation convention is used, that is, the right-hand side actually means that $\sum_{I=1}^n v^I e_I^0 = v^1 e_1^0 + \cdots + v^n e_n^0$.) Every frame determines a homeomorphism $A^n \to \mathbb{R}^n$, $x \mapsto (x^1, \ldots, x^n)$ with $\vec{ox} = x^I e_I^0$.

Considering the set of all frames (x, ε) of A^n one defines a (right) action of the general linear group $GL(n, \mathbb{R})$ (i.e., the multiplicative group of all real nonsingular $n \times n$-matrices $A = (A_I^J)$) on this set by $(x, \varepsilon) \circ A = (x, \varepsilon A)$; here εA is the product

[1] Note that some authors use slightly different terminology, e.g., in [Ra 53] (pseudo-)Euclidean spaces are called Euclidean, Euclidean spaces are called properly Euclidean; in [KN 63, 69] pseudo-Euclidean spaces are called indefinite Euclidean.

of $1 \times n$- and $n \times n$- matrices ε and A, i.e., $(\varepsilon A)_I = e_J A^J_I$. This introduces on this set a principal bundle structure with base A^n and structural group $GL(n, \mathbb{R})$ (see [KN 63], Chapter I, Section 5). Every fibre (i.e., orbit of the action) is the set of all frames having the same origin x. This principal bundle is called the *frame bundle* of A^n and realizes the idea of a *moving frame* of É. Cartan (an arbitrary element of this bundle is considered here as a moving frame in A^n; see [IL 2003]).

With respect to a fixed frame ε^0, every moving frame ε in A^n is determined by the coordinates x^I of its origin x, according to $\vec{ox} = x^I e^0_I$, and by the elements X^J_I of the matrix in $e_I = e^0_J X^J_I$, where $I, J, \dots \in \{1, \dots, n\}$.

Note that the differential $d(\vec{ox})$ does not depend on the choice of the origin o, because $\vec{\acute{o}x}$ and $\vec{ox}$ differ only by the constant vector $\vec{o\acute{o}}$. Therefore, $d(\vec{ox})$ can be denoted simply by dx. Also, the point $x \in A^n$ can be identified with its radius vector $\vec{ox}$ from the fixed origin o.

One can calculate

$$dx = e_I \omega^I, \quad de_I = e_J \omega^J_I, \tag{1.2.1}$$

where

$$\omega^I = (X^{-1})^I_J dx^J, \quad \omega^J_I = (X^{-1})^K_I dX^J_K, \quad (X^{-1})^K_J X^J_I = \delta^K_I. \tag{1.2.2}$$

The formulas (1.2.1) are called the *infinitesimal displacement equations* of the moving frame. The differential 1-forms (1.2.2), called the *infinitesimal displacement* 1*-forms*, satisfy the equations

$$d\omega^I = \omega^J \wedge \omega^I_J, \quad d\omega^J_I = \omega^K_I \wedge \omega^J_K, \tag{1.2.3}$$

which are obtained by exterior differentation from (1.2.1) (see [Ste 64], Chapter III, Section 1; [IL 2003], Section B.2) and thus are necessary and sufficient conditions for the complete integrability of (1.2.1). Here (1.2.3) are called the *structure equations* of A^n.

In a real space with bilinear metric one can introduce for every frame the matrix $g = (g_{IJ})$, where $g_{IJ} = \langle e_I, e_J \rangle$. By differentiation, one obtains the relation

$$dg_{IJ} = g_{KJ} \omega^K_I + g_{IK} \omega^K_J. \tag{1.2.4}$$

If the space is ${}_sE^n$, the frame bundle can be reduced to the principal bundle of *orthonormal frames*, characterized by $g_{IJ} = \epsilon_I \delta_{IJ}$, where ϵ_I is -1 for s values of I and 1 for the remaining $n - s$ values of I, and δ_{IJ} is the Kronecker delta. The structural group of the above bundle is the pseudo-orthogonal group ${}_sO(n, \mathbb{R})$; in the cases $s = 0$, $s = 1$, and $s = n-1$, respectively, this is the orthogonal group $O(n, \mathbb{R})$, and the Lorentz groups ${}_1O(n, \mathbb{R})$ and ${}_{n-1}O(n, \mathbb{R})$ (the last two are isomorphic; see [Wo 72], Section 2.4).

For the bundle of orthonormal frames, the relation (1.2.4) reduces to

$$\varepsilon_J \omega^J_I + \varepsilon_I \omega^I_J = 0 \qquad \text{(no sum!)}. \tag{1.2.5}$$

In the case of E^n, i.e., when $s = 0$, the matrix ω^J_I is skew-symmetric and gives an arbitrary element of the Lie algebra of the orthogonal group $O(n, \mathbb{R})$. For ${}_1E^n$, when $s = 1$, one obtains the same for the Lorentz group ${}_1O(n, \mathbb{R})$.

1.3 (Pseudo-)Riemannian Manifolds

The (pseudo-)Euclidean space ${}_sE^n$ is a special case of the more general concept of a (pseudo-)Riemannian manifold ${}_sN^n$. This is a real n-dimensional differentiable manifold with a smooth field g of symmetric scalar products in the tangent vector spaces. Here the constant natural number s has the same meaning as in ${}_sE^n$. For a local section

$$(x, \varepsilon) = (x; e_1, \dots, e_n)$$

of the frame bundle on ${}_sN^n$ and two tangent vector fields, $X = e_I X^I$ and $Y = e_J Y^J$, one has $\langle X, Y\rangle = g_{IJ} X^I Y^J$, where $g_{IJ} = \langle e_I, e_J\rangle$ are the components of the metric tensor field on ${}_sN^n$, denoted also by g. In the particular case when $\langle X, X\rangle$ is positive definite, the (pseudo-)Riemannian manifold $N^n (= {}_0N^n)$ is called a *Riemannian manifold*, otherwise, a *pseudo-Riemannian manifold* (cf. footnote 1).

