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Anton Alekseev, Edward Frenkel,
Marc Rosso, Ben Webster,
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Representation Theory, Mathematical Physics, and Integrable Systems

In Honor of Nicolai Reshetikhin

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Nicolai Reshetikhin

Preface

Anton Alekseev, Edward Frenkel, Marc Rosso,

Ben Webster, and Milen Yakimov

It is a pleasure to present this volume dedicated to Nicolai Reshetikhin, mathematician and friend we admire. Kolya, as he is affectionately known, has made a number of groundbreaking contributions in representation theory, integrable systems, and topology. His ideas have profoundly influenced the evolution of these fields in the past 40 years and will certainly continue to do so for many years to come. This book is a collection of chapters by distinguished mathematicians and physicists, many of them Kolya's students and collaborators, who develop further the themes of his research. Some of these chapters are based on the talks given by their authors at a conference in honor of Kolya's 60th birthday held at CIRM, Luminy, in June 2018.

In this preface, we present a brief summary of some of Kolya's discoveries.

1 Quantum Groups

Kolya's scientific career began in St. Petersburg, then known as Leningrad, in the late 1970s. He was a member of the famous Leningrad School led by Ludvig Faddev. Kolya joined the Leningrad School at an opportune moment, when Faddeev, Sklyanin, and Takhtajan were developing the quantum inverse scattering method and applying it to statistical models such as the Heisenberg XYZ spin chain [51, 52]. Their work, synthesizing the classical inverse scattering method used in soliton theory and the results of Bethe, Baxter, and others on the exactly solved models of statistical mechanics, heralded a revolution in quantum integrable systems. New powerful algebraic structures were emerging, including what came to be known as *quantum groups*.

Kolya was at the forefront of this research from the start. In fact, one of his first scientific papers [32], joint with Kulish, introduced the first quantum group, now known as $U_q(sl_2)$, as the algebraic structure behind the higher spin generalization of the quantum models such as the sine-Gordon and the Heisenberg XXZ spin chain. Soon after that Sklyanin endowed the algebra $U_q(sl_2)$ with a Hopf algebra structure [55], and just a few years later Drinfeld [12] and Jimbo [24] generalized the construction from sl_2 to an arbitrary Kac–Moody algebra. Thus, quantum groups were born.

Since then, they have become as ubiquitous as Lie groups and Lie algebras in many areas of mathematics and mathematical physics, far beyond the theory of integrable systems where they originated.

Kolya played a big role in these developments. His works, such as the influential papers [13] with Faddeev and Takhtajan and [44] with Semenov-Tian-Shansky, elucidated the algebraic structure of quantum groups. And he also pioneered many exciting applications of quantum groups in other fields: integrable systems, representation theory, combinatorics, and topology. We discuss some of these works in the next sections.

After obtaining his doctorate degree in 1984, Kolya joined the Leningrad Branch of the Steklov Institute of the Academy of Sciences, where he worked for 5 years. In 1989, he came to Harvard University as a winner of the Harvard Prize Fellowship. Two years later, he joined the faculty at University of California, Berkeley. In 2021, after 30 years of service, he retired from Berkeley and joined the Yau Center for Mathematical Sciences at Tsinghua University, Beijing. Kolya has also held numerous visiting positions, such as Niels Bohr Visiting Professorship at the Aarhus University and Humboldt Visiting Professorship at the Technical University of Berlin and the Max Planck Institute for Gravitational Physics. He is currently affiliated with the KdV Institute of the University of Amsterdam and the St. Petersburg State University.

2 Quantum Integrable Systems

The quantum inverse scattering method was initially applied to models related to the Lie algebra sl_2, such as the XXX and XXZ model. As far as we know, Kolya was the first to systematically extend this method to quantum models associated to other Lie algebras. In modern language, these models correspond to finite-dimensional representations of the Yangians $Y(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra, and quantum affine algebras $U_q(\widehat{\mathfrak{g}})$, where $\widehat{\mathfrak{g}}$ is an affine Kac–Moody algebra (including the twisted ones). The commuting Hamiltonians acting on these representations come from a family of commuting transfer-matrices in these algebras. Thus, one gets a vast collection of quantum integrable systems associated to these Lie algebras.