A linear connection ∇ on a (pseudo-)Riemannian manifold (see, e.g., [KN 63], Chapter III), which has the property that g is covariantly constant with respect to ∇, i.e., $\nabla g = 0$, is called a (*pseudo-*)*Riemannian* (in particular *Riemannian*, or *pseudo-Riemannian*) *connection*. Componentwise, the last condition is

$$\nabla g_{IJ} \equiv dg_{IJ} - g_{KJ}\omega_I^K - g_{IK}\omega_J^K = 0, \tag{1.3.1}$$

where $\omega = (\omega_I^J)$ is the matrix field of connection 1-forms of ∇. It is well known that every (pseudo-)Riemannian manifold has a unique (pseudo-)Riemannian connection ∇ without torsion, called the *Levi-Civita connection*.[2]

The elements $\{\omega^I\}$ of the coframe bundle on ${}_sN^n$ and the connection 1-forms ω_I^J of the Levi-Civita connection satisfy the structure equations

$$d\omega^I = \omega^J \wedge \omega_J^I, \quad d\omega_I^J = \omega_I^K \wedge \omega_K^J + \Omega_I^J, \tag{1.3.2}$$

where

$$\Omega_I^J = -R_{I,KL}^J \omega^K \wedge \omega^L \tag{1.3.3}$$

are the *curvature* 2-*forms* of the Levi-Civita connection ∇. Here the coefficients $R_{I,KL}^J$ are the components of the *curvature tensor field* R of ∇. By exterior differentiation of (1.3.1), one obtains via (1.3.2) the equality

$$\Omega_{IJ} + \Omega_{JI} = 0, \tag{1.3.4}$$

where $\Omega_{IJ} = g_{IP}\Omega_J^P = -R_{IJ,KL}\omega^K \wedge \omega^L$ and so $R_{IJ,KL} = g_{IP}R_{J,KL}^P$. Then exterior differentiation of (1.3.2) yields the relations

$$\omega^J \wedge \Omega_J^I = 0, \quad d\Omega_I^J = \Omega_K^J \wedge \omega_I^K - \omega_K^J \wedge \Omega_I^K, \tag{1.3.5}$$

[2] In some recent books, e.g., [Pe 98], historical terminology is disregarded and the Levi-Civita connection is called simply the (pseudo-)Riemannian connection. Sometimes these two terms are considered as equivalent, and then the (pseudo-)Riemannian connection as defined above is called the *metric connection* (see, e.g., [KN 63], Chapter IV; also [Li 55], Section 52; [He 62], Chapter I, Section 9).

which are equivalent to the identities

$$R^I_{J,KL} + R^I_{K,LJ} + R^I_{L,JK} = 0, \quad \nabla_P R^I_{J,KL} + \nabla_K R^I_{J,LP} + \nabla_L R^I_{J,PK} = 0 \quad (1.3.6)$$

(the *Bianchi identities*), where $(\nabla_P R^I_{J,KL})\omega^P = \nabla R^I_{J,KL}$ and

$$\nabla R^I_{J,KL} = dR^I_{J,KL} - R^I_{P,KL}\omega^P_J - R^I_{J,PL}\omega^P_K - R^I_{J,KP}\omega^P_L + R^P_{J,KL}\omega^I_P. \quad (1.3.7)$$

The first identities (1.3.6) and the consequences $R_{IJ,KL} + R_{JI,KL} = 0$ from (1.3.4) imply

$$R_{IJ,KL} = R_{KL,IJ}. \quad (1.3.8)$$

A (pseudo-)Riemannian manifold ${}_sN^n$ of dimension $n > 2$ is said to be a manifold of *constant curvature* if its curvature forms can be represented as $\Omega^J_I = cg_{IK}\omega^J \wedge \omega^K$. Then from (1.3.5) it follows that $dc \wedge \omega^J \wedge \omega^K = 0$, and since $dc = c_I\omega^I$ this gives $c_I\omega^I \wedge \omega^J \wedge \omega^K = 0$. Due to the supposition $n > 2$, for every value of I there exist values of J and K such that $\omega^I \wedge \omega^J \wedge \omega^K \neq 0$. Therefore, $c_I = 0$, and thus $c = \text{const}$. This constant c is called the *curvature* of such a ${}_sN^n$ (cf. [Wo 72], 2.2.7).

The structure equations for a Riemannian manifold of constant curvature c are, due to (1.3.2),

$$d\omega^I = \omega^J \wedge \omega^I_J, \quad d\omega^J_I = \omega^K_I \wedge \omega^J_K + cg_{IK}\omega^J \wedge \omega^K. \quad (1.3.9)$$

1.4 Standard Models of Space and Spacetime Forms

The space ${}_sE^n$ is the simplest n-dimensional (pseudo-)Riemannian manifold of zero curvature.

A connected complete Riemannian manifold of constant curvature c is called a *space form* (see [Wo 72], Section 2.4). Their standard models, denoted by $N^n(c)$, are as follows:

- for $c = 0$, the Euclidean space E^n,
- for $c > 0$,
$$S^n(c) = \{x \in E^{n+1} \mid \langle \vec{ox}, \vec{ox} \rangle = r^2\},$$
which is the sphere with a real radius $r = 1/\sqrt{c}$ and with center at the origin o,
- for $c < 0$, a connected component of
$$H^n(c) = \{x \in {}_1E^{n+1} \mid \langle \vec{ox}, \vec{ox} \rangle = -r^2\},$$
which is the sphere in Lorentz space ${}_1E^{n+1}$ with imaginary radius $r = i/\sqrt{|c|}$ and with center at the origin o.

Note that $H^n(c)$ consists of two connected components, each of which is a *hyperbolic* (or *Lobachevsky–Bolyai*) space.

The Minkowski space ${}_1E^4$ (the special case of Lorentz space for $n+1=4$), which is the spacetime of the special relativity theory, is a simple case of pseudo-Riemannian space ${}_sN^n$ of constant curvature c, namely, the case of $s=1$, $n=4$, $c=0$.

In general a connected complete pseudo-Riemannian space ${}_sN^n$ of constant curvature c is called a *spacetime form* and is denoted by ${}_sN^n(c)$. The standard models are ${}_sE^n$ and the connected components of ${}_sS^n(c)$ and ${}_sH^n(c)$, where the latter two are defined similarly to $S^n(c)$ and $H^n(c)$, with E^{n+1} and ${}_1E^{n+1}$ replaced, respectively, with ${}_sE^{n+1}$ and ${}_{s+1}E^{n+1}$ (see [Wo 72], Section 2.4).

Here the special cases are *de Sitter spacetime* ${}_sS^n(c)$ and *anti-de Sitter spacetime* ${}_sH^n(c)$, which for $n=4$ and $s=1$ (resp. $s=2$) are the simplest nonflat spacetime models for general relativity theory (see [HE 73], [PR 86]).

The moving frame bundle of ${}_\sigma E^{n+1}$, where σ is s or $s+1$, can be adapted to a standard (pseudo-)Riemannian model ${}_sN^n(c)$ as follows.

For every frame it is supposed that

(1) $x \in {}_sN^n(c)$, i.e., $\langle \vec{ox}, \vec{ox}\rangle = c^{-1} = \text{const}$,
(2) $e_{n+1} \parallel \vec{ox}$, i.e., $e_{n+1} = -\sqrt{|c|}\vec{ox}$ and therefore $g_{n+1,n+1} = \langle e_{n+1}, e_{n+1}\rangle = |c|c^{-1} = \operatorname{sign} c$,
(3) $e_1, \dots, e_n$ are orthogonal to e_{n+1}, therefore tangent to ${}_sN(c)$, so that $g_{I,n+1} = 0$ $(I = 1, \dots, n)$.

Differentiation of the equality in (1) gives $\langle dx, e_{n+1}\rangle = 0$; hence $\omega^{n+1} = 0$. Similarly from the equalities in (2) and (3) one obtains

$$\omega_{n+1}^{n+1} = 0, \quad \omega_{n+1}^{I} = -\sqrt{|c|}\omega^I, \quad \omega_I^{n+1} = \operatorname{sign} c\sqrt{|c|}g_{IK}\omega^K, \tag{1.4.1}$$

where $I, J, \dots$ are in $\{1, \dots, n\}$ and the last relation holds due to (1.2.4) and the equality in (3).

For such a frame bundle adapted to ${}_sN^n(c)$, the relations (1.2.1) and (1.2.3) imply (writing them for dimension $n+1$ and using (1.4.1)) that

$$dx = e_I\omega^I, \quad de_I = e_J\omega_I^J - xcg_{IK}\omega^K, \tag{1.4.2}$$

$$d\omega^I = \omega^J \wedge \omega_J^I, \quad d\omega_I^J = \omega_I^K \wedge \omega_K^J + cg_{IK}\omega^J \wedge \omega^K, \tag{1.4.3}$$

where now $I, J, \dots \in \{1, \dots, n\}$ and (1.2.4) hold. Recall that the radius vector $\vec{ox}$ from the center o of the sphere ${}_sS^n(c)$ (resp. ${}_sH^n(c)$) is being denoted here simply by x, and so $\langle dx, dx\rangle = g_{IJ}\omega^I\omega^J$.

For the (pseudo-)Euclidean space ${}_sE^n \subset {}_sE^{n+1}$ one must take $e_{n+1} = \text{const}$. This leads to the particular case of the formulas (1.4.2) and (1.4.3), obtained by $c=0$ (and thus to (1.2.1) and (1.2.3)). So the formulas above are universal for all standard models of space and spacetime forms.

Remark 1.4.1. The standard models of spacetime forms ${}_sN^n(c)$ can also be treated by means of projective geometry as follows.

Every such model lies in ${}_\sigma E^{n+1}$ with fixed origin at the center of the model ${}_sN^n(c)$; here $\sigma = s$ or $s+1$. There is a one-to-one correspondence between $\mathbb{R}^{n+1}$

and ${}_\sigma E^{n+1}$. The projectivization of $\mathbb{R}^{n+1}$ gives the real projective space $P^n(\mathbb{R})$ and then the asymptotic cone of ${}_sN^n(c)$ gives the absolute quadric ${}_sQ^{n-1} \subset P^n(\mathbb{R})$, which determines the projective metric of curvature c. Two vectors of ${}_\sigma E^{n+1}$ are orthogonal iff the corresponding points of $P^n(\mathbb{R})$ are polar with respect to ${}_sQ^{n-1}$. The q-dimensional totally geodesic submanifolds of ${}_sN^n(c)$ (the q-dimensional great spheres) can then be interpreted as projective q-planes of $P^n(\mathbb{R})$. This simplifies the understanding of the geometry of ${}_sN^n(c)$ and will be used often below. Note that a projective q-plane is the intersection of the model sphere ${}_sN^n(c)$ with a $(q+1)$-plane through the origin in ${}_\sigma E^{n+1}$.

1.5 Symmetric (Pseudo-)Riemannian Manifolds

A vector field $X = e_I X^I$on a (pseudo-)Riemannian manifold ${}_sN^n$ is said to be *parallel* along a curve in ${}_sN^n$, if $\nabla X = 0$ on this curve, where $\nabla X = e_I(\nabla X^I)$ and $\nabla X^I = dX^I + X^J \omega^I_J$. A curve in ${}_sN^n$ is a *geodesic* if its tangent vector field is parallel along the curve. It is well known that a geodesic with nonzero arclength s, defined by $ds^2 = g_{IJ}\omega^I\omega^J$, is locally a curve of stationary length between any two of its points.

A (pseudo-)Riemannian manifold ${}_sN^n$ is said to have *parallel curvature tensor field R* if $\nabla R = 0$ on ${}_sN^n$, or more explicitly, if $\nabla R^I_{J,KL} = 0$ (i.e., if (0.1) is satisfied), where the left side is defined by (1.3.7).

Let U_{x_0} be a normal neighborhood of a point $x_0 \in_s N^n$, i.e., every point $x \in U_{x_0}$ is connected to x_0 by only one geodesic of ${}_sN^n$ which lies in U_{x_0}. Suppose this curve to be nonisotropic (i.e., with nonzero arclength) and take on it the point x', which is at the same real or imaginary distance from x_0 as x, but on the other side, one gets the *geodesic symmetry map* with respect to x_0. A pseudo-Riemannian manifold ${}_sN^n$ is *locally symmetric* if each of its points x_0 has a normal neighborhood whose geodesic symmetry map with respect to x_0 is an isometry.

É. Cartan proved the following relationship between these properties (also in the more general case of affinely connected manifolds).

Theorem 1.5.1 **(**[Ca 26] and [He 62], Chapter IV, Section 1**).** *A (pseudo-)Riemannian manifold ${}_sN^n$ is locally symmetric if and only if its curvature tensor field R is parallel on ${}_sN^n$.*

A Riemannian manifold N^n is said to be globally symmetric if every point x_0 is an isolated fixed point of an involutive isometry s_{x_0} of N^n; here involutive means that $s^2_{x_0} = \mathrm{Id}$. It follows that x_0 has a normal neighborhood on which s_{x_0} is a geodesic symmetry map (see [He 62], Chapter IV, Section 3). Thus a globally symmetric N^n is also locally symmetric, and vice versa, every complete simply connected locally symmetric Riemannian manifold is globally symmetric (see [He 62], Chapter IV, Section 5). More generally, for every point x_0 of a locally symmetric Riemannian manifold N^n there exist a globally symmetric Riemannian manifold $\tilde{N}^n$, an open neighborhood U_{x_0} of x_0 in N^n, and an isometry φ mapping U_{x_0} onto an open neighborhood of the point $\varphi(x_0)$ in $\tilde{N}^n$.

The manifold $\tilde{N}^n$ is diffeomorphic to the homogeneous space G/K, where G is the identity component of the Lie group of isometries of $\tilde{N}^n$ and K is the compact subgroup of isometries with fixed point x_0; the diffeomorphism $G/K \to \tilde{N}^n$ is given by $gK \mapsto g \circ x_0$, $g \in G$ (see [He 62], Chapter IV, Section 3).

In turn, let G be a connected Lie group, K a closed subgroup with compact $\mathrm{Ad}_G(K)$, and γ an analytic automorphism of G such that $(K_\gamma)_0 \subset K \subset K_\gamma$, where K_γ is the set of fixed points of γ and $(K_\gamma)_0$ is its identity component. Then for every G-invariant Riemannian structure on G/K this G/K is a globally symmetric Riemannian manifold (see [He 62], Chapter IV, Section 3). In this case (G, K) is called a *Riemannian symmetric pair*.

These results reduce the study of globally symmetric Riemannian manifolds $\tilde{N}^n$ to the study of Riemannian symmetric pairs (G, K) by means of Lie group theory.