In the case the XXX and XXZ models, which correspond to $Y(sl_2)$ and $U_q(\widehat{sl}_2)$, respectively, the spectra of these Hamiltonians can be computed using the algebraic Bethe Ansatz method, introduced by Bethe and developed further by the Leningrad School. The problem of generalizing this method to the integrable systems associated to other Lie algebras is highly non-trivial. In a series of papers [36–38], Kolya proposed an analogue of this method, which he dubbed *analytic Bethe Ansatz*, for the quantum integrable systems associated to the Yangians and quantum affine algebras. A crucial role in it was played by an elegant generalization of the famous Baxter relations observed in the XXX and XXZ model.

The resulting Bethe Ansatz equations [38, 50] have been widely used in the subject, even though the method itself remained something of a mystery. It was finally put on a firm foundation with the advent of the theory of *q-characters* developed by Kolya and Edward Frenkel [20] (see Sect. 4 below). Using this theory, Frenkel and Hernandez [17] proved the generalized Baxter relations and gained new insights into the analytic Bethe Ansatz.

The quantum integrable systems associated to $U_q(\widehat{\mathfrak{g}})$ have a limit, in which the symmetry algebra becomes the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ itself. Together with Feigin and E. Frenkel, Kolya showed how to obtain the quantum Hamiltonians of the corresponding integrable system, called the *Gaudin model*, from the center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level [16]. Moreover, they were able to describe the spectrum of the Hamiltonians of the Gaudin model in terms of the geometric objects called *opers*. An interesting aspect of this construction is that the opers are associated not to $\mathfrak{g}$ but to the Langlands dual Lie algebra ${}^L\mathfrak{g}$ of $\mathfrak{g}$. The appearance of the *Langlands duality* here is important. It manifests a deep connection between the Gaudin model (and its generalizations) and the geometric Langlands correspondence developed by Beilinson and Drinfeld [3]. As far as we know, the paper [16] was the first case study of the Langlands duality in quantum integrable systems. Since then, dualities of this kind have been extensively studied and connected to various dualities of quantum field theories.

Around the same time, together with Varchenko, Kolya established a link between the Gaudin model and the critical level limit of the solutions of the Knizhnik–Zamolodchikov (KZ) equations [49].

Closely related to this topic is another groundbreaking work [22], in which Kolya and Igor Frenkel introduced the celebrated *qKZ equations* (in a special case, this equation was also introduced by Fedor Smirnov, another alumnus of the Leningrad School [56]). The qKZ equations have roughly the same relationship with the quantum integrable models associated to $U_q(\widehat{\mathfrak{g}})$ as the KZ equations with the Gaudin models. In particular, the critical level limits of the solutions of the qKZ equations give rise to the eigenvectors of the quantum integrable systems associated to $U_q(\widehat{\mathfrak{g}})$, see [40]. Together with Jasper Stokman and Bart Vlaar, Kolya explored similar structures for the so-called boundary qKZ equations [45, 46].

3 Topology

One of the most striking and influential applications of quantum groups is low-dimensional topology. In the mid-1980s, Vaughan Jones and others defined stunning new invariants of knots—most famously, the Jones polynomial, HOMFLYPT polynomial, and Kauffman polynomial. In a groundbreaking work [39, 57], Kolya and Vladmir Turaev showed that these invariants were special cases of a much more general, categorical construction that attached a knot invariant to each representation of the quantum group associated to a simple Lie algebra. These are now called *Reshetikhin–Turaev invariants*.

This work supplied an enormous family of new powerful invariants, and its novel categorical approach opened the door to defining invariants of 3-manifolds. This was done by Kolya with Turaev in [48]. Among other things, this gave a mathematical interpretation of the invariants that Edward Witten had proposed earlier in the context of the Chern-Simons theory [58] (from Witten's perspective, the invariant arises as the partition function of the Chern-Simons theory on the corresponding 3-manifold). These *Witten–Reshetikhin–Turaev invariants* have had tremendous influence in 3-manifold theory, topological quantum field theory (including topological quantum computing), tensor categories, and beyond. This is one of Kolya's best known works.

Since then, Kolya continued making new strides in topology, making new advances such as defining new invariants of links and tangles in 3-manifolds with the additional datum of a flat connection in the complement [25], the recursion formulas for knot invariants [10], and the extension of the Reshetikhin–Turaev invariants to braided categories with weaker properties than the ribbon property usually required [4].

4 Representation Theory and Combinatorics

Kolya has made a number of important contributions to representation theory of quantum groups. For example, in [29], he and Anatol Kirillov introduced a family of what came to be known as *Kirillov–Reshetikhin modules* over the Yangians $Y(\mathfrak{g})$. These modules have been used extensively in representation theory and combinatorics. Another important result, a bijection between semi-standard Young tableaux and rigged configurations, was obtained by Kolya with Kerov and Kirillov from the study of asymptotic completeness of the Bethe Ansatz equations [26]. This led Kolya and Kirillov to new formulas for the Kostka polynomials [27, 28].

Another direction concerns *deformations of $\mathcal{W}$-algebras*. According to the results of the paper [16] mentioned earlier, quantum Hamiltonians of the Gaudin model could be obtained from the center of the enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level. The center, however, is not just a commutative algebra. It also has a Poisson algebra structure due to the possibility of deforming the level away from the critical level. It is known from [14] that this Poisson algebra is isomorphic to the classical $\mathcal{W}$-algebra associated to ${}^L\mathfrak{g}$ (and this is in fact the root of the Langlands duality mentioned earlier). Now consider the $U_q(\widehat{\mathfrak{g}})$-analogue of the XXZ model, which is a q-deformation of the Gaudin model. It turns out that the quantum Hamiltonians of this model may also be obtained from the center at the critical level, but now of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. And this q-deformed center also has a Poisson algebra structure. This way, Kolya and E. Frenkel discovered in [18] q-deformations of the classical $\mathcal{W}$-algebras, in particular, a q-deformation of the classical Virasoro algebra (corresponding to $\mathfrak{g} = sl_2$). Together with Semenov-Tian-Shansky, they were able to obtain this Poisson algebra by means of a q-deformed Drinfeld–Sokolov reduction [21] (this construction was generalized to other Lie algebras in [53]).

Two years later, Frenkel and Reshetikhin quantized these Poisson algebras [19]. Namely, they introduced the deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ for an arbitrary simple Lie algebra $\mathfrak{g}$ (in the case of $\mathfrak{g} = sl_n$ this algebra was constructed earlier in [2, 15, 54]). The deformed $\mathcal{W}$-algebra depends on two parameters, q and t, and one recovers the center of $U_q(\widehat{\mathfrak{g}})$ at the critical level in the limit $t \to 1$. Other limits also give rise to interesting algebras. Frenkel and Reshetikhin used the $t \to 1$ limit in [20] to introduce the theory of *q-characters* of finite-dimensional representations of quantum affine algebras. The q-characters proved to be a powerful tool in the study of these representations and beyond.

Recently, the deformed $\mathcal{W}$-algebras and related structures appeared in the study of four-dimensional gauge theories associated to Nakajima quiver varieties [1, 30].

5 Poisson Algebras, BV Theory, and Quantization

Poisson geometry, quantization, and Batalin–Vilkovisky (BV) structures are closely related topics. Their connection was established in the work of Kontsevich on quantization of Poisson manifolds using Feynman path integral [31] and in the work of Cattaneo–Felder [5] which explained the Feynman rules used by Kontsevich in terms of a BV structure of the Poisson σ-model.

Kolya's interest in the topic dates back to his joint work with Takhtajan [47], where they gave a simple formula for quantization of Kähler manifolds in terms of Feynman graphs. Recently, in collaboration with Alberto Cattaneo and Pavel Mnev, Kolya put forward an ambitious *Cattaneo–Mnev–Reshetikhin program* which bridges the BV quantization and low dimensional topology. More precisely, the aim of this program is to upgrade low dimensional algebraic topology to quantum algebraic topology in order to define and compute quantum invariants of manifolds in dimensions 2, 3, and 4 by gluing them from simple pieces (e.g., tetrahedra or cubes).

This program is still under development, but significant progress has already been achieved. The first important result is a better formulation of quantum field theory (QFT) on manifolds with boundary. While BV formalism is the standard tool of treating symmetries in the bulk, the Batalin–Fradkin–Vilkovisky formalism (BFV) is needed to understand the symmetry at the boundary. The combined BV-BFV theory was stated at the classical level in [6], and the quantum version was addressed in [7]. Interesting partial results include applications to integrable systems [8] and an example of a BF theory verifying Atiyah–Segal type gluing axioms [9].

6 Other Works

The scope of Kolya's research has been extremely broad, and he has made substantial contributions in numerous directions in addition to those described in the previous sections.