Remark 1.5.2. In general, symmetric pseudo-Riemannian manifolds have not been studied so thoroughly as the Riemannian ones. É. Cartan [Ca 26] had noted that these types of manifolds with solvable isometry group exist. The case of dimension 4 was then studied in [Wal 46] and [Wal 50] (see also [Ab 71]). In [Ro 49b], [Fed 56], [Fed 59] symmetric pseudo-Riemannian manifolds with simple groups of isometries were classified; in [Be 57] the case of semisimple groups was also included. The classification problem for four-dimensional symmetric Einsteinian spaces with Lorentzian signature and of the first type was solved by A.Z. Petrov [Pe 66]. In [CML 68], all symmetric four-dimensional spaces of signature ± 2 were listed. A complete classification of the spaces of signature 2 with solvable transvection group was given in [CP 70]; see also [Ast 73].

Example 1.5.3. Comparing the structure equations (1.3.2) and (1.4.3), one sees that for the standard models ${}_sN^n(c)$ of spacetime forms the curvature 2-forms are

$$\Omega_I^J = c g_{IK} \omega^J \wedge \omega^K, \quad c = \text{const}.$$

Thus $R^J_{I,KL} = -c g_{IK} \delta^J_L$. From (1.2.4) $\nabla g_{IK} = 0$; also $\nabla \delta^J_L = d\delta^J_L - \delta^J_P \omega^P_L + \delta^P_L \omega^J_P = 0$, and this leads to $\nabla R^J_{I,KL} = 0$. Hence, every ${}_sN^n(c)$ is a locally symmetric (pseudo-)Riemannian manifold; actually it is also globally symmetric (see [Wo 72], Chapter 11).

Example 1.5.4. The manifold of all q-dimensional vector subspaces of a p-dimensional real vector space $\mathbb{R}^p$ is called the *Grassmann manifold* and denoted by $G(q, \mathbb{R}^p)$ (see, e.g., [Sha 88], Chapter 1, Section 4). Let a pseudo-Euclidean metric of index k be given in $\mathbb{R}^p$ by the metric tensor $g_{\lambda\mu}$ and consider the manifold of all regular q-dimensional vector subspaces of index l. This manifold is called the *Grassmann manifold of regular subspaces* and is denoted by ${}_{l,k}G^{q,p}$.

For an element of ${}_{l,k}G^{q,p}$ considered as a subspace, the orthonormal basis $\{e_\lambda\}$ in $\mathbb{R}^p$ $(1 \le \lambda \le p)$ can be chosen so that e_a and e_u $(1 \le a \le q;\, q+1 \le u \le p)$ are vectors belonging to this subspace and to its orthogonal complement, respectively. Thus the subspace is determined by the simple q-vector $e_1 \wedge e_2 \wedge \cdots \wedge e_q$.

Let $\wedge^q(\mathbb{R}^p)$ be the space of antisymmetric $(q, 0)$-tensors (see [Ste 64], Chapter I, Section 4). This $\wedge^q(\mathbb{R}^p)$ is a vector space, for which the simple q-vectors $e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_q}$

with $\lambda_1 < \cdots < \lambda_q$ form a basis. From the infinitesimal displacement equations (1.2.1) it follows that

$$d(e_1 \wedge e_2 \wedge \cdots \wedge e_q) = \sum_{a,u} (e_1 \wedge \cdots \wedge e_{a-1} \wedge e_u \wedge e_{a+1} \wedge \cdots \wedge e_m)\omega_a^u, \quad (1.5.1)$$

because for an orthonormal basis $\omega_a^a = 0$ (no sum; see (1.2.5)), so that the 1-forms ω_a^u play the role of ω^I in the first formula of (1.2.1). Now the argument used in [Lu 92a], [Maa 74] can be applied. There it is shown that the pseudo-Riemannian structure on ${}_{l,k}G^{q,p}$ is given by

$$ds^2 = g^{ab} g_{uv} \omega_a^u \omega_b^v,$$

where $1 \le a, b \le q$; $q + 1 \le u, v \le p$ (see also [Ha 65]) and that it is Einstein of constant scalar curvature; for $k = l = 0$ this is established in [Le 61]. Thus ω_a^u generates a moving coframe on ${}_{l,k}G^{q,p}$ and for this the first structure equations (1.3.2) must hold. On the other hand,

$$d\omega_a^u = \omega_a^b \wedge \omega_b^u + \omega_a^v \wedge \omega_v^u = \omega_b^v \wedge (-\omega_a^b \delta_v^u + \delta_a^b \omega_v^u),$$

so that the role of ω_J^I in (1.3.2) is played by the 1-forms in the last parentheses above. Since $d\omega_J^I$ is now

$$d(-\omega_a^b \delta_v^u + \delta_a^b \omega_v^u) = -(\omega_a^c \wedge \omega_c^b + \omega_a^w \wedge \omega_w^b)\delta_v^u + \delta_a^b(\omega_v^c \wedge \omega_c^u + \omega_v^w \wedge \omega_w^u)$$

and $\omega_J^K \wedge \omega_K^I$ is

$$(-\omega_c^b \delta_v^w + \delta_c^b \omega_v^w) \wedge (-\omega_a^c \delta_w^u + \delta_a^c \omega_w^u) = -\omega_a^c \wedge \omega_c^b \delta_v^u + \omega_v^w \wedge \omega_w^u \delta_a^b,$$

the curvature 2-forms Ω_J^I in (1.3.2) for ${}_{l,k}G^{q,p}$ are

$$-\omega_a^w \wedge \omega_w^b \delta_v^u + \delta_a^b \omega_v^c \wedge \omega_c^u.$$

Now (1.2.5) implies that $\omega_w^b = -\varepsilon_b \varepsilon_w \omega_b^w$, so that these curvature 2-forms are

$$(\delta_a^d g^{bc} g_{wx} \delta_v^u - \delta_a^b g^{cd} g_{vw} \delta_x^u)\omega_d^w \wedge \omega_c^x,$$

where $g^{bc} = \varepsilon_b \delta^{bc}$, $g_{vw} = \varepsilon_v \delta_{vw}$, etc., and the indices w, x and c, d run through the same values as u, v and a, b, respectively. The reduced coefficients are the components of the curvature tensor $R^I_{J,KL}$ of ${}_{l,k}G^{q,p}$. This and the above expressions for ω_J^I imply that for the pseudo-Riemannian connection ∇^G of ${}_{l,k}G^{q,p}$ the equations $\nabla^G R^I_{J,KL} = 0$ hold (cf. (1.3.7)).

Consequently, the following statement holds.

Theorem 1.5.5. *The Grassmann manifold ${}_{l,k}G^{q,p}$ of regular subspaces is a locally symmetric pseudo-Riemannian manifold, which is Einstein of constant scalar curvature.*

In the particular case $q = 2$, this ${}_{l,k}G^{2,p}$ is called the *Plücker manifold*.