Throughout his career, he has actively worked on the semiclassical structures behind quantum groups. He has made key contributions to the geometry of *Poisson–Lie groups* and the dynamics of related integrable systems, such as the ones in [23] on the symplectic foliation of the standard Poisson–Lie groups and the associated Coxeter–Toda lattices. He introduced powerful representation theoretic methods to study degenerate integrability on non-Coxeter symplectic leaves [42] and spin Calogero–Moser integrable systems [43].

Kolya has done extensive research on quantum groups at roots of unity. In his solo paper [41], he constructed quasitriangular structures on the unrestricted quantum groups at roots of unity. In a joint paper with De Concini et al. [11], he described the tensor product structure of the finite-dimensional representations of these algebras and developed a general axiomatic setting of Cayley–Hamilton Hopf algebras.

He was also very much involved in the study of *random matrices, random processes, and random surfaces*. Jointly with Andrei Okounkov, he obtained integral

Reshetikhin's mathematical descendants at the Luminy conference. From left to right: David Keating, Noah Snyder, Sevak Mkrtchyan, Qingtao Chen, Aaron Brookner, Alexander Shapiro, Ben Webster, Peter Tingley, Nicolai Reshetikhin, Theo Johnson-Freyd, Olya Mandelshtam, Harold Williams, Kurt Trampel, Meredith Shea, Kent Vashaw, Raeez Lorgat, Gus Schrader, and Kai Chieh Chen representations of the correlation functions of the Schur process and used them to obtain explicit formulas for the asymptotic correlation functions for 3D Young diagrams in the bulk limit [33]. This paper inspired much subsequent research on determinantal processes and their asymptotic behavior. In another paper [34], Kolya and Okounkov carried out a detailed study of random skew 3D partitions and their various asymptotics expressed in terms of Airy and Pearcey kernels. These ideas led to a remarkable duality that was proposed by Kolya, Okounkov, and Vafa between the topological A-model in string theory and a classical statistical mechanical model of crystal melting [35].

Kolya has brought up many outstanding graduate students. According to the Mathematics Genealogy Project, as of the summer of 2020, Kolya had 20 students and 33 descendants. He has said that the opportunity to work with graduate students is one of the joys of being in academia.

A conference to mark Kolya's 60th birthday took place at Centre International de Rencontres Mathématiques, Luminy, in June 2018. It was funded by the National Science Foundation grant DMS-1803265 and the European Research Council project MODFLAT.

In conclusion, we wish Kolya many more years of health, productivity, success, and inspiration.

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References

1. M. Aganagic, E. Frenkel, and A. Okounkov, *Quantum q-Langlands Correspondence*, Trans. Moscow Math. Soc. **79** (2018) 1–83.
2. H. Awata, H. Kubo, S. Odake, and J. Shiraishi, *Quantum $\mathcal{W}_N$ algebras and Macdonald polynomials,* Comm. Math. Phys. **179** (1996) 401–416.
3. A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, Preprint.
4. C. Blanchet, N. Geer, B. Patureau-Mirand, and N. Reshetikhin. *Holonomy braidings, biquandles and quantum invariants of links with $\mathsf{SL}_2(\mathbb{C})$-flat connections.* Selecta Math. **26**, no. 2 (2020) 1–58.
5. A. Cattaneo, G. Felder, *A path integral approach to the Kontsevich quantization formula*, Comm. Math. Phys. **212**, no. 3 (2000) 591–611.

6. A. Cattaneo, P. Mnev, N. Reshetikhin. *Classical BV theories on manifolds with boundary*, Comm. Math. Phys. **332**, no. 2 (2014) 535–603.
7. A. Cattaneo, P. Mnev, N. Reshetikhin. *Perturbative quantum gauge theories on manifolds with boundary*, Comm. Math. Phys. **357**, no. 2 (2018) 631–730.
8. A. Cattaneo, P. Mnev, N. Reshetikhin, *Poisson sigma model and semiclassical quantization of integrable systems*, Rev. Math. Phys. **30**, no. 6 (2018) 1840004, 26 pp.
9. A. Cattaneo, P. Mnev, N. Reshetikhin, *A Cellular Topological Field Theory*, Comm. Math. Phys. **374**, no. 2 (2020) 1229–1320.
10. Q. Chen, N. Reshetikhin, *Recursion formulas for HOMFLY and Kauffman invariants*, J. Knot Theory Ramifications **23**, no. 5 (2014) 1450024, 23 pp.
11. C. De Concini, C. Procesi, N. Reshetikhin, and M. Rosso, *Hopf algebras with trace and representations*, Invent. Math. **161**, no. 1 (2005) 1–44.
12. V.G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Dokl. **32** (1985) 254–258.
13. L.D. Faddeev, N.Yu. Reshetikhin, and L.A. Takhtajan, *Quantization of Lie Groups and Lie Algebras*, Leningrad Math. J. **1** (1990) 193–225.
14. B. Feigin and E. Frenkel, *Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras*, in *Infinite Analysis*, eds. A. Tsuchiya, T. Eguchi, M. Jimbo, Adv. Ser. in Math. Phys. **16**, 197–215, World Scientific, 1992.
15. B. Feigin and E. Frenkel, *Quantum $\mathcal{W}$–algebras and elliptic algebras*, Comm. Math. Phys. **178** (1996) 653–678.
16. B. Feigin, E. Frenkel, and N. Reshetikhin, *Gaudin model, Bethe ansatz and critical level*, Comm. Math. Phys. **166** (1994) 27–62.
17. E. Frenkel and D. Hernandez, *Baxter's Relations and Spectra of Quantum Integrable Models*, Duke Math. J. **164** (2015) 2407–2460.
18. E. Frenkel and N. Reshetikhin, *Quantum affine algebras and deformations of the Virasoro and $\mathcal{W}$-algebras*, Comm. Math. Phys. **178** (1996) 237–264.
19. E. Frenkel and N. Reshetikhin, *Deformations of $\mathcal{W}$–algebras associated to simple Lie algebras*, Comm. Math. Phys. **197** (1998) 1–32.
20. E. Frenkel and N. Reshetikhin, *The q–characters of representations of quantum affine algebras and deformations of $\mathcal{W}$–algebras*, in Recent Developments in Quantum Affine Algebras and Related Topics, N. Jing and K. Misra (eds.), Contemporary Mathematics **248**, pp. 163–205, AMS, 1999.
21. E. Frenkel, N. Reshetikhin, and M.A. Semenov-Tian-Shansky, *Drinfeld-Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras I.*, Comm. Math. Phys. **192** (1998) 605–629.
22. I.B. Frenkel and N.Yu. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992) 1–60.
23. T. Hoffmann, J. Kellendonk, N. Kutz, and N. Reshetikhin, *Factorization dynamics and Coxeter-Toda lattices*, Comm. Math. Phys. **212**, no. 2 (2000) 297–321.
24. M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985) 63–69.

25. R. Kashaev and N. Reshetikhin, *Invariants of tangles with flat connections in their complements*, in Graphs and patterns in mathematics and theoretical physics, pp. 151–172, Proc. Sympos. Pure Math. **73**, AMS, 2005.
26. S.V. Kerov, A.N. Kirillov, and N.Yu. Reshetikhin, *Combinatorics, the Bethe ansatz and representations of the symmetric group*, J. Soviet Math. **41**, no. 2 (1988) 916–924.
27. A.N. Kirillov and N.Yu. Reshetikhin, *The Bethe ansatz and the combinatorics of Young tableaux*, J. Soviet Math. **41**, no.2 (1988) 925–955.
28. A.N. Kirillov and N.Yu. Reshetikhin, *Multiplicity of weights in irreducibile tensor representations of a complete linear superalgebra*, Funct. Anal. Appl. **22**, no. 4 (1988) 328–330.
29. A.N. Kirillov and N.Yu. Reshetikhin, *Representations of Yangians and multiplicities of occurrence of the irreducible components of the tensor product of representations of simple Lie algebras*, J. Soviet Math. **52** (1990) 3156–3164.
30. T. Kimura and V. Pestun, *Quiver W-algebras*, Lett. Math. Phys. **108** (2018) 1351–1381.
31. M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66**, no. 3 (2003) 157–216.
32. P.P. Kulish and N.Yu. Reshetikhin, *Quantum linear problem for the sine-Gordon equation and higher representations*, Zapiski Nauchn. Sem. LOMI **101** (1981) 101–110. English translation: J. Soviet Math. **23** (1983) 2435–2441.
33. A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16**, no. 3 (2003) 581–603.
34. A. Okounkov and N. Reshetikhin, *Random skew plane partitions and the Pearcey process*, Comm. Math. Phys. **269**, no. 3 (2007) 571–609.
35. A. Okounkov, N. Reshetikhin, and C. Vafa, *Quantum Calabi-Yau and classical crystals*, in The unity of mathematics, Progr. Math. **244**, pp. 597–618, Birkhüser Boston, 2006.
36. N.Yu. Reshetikhin, *A Method Of Functional Equations In The Theory Of Exactly Solvable Quantum Systems*, Lett. Math. Phys. **7** (1983) 205–213.
37. N.Yu. Reshetikhin, *Integrable Models of Quantum One-dimensional Magnets With $O(N)$ and $Sp(2k)$ Symmetry*, Theor. Math. Phys. **63** (1985) 555–569.
38. N.Yu. Reshetikhin, *The spectrum of the transfer matrices connected with Kac–Moody algebras*, Lett. Math. Phys. **14** (1987) 235–246.
39. N.Yu. Reshetikhin, *Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links I., II.* LOMI Preprints E–4–87, E–17–87 (1987).
40. N. Reshetikhin, *Jackson-type integrals, Bethe vectors, and solutions to a difference analog of the Knizhnik-Zamolodchikov system*, Lett. Math. Phys. **26** (1992) 153–165.
41. N. Reshetikhin, *Quasitriangularity of quantum groups at roots of 1*, Comm. Math. Phys. **170**, no. 1 (1995) 79–99.