The same conclusion also holds for $k = 0$ (thus also $l = 0$); then "pseudo-" is to be omitted and the corresponding Grassmann manifold is denoted simply by $G^{q,p}$. A projective space treatment of most of these results for Grassmann manifolds with polar normalization can be found in [AG 96], Chapter 6, Section 6.6; see also [Ro 49a].

Grassmann manifolds are also globally symmetric, as shown in [Wo 72] (for the Riemannian case, see Section 9.2, where a corresponding Riemannian symmetric pair is used; for the pseudo-Riemannian case, cf. Section 12.2).

Remark 1.5.6. Some generalizations of symmetric Riemannian spaces have been made by Fedenko [Fed 77] and Kowalski [Kow 80]. In [KoK 87] Kowalski's approach is transferred to the geometry of submanifolds M^m in E^n; in [CMR 94] the same is for M^m in $N^n(c)$.

Another generalization is made by Deszcz in [Des 92] and for submanifolds in [ALM 99, 2002], [LT 2006] (see Section 12.4).

1.6 Semisymmetric (Pseudo-)Riemannian Manifolds

According to Theorem 1.5.1 the class of locally symmetric pseudo-Riemannian manifolds is analytically characterized by the system of differential equations $\nabla R = 0$ for the components of the curvature tensor field R (cf. with (0.1)). More explicitly, due to (1.3.7) this system is

$$dR^I_{J,KL} - R^I_{P,KL}\omega^P_J - R^I_{J,PL}\omega^P_K - R^I_{J,KP}\omega^P_L + R^P_{J,KL}\omega^I_P = 0. \qquad (1.6.1)$$

The integrability condition of this system can be obtained by exterior differentiation, using the structure equations (1.3.2), which leads to the equations

$$R^I_{P,KL}\Omega^P_J + R^I_{J,PL}\Omega^P_K + R^I_{J,KP}\Omega^P_L - R^P_{J,KL}\Omega^I_P = 0. \qquad (1.6.2)$$

Replacing Ω^P_J with the expressions $R^P_{J,QS}\omega^Q \wedge \omega^S$ given in (1.3.3), and collecting the terms before $\omega^Q \wedge \omega^S$, one obtains a system of purely algebraic (quadratic) equations for the components of R. Contracting the left sides of these equations with coordinates of two linearly independent tangent vectors $X = e_Q X^Q$, $Y = e_S Y^S$ and considering $R^P_{J,QS} X^Q Y^S = R^P_J(X, Y)$ as the entries of the matrix of a linear operator $R(X, Y)$ acting on R, this algebraic system can be written concisely as (cf. with (0.3))

$$R(X, Y) \cdot R = 0. \qquad (1.6.3)$$

The system (1.6.2), or the equivalent system (1.6.3), was already found as the integrability condition of (1.6.1) in the first investigations by P. A. Shirokov and É. Cartan about symmetric spaces (see [Shi 25], [Ca 26]). A natural generalization of these spaces was considered by É. Cartan, who in [Ca 46] introduced the Riemannian manifolds satisfying (1.6.3). His investigations were continued by A. Lichnerowicz [Li 52], [Li 58] and R. Couty [Co 57].

What follows is a short survey of the results about the Riemannian manifolds satisfying (1.6.3). More detailed information can be found in the monograph [BKV 96].

The term *semisymmetric* for manifolds satisfying the condition (1.6.3) was introduced by N. S. Sinyukov [Si 56, 62], who showed the importance of this condition in the theory of geodesic mappings of Riemannian manifolds (see [Si 79], Chapter 2, Section 3).

A fruitful impulse for investigations of manifolds of this class was given by K. Nomizu, who in [No 68] conjectured that all complete, irreducible n-dimensional Riemannian manifolds ($n \geq 3$) satisfying $R(X, Y) \cdot R = 0$ are locally symmetric, i.e., they also satisfy $\nabla R = 0$. This conjecture was supported by the result that for a Riemannian manifold $\nabla^k R = 0$ yields $\nabla R = 0$, which was proved for the compact case in [Li 58] and for the complete case in [NO 62] (and it is also valid in general; cf. [KN 63], Vol. 1, Remark 7). However, Nomizu's conjecture was eventually refuted. Namely, in [Ta 72] a hypersurface in E^4 was constructed satisfying $R(X, Y) \cdot R = 0$ but not $\nabla R = 0$; and a counterexample of arbitrary dimension was given in [Sek 72]. Nevertheless, by adding some further conditions to $R(X, Y) \cdot R = 0$, the conjecture becomes true; such additional conditions were given in [ST 70], [Tan 71], [Fu 72]. For instance, it is shown in these papers that it suffices to add $\nabla C = 0$, $S = \text{const}$, where C is the tensor of conformal curvature and S is the scalar curvature (cf. also [Sek 75], [Sek 77]).

For pseudo-Riemannian manifolds the term *semisymmetric* was used (for the case of Lorentzian signature) by V. R. Kaigorodov [Kai 78] in the course of investigations on the curvature structure of spacetime (cf. also [Kai 83]).

Let a (pseudo-)Riemannian space be a direct product of the same kind of spaces. Then the frame bundle can be adapted so that the basis vectors are successively tangent to the mutually orthogonal components of the product. Then $R^I_{J,KL}$ are zero if two of the indices I, J, K, L are indices of basis vectors tangent to different components. A straightforward calculation shows that if (1.6.2) is satisfied for every component, then it is also satisfied for the direct product. The same holds if (1.6.2) replaced by $\nabla R = 0$, i.e., by (1.6.1). Thus the direct product of semisymmetric (resp. symmetric) (pseudo-)Riemannian manifolds is a semisymmetric (resp. symmetric) (pseudo-)Riemannian manifold.

The local classification of semisymmetric Riemannian manifolds was given by Z. I. Szabó, locally in [Sza 82] and then globally in [Sza 85]. First he proved by means of the infinitesimal or the local holonomy group that for every semisymmetric Riemannian manifold M^m there exists a dense open subset U such that around the points of U the manifold M^m is locally isometric to a direct product of semisymmetric manifolds $M_0 \times M_1 \times \cdots \times M_r$, where M_0 is an open part of a Euclidean space and the manifolds M_i, $i > 0$, are infinitesimally irreducible simple semisymmetric leaves. Here a semisymmetric M is called a *simple leaf* if at each of its points x the primitive holonomy group determines a simple decomposition $T_x M = V_x^{(o)} + V_x^{(1)}$, where this group acts trivially on $V_x^{(0)}$ and there is only one subspace $V_x^{(1)}$ that is invariant for this group. A simple leaf is said to be infinitesimally irreducible if at least at one point the infinitesimal holonomy group acts irreducibly on $V_x^{(1)}$.

The dimension $\nu(x) = \dim V_x^{(0)}$ is called the *index of nullity* at x and $u(x) = \dim M - \nu(x)$ the *index of conullity* at x.

The classification theorem of Szabó asserts the following (according to the formulation given in [BKV 96]).

Theorem 1.6.1. *For every semisymmetric Riemannian manifold there exists an everywhere dense open subset U such that around every point of U the manifold is locally isometric to a space that is the direct product of an open part of a Euclidean space and of infinitesimally irreducible simple semisymmetric leaves, each of which is one of the following:*

(a) *if $\nu(x) = 0$ and $u(x) > 2$, then locally symmetric (hence locally isometric to a symmetric space);*
(b) *if $\nu(x) = 1$ and $u(x) > 2$, then locally isometric to an elliptic, a hyperbolic or a Euclidean cone;*
(c) *if $\nu(x) = 2$ and $u(x) > 2$, then locally isometric to a Kählerian cone;*
(d) *if $\nu(x) = \dim M - 2$ and $u(x) = 2$, then locally isometric to a space foliated by Euclidean leaves of codimension* 2 (*or to a two-dimensional manifold, this in the case when* $\dim M = 2$).

The following examples give more detailed descriptions (according to [Sza 82] and [BKV 96]) of these product components, some of them in the general (pseudo)-Riemannian situation.

Example 1.6.2 (*for case* (a)). Every symmetric (pseudo)-Riemannian space is also semisymmetric.

Indeed, the condition (1.6.1) yields (1.6.2), and thus (1.6.3) too, because they are the integrability conditions of (1.6.1).

Example 1.6.3 (*for case* (b)). Consider $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ with the standard coordinate system $(x^0, x^1, \dots, x^{n-1})$ and the Riemannian metric given by

$$ds^2 = (dx^0)^2 + (x^0 + C)^2[(dx^1)^2 + \cdots + (dx^{n-1})^2].$$

This Riemannian manifold is called the *Euclidean cone* and it is semisymmetric.

Example 1.6.4 (*for case* (b)). Let $S^{n-1}(c)$ be a sphere with center o in E^n and v a point in E^{n+1} such that the straight line ov is orthogonal to $E^n \subset E^{n+1}$. The *elliptic cone* is a hypersurface in E^{n+1} described by the straight half-lines emanating from v (the vertex) and intersecting the points of $S^{n-1}(c)$. With its induced metric, this hypersurface is intrinsically an n-dimensional Riemannian manifold that turns out to be semisymmetric. This induced metric has an expression similar to the metric in the previous example: in the square brackets one takes here the standard metric of $S^{n-1}(c)$.

Example 1.6.5 (*for case* (b)). Let the expression in square brackets be replaced by the standard metric of the $(n-1)$-dimensional hyperbolic space. Then the Riemannian manifold with this metric is called the *hyperbolic cone*.

This space can also be represented in the following way. Let $H^{n-1}(c)$ be a sphere with center o and imaginary radius $i/\sqrt{|c|}$ in ${}_1E^n \subset {}_1E^{n+1}$. Its connected component is intrinsically a hyperbolic space (see Section 1.3). Choose a point v in ${}_1E^{n+1}$ so that the straight line ov is orthogonal to ${}_1E^n$, and consider the hypersurface in ${}_1E^{n+1}$ defined by the straight half-lines emanating from v and intersecting the points of a connected component of $H^{n-1}(c)$. This hypersurface is in fact a hyperbolic cone.

Example 1.6.6 (*for case* (c)). The Kählerian cones are the complex analogs of Examples 1.6.4, 1.6.5 and 1.6.6 (see [BKV 96]).

Example 1.6.7 (*for case* (d)). Every two-dimensional (pseudo-)Riemannian manifold is semisymmetric.

Indeed, if $n = 2$, then (1.3.4) and (1.3.8) imply that $R_{IJ,KL}$ is nonzero only if $(IJ, KL) = (12, 12)$ or a permutation of (12, 12). This easily yields (1.6.3).

Example 1.6.8 (*case* (d) *in general*). In the Szabó classification a special role is played by the n-dimensional Riemannian manifolds foliated by $(n-2)$-dimensional Euclidean spaces. They are characterized as those manifolds, whose tangent vector spaces are orthogonal products $V_x^{(0)} + V_x^{(1)}$, where $V_x^{(1)}$ are of dimension 2 at every point, and $V_x^{(0)}$ define a foliation of dimension $(n-2)$ with Euclidean spaces as leaves.

These foliated manifolds were treated implicitly in [Sza 82], without considering any explicit expressions for their metrics. They were considered as solutions of a certain integrable system of nonlinear partial differential equations. A more detailed analysis was given by Kowalski in [Kow 96] for the three-dimensional case ($n = 3$); see also [BKV 96].

In [BKV 96] (cf. the remark concluding Chapter 8) the situation was described as follows. The general solution of the basic system of partial differential equations given by Szabó depends formally on $\frac{1}{2}(n-2)(n+3)+4$ arbitrary functions of two variables and $\frac{1}{2}(n-2)(n+3)$ arbitrary functions of one variable. In dimension 3, this means seven functions of two variables and three functions of one variable. But a more detailed analysis has shown that in fact three arbitrary functions of two variables suffice to parametrise the corresponding spaces. The exact number of arbitrary functions of two variables that parametrise local isometry classes of foliated semisymmetric manifolds in dimension n remained an unsolved problem in [BKV 96]. It was noted only that the first explicit examples depending on one arbitrary function of two variables were constructed in [KoTV 90] and [KoTV 92]. The new approach, given by O. Kowalski in dimension 3, was then generalized by E. Boeckx [Bo 95] to arbitrary dimension n. These results are summarized in [BKV 96], where these manifolds are called Riemannian manifolds *of conullity two*, motivated by case (d).

Remark 1.6.9. For four-dimensional semisymmetric Riemannian manifolds an elementary classification can be given independently from Szabó's (which is indirect and relies on some essential results from other sources, for instance, a theorem of Kostant). This elementary classification is given in [Lu 96e], [Lu 96f]. The result is as follows.

Theorem 1.6.10. *Locally, every four-dimensional semisymmetric Riemannian manifold is one of the following:*

(a) *a locally Euclidean manifold,*
(b) *a space of nonzero constant curvature,*
(c) *a locally symmetric space other than* (a) *and* (b)*,*
(d) *the direct product of two two-dimensional spaces,*
(e) *locally isometric to an elliptic or hyperbolic cylinder* (*i.e., direct product* $S^3 \times \mathbb{R}$ *or* $H^3 \times \mathbb{R}$) *or to a Euclidean, elliptic or hyperbolic cone,*
(f) *space foliated by two-dimensional totally geodesic and locally Euclidean leaves that are transversally flat along themselves and normally flat in sections normal to them.*

The proof is given in [Lu 96f] and is based on an elementary algebraic classification of semisymmetric curvature operators [Lu 96e]. Chern bases are used to minimize the number of nonzero components of the curvature tensor. This considerably simplifies the semisymmetric condition as a system of quadratic equations on these components.

Note that in the pseudo-Riemannian case the problem of detailed classification of semisymmetric manifolds is currently still open, to the author's knowledge.

Remark 1.6.11. In [Kow 96] (and then in [BKV 96]) the following terminology is used for semisymmetric Riemannian manifolds of types (a)–(d): the manifolds of type (a) are said to be of "trivial" class, types (b) and (c) of "exceptional" class, and of type (d) "typical" class.

For three-dimensional manifolds of this last class, O. Kowalski introduced (in a preprint of 1991 and published afterwards in [Kow 96]) the geometric concept of *asymptotic foliation*, which was generalized by E. Boeckx [Bo 95] to arbitrary dimensions.

An $(m-1)$-dimensional submanifold M^{m-1} of a manifold M^m of conullity two is called an *asymptotic leaf* if it is generated by $(m-2)$-dimensional Euclidean leaves of M^m and if its tangent spaces are parallel along each Euclidean leaf with respect to the Levi-Civita connection ∇ of M^m.

An *asymptotic distribution* on M^m is an $(m-1)$-dimensional distribution that is integrable and whose integral submanifolds are asymptotic leaves. The integral manifolds of an asymptotic distribution determine a foliation of M^m, called an *asymptotic foliation*.

For an M^m of conullity two, the adapted frame bundle and corresponding coframes can be chosen so that the Euclidean leaves are determined by $\omega^a = 0$, $a, b, \dots \in \{1, 2\}$. Since this last differential system is totally integrable, $d\omega^1$ and $d\omega^2$ must vanish as an algebraic consequence of $\omega^1 = \omega^2 = 0$ (due to the Frobenius theorem, second version; see [Ste 64]). This together with the fact that Euclidean leaves are totally geodesic, because M is a simple leaf, yields, due to (1.3.2),

$$\omega_u^1 = A_u\omega^1 + B_u\omega^2, \quad \omega_u^2 = C_u\omega^1 + F_u\omega^2; \tag{1.6.4}$$

where $u, v, \dots \in \{3, \dots, m\}$.

Let the unit vector $X = e_1 \cos\varphi + e_2 \sin\varphi$ be taken so that $\operatorname{span}\{X, e_3, \dots, e_m\}$ is the tangent plane of an asymptotic leaf. Then, $\nabla_{e_u} X = \nabla_X e_u + [e_u, X]$ must belong to the tangent plane of this asymptotic leaf for every value of u. Since the tangent distribution of these leaves is a foliation, this tangent plane contains $[e_u, X]$. Thus this plane must also contain

$$\nabla_X e_u = \nabla_{e_1} e_u \cos\varphi + \nabla_{e_2} e_u \sin\varphi = (\omega_u^k(e_1)e_k)\cos\varphi + (\omega_u^k(e_2)e_k)\sin\varphi.$$

Hence

$$(A_u e_1 + C_u e_2)\cos\varphi + (B_u e_1 + F_u e_2)\sin\varphi$$

must belong to $\operatorname{span}\{X, e_3, \dots, e_m\}$ and therefore must be a multiple of $X = e_1 \cos\varphi + e_2 \sin\varphi$. This last condition is equivalent to

$$B_u \sin^2\varphi + (A_u - F_u)\cos\varphi \sin\varphi - C_u \cos^2\varphi = 0.$$

But along the asymptotic leaf, $\omega^1 \sin\varphi = \omega^2 \cos\varphi$, so that the above condition reduces to

$$C_u(\omega^1)^2 + (F_u - A_u)\omega^1\omega^2 - B_u(\omega^2)^2 = 0.$$

According to [Kow 96], [BKV 96] a foliated M is said to be *planar* if it admits infinitely many asymptotic foliations. If it admits just two (or one, or none, respectively) asymptotic foliations, it is said to be *hyperbolic* (or *parabolic*, or *elliptic*, respectively).

2

Submanifolds in Space Forms

2.1 A Submanifold and Its Adapted Frame Bundle

Submanifolds will be considered in the context of differentiable manifolds of class C^∞ (see [Ste 64], Chapter II; [KN 63], Chapter I), or more precisely, in the context of (pseudo-)Riemannian manifolds (see [KN 69], Chapter VII; [Ch 73b], [Ch 2000], [BCO 2003]). It is worth mentioning that the introduction of [Ch 2000] contains a brief survey of the long history of the differential geometry of submanifolds.[1] Recent developments in submanifold theory are described in the introduction of [BCO 2003].

Let $f : M^m \to {}_sN^n(c)$ be an isometric immersion of class C^∞ of an m-dimensional (pseudo-)Riemannian manifold into an n-dimensional space form (or spacetime form, if $s > 0$), $n > m$, taken as the standard model ${}_sN^n(c)$ (see Section 1.4). Then $f(M^m)$ is a *submanifold* in ${}_sN^n(c)$ (see [KN 63], Chapter VII, also [Ch 73b] and [Ch 2000], for the case of Riemannian manifolds). Such a submanifold will be denoted simply by M^m, i.e., f is considered as the inclusion map.

For such a submanifold M^m its *tangent vector space* T_xM^m at an arbitrary point $x \in M^m$ is a regular vector subspace of $T_x({}_sN^n(c))$ and therefore has an orthogonal complement $T_x^\perp M^m$ in the latter, which is an $(n-m)$-dimensional regular vector space, called the *normal vector space* of the submanifold M^m at x.

If, for the case $s > 0$ and at an arbitrary point $x \in M^m$, the tangent vector space T_xM^m is spacelike (resp. timelike), then the submanifold M^m in ${}_sN^n(c)$ is also said to be spacelike (resp. timelike).

If $c \neq 0$ then ${}_sN^n(c) \subset {}_\sigma E^{n+1}$ (recall that $\sigma = s$ for $c > 0$ and $\sigma = s+1$ for $c < 0$). Thus an orthogonal complement $T_x^{*\perp}M^m$ of T_xM^m in $T_x({}_\sigma E^{n+1})$ is defined, called the *outer normal vector space* of M^m at x; obviously $T_x^{*\perp}M^m$ is the span of $T_x^\perp M^m$ and of $x = -(\sqrt{|c|})^{-1}e_{n+1}$, which are mutually orthogonal. If $c = 0$, then $N^n(0) = E^n$ and the designation *outer* is superfluous.

[1] One should note, however, that omitted in this survey are some newer historical investigations shedding light, in particular, on the emerging role of M. Bartels, F. Minding, and K. Peterson of the 19th century differential geometric school at the University of Tartu (Dorpat) (see [Stru 33], [GLOP 70], [Rei 73], [Ph 79], [Lu 96g], [Lu 97a], [Lu 99b]).

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The union of all tangent (normal or outer normal) vector spaces constitutes the *tangent* (resp. *normal* or *outer normal*) *vector bundle* of M^m, denoted by TM^m (resp. $T^{\perp}M^m$ or $T^{*\perp}M^m$). Its sections are the *tangent* (resp. *normal* or *outer normal*) *vector fields* on M^m.

In this book the method of frame bundles and exterior differential calculus is used. For a submanifold M^m in ${}_sN^n(c)$ the frame bundle adapted to ${}_sN^n(c)$ can be reduced to the subbundle of frames adapted to M^m as follows (see [KN 69], Chapter VII, Section 1). Let $x \in M^m$, let the first m basis vectors $e_1, \dots, e_m$ (in general, e_i, where $i, j, \dots \in \{1, \dots, m\}$) belong to T_xM^m and the next $n-m$ basis vectors $e_{m+1}, \dots, e_n$ (in general, e_α, where $\alpha, \beta, \dots \in \{m+1, \dots, n\}$) to $T_x^{\perp}M^m$. Then $g_{i\alpha} = 0$, and due to (1.2.4)

$$g_{\beta\alpha}\omega_i^\beta + g_{ik}\omega_\alpha^k = 0, \tag{2.1.1}$$

$$dg_{ij} = g_{kj}\omega_i^k + g_{ik}\omega_j^k, \quad dg_{\alpha\beta} = g_{\gamma\beta}\omega_\alpha^\gamma + g_{\alpha\gamma}\omega_\beta^\gamma. \tag{2.1.2}$$

Since the differential dx of the radius vector of the point $x \in M^m$ (recall that it is also denoted by x) belongs to T_xM^m, the first equation of (1.4.2) reduces to $dx = e_i\omega^i$, which means

$$\omega^\alpha = 0. \tag{2.1.3}$$

The submanifold M^m can be considered as an integral submanifold in ${}_sN^n(c)$ of this differential system (2.1.3).

From (1.4.3) and (2.1.3) it follows that $\omega^i \wedge \omega_i^\alpha = 0$, and now Cartan's lemma (see [Ste 64], Chapter I, Section 4; [BCGGG 91], p. 320; [IL 2003], p. 314) gives

$$\omega_i^\alpha = h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.1.4}$$

Therefore, from (1.4.2)

$$de_i = e_j\omega_i^j + (e_\alpha h_{ij}^\alpha - xcg_{ij})\omega^j, \tag{2.1.5}$$

and so for an arbitrary vector field $X = e_iX^i$ in the tangent vector bundle TM^m one has

$$dX = e_i(dX^i + X^j\omega_j^i) + (e_\alpha h_{ij}^\alpha - xcg_{ij})X^i\omega^j,$$

where the right-hand side is a sum of a tangent component and an outer normal component. Now if the point x is considered fixed, so that $dx = 0$ and thus all $\omega^i = 0$, one must also have $dX = 0$. Hence in the tangent component, the expressions in the first set of parentheses must be linear combinations of these ω^j. In other words, $dX^i + X^j\omega_j^i = \nabla_jX^i\omega^j$. The expression ∇_jX^i is the covariant derivative of the (1,0)-tensor field X^i on M^m with respect to the Levi-Civita connection ∇ of M^m, and thus ω_j^i are the connection 1-forms of ∇.

In the normal component, the coefficients h_{ij}^α, taken from (2.1.4), constitute a mixed tensor field, called the *second fundamental tensor* of M^m in ${}_sN^n(c)$. This mixed tensor field determines the *second fundamental form* (denoted by h) of M^m in ${}_sN^n(c)$ with values in $T_x^{\perp}M^m$.

To describe the relationship between this tensor and form, let another tangent vector field $Y = e_j Y^j$ be given on M^m and let t be the parameter of its integral curve such that $dx/dt = Y$. Then $\omega^j = Y^j dt$, and in the normal component of dX/dt with respect to ${}_sN^n(c)$ one has $h(X, Y) = e_\alpha h^\alpha_{ij} X^i Y^j$ (cf. [KN 69], Chapter VII, Section 3 and [Ch 73b], Chapter 2, Section 1).

With respect to ${}_\sigma E^{n+1}$ the normal component of de_i has the vector-valued coefficients

$$h_{ij} - xcg_{ij} = h^*_{ij}, \tag{2.1.6}$$

where $h_{ij} = e_\alpha h^\alpha_{ij}$, so that (2.1.5) is

$$de_i = e_j \omega^j_i + h^*_{ij} \omega^j. \tag{2.1.7}$$

The coefficients h^*_{ij} define a bilinear symmetric form with values in $T^{*\perp}_x M^m$, called the *outer second fundamental form* of M^m and denoted by h^*, i.e.,

$$h^*(X, Y) = h(X, Y) - xc\langle X, Y\rangle,$$

where $\langle X, Y\rangle = g_{ij} X^i Y^j$ is the scalar product of X and Y.

The usual lowering and raising of indices can be used by means of g_{ij}, $g_{\alpha\beta}$ and g^{ij}, $g^{\alpha\beta}$, where $g^{ik} g_{kj} = \delta^i_j$, $g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$. For instance, if one defines $h^i_{\alpha k} = g^{ij} h^\beta_{jk} g_{\beta\alpha}$, then $h^i_{\alpha k}\xi^\alpha$ gives the *shape (or Weingarten) operator* A_ξ of M^m in $N^n(c)$, which can also be defined by $\langle A_\xi(X), Y\rangle = \langle \xi, h(X, Y)\rangle$ (see, e.g., [KN 69], Chapter VII, Section 3, [BCO 2003], 2.1).

For the normal basis vectors e_α of the frame adapted to M^m one has, due to (1.4.2), (2.1.1), and (2.1.4),

$$de_\alpha = e_i(-h^i_{\alpha k}\omega^k) + e_\beta \omega^\beta_\alpha;$$

hence, for a normal vector field $\xi = e_\alpha \xi^\alpha$,

$$d\xi = e_\alpha(d\xi^\alpha + \xi^\beta \omega^\alpha_\beta) - e_i h^i_{\alpha k} \omega^k \xi^\alpha.$$

Here in the normal component of $d\xi$ the coefficients $d\xi^\alpha + \xi^\beta \omega^\alpha_\beta = \nabla^\perp \xi^\alpha$ give the covariant derivative of the normal (1,0)-tensor field ξ^α on M^m with respect to the *normal connection* $\nabla^\perp$ of M^m in $N^n(c)$, with the *connection* 1*-forms* ω^α_β. From (1.4.3) one obtains

$$d\omega^j_i = \omega^k_i \wedge \omega^j_k + \Omega^j_i, \quad d\omega^\beta_\alpha = \omega^\gamma_\alpha \wedge \omega^\beta_\gamma + \Omega^\beta_\alpha, \tag{2.1.8}$$

where

$$\Omega^j_i = \omega^\alpha_i \wedge \omega^j_\alpha + cg_{ik}\omega^j \wedge \omega^k, \quad \Omega^\beta_\alpha = \omega^i_\alpha \wedge \omega^\beta_i \tag{2.1.9}$$

are called the *curvature* 2*-forms* of ∇ and $\nabla^\perp$, respectively. Making substitutions from (2.1.1) and (2.1.4) and denoting

$$R^j_{i,pq} = (\langle h_{i[p}, h^j_{q]}\rangle + cg_{i[p}\delta^j_{q]}) = \langle h^*_{i[p}, h^{*j}_{q]}\rangle, \quad R^\beta_{\alpha,pq} = h^i_{\alpha[p} h^\beta_{q]i} \tag{2.1.10}$$