42. N. Reshetikhin, *Integrability of characteristic Hamiltonian systems on simple Lie groups with standard Poisson Lie structure*, Comm. Math. Phys. **242**, no. 1 (2003) 1–29.
43. N. Reshetikhin, *Degenerate integrability of the spin Calogero-Moser systems and the duality with the spin Ruijsenaars systems*, Lett. Math. Phys. **63** (2003), no. 1, 55–71.
44. N.Yu. Reshetikhin and M.A. Semenov-Tian-Shansky, *Quantum R-matrices and factorization problems*, J. Geom. Phys. **5**, no. 4 (1988) 533–550.
45. N. Reshetikhin, J. Stokman, and B. Vlaar, *Boundary quantum Knizhnik-Zamolodchikov equations and Bethe vectors*, Comm. Math. Phys. **336** (2015) 953–986.
46. N. Reshetikhin, J. Stokman, and B. Vlaar, *Integral solutions to boundary quantum Knizhnik-Zamolodchikov equations*, Adv. in Math. **323** (2018) 486–528.
47. N. Reshetikhin and L.A. Takhtajan, *Deformation quantization of Kähler manifolds*, L.D. Faddeev seminar on Mathematical Physics, Amer. Math. Soc. Transl. Ser. 2, **201**, pp. 257–276, AMS, 2000.
48. N. Reshetikhin and V. G. Turaev. *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103**, no. 1 (1991) 547–597.
49. N. Reshetikhin and A. Varchenko, *Quasiclassical asymptotics of solutions to the KZ equations*, in Geometry, topology, and physics, Conf. Proc. Lect. Notes Geom. Topology, IV, pp. 293–322, Int. Press, Cambridge, MA, 1995.
50. N. Reshetikhin and P. Wiegmann, *Towards the classification of completely integrable quantum field theories (the Bethe-Ansatz associated with Dynkin diagrams and their automorphisms)*, Physics Letters B **189** (1987) 125–131.
51. E.K. Sklyanin, L.A. Takhtajan, and L.D. Faddeev, *Quantum inverse problem method. I.*, Theor. Math. Phys. **40** (1979) 688–706.
52. L.A. Takhtajan and L.D. Faddeev, *The quantum method of the inverse problem and the Heisenberg XYZ model*, Russ. Math. Surv. **34**, no. 5 (1979) 11–68.
53. M.A. Semenov-Tian-Shansky and A.V. Sevostyanov, *Drinfeld-Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras. II.*, Comm. Math. Phys. **192** (1998) 631–647.
54. J. Shiraishi, H. Kubo, H. Awata, and S. Odake, *A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions*, Lett. Math. Phys. **38** (1996) 33–51.
55. E.K. Sklyanin, *On an algebra generated by quandratic relations*, Uspekhi Mat. Nauk **40**, no. 2 (1985) 214.
56. F. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, Advanced Series in Mathematical Physics **14**, World Scientific, 1992.
57. V. G. Turaev, *The Yang-Baxter equation and invariants of links*, Invent. Math. **92**, no. 3 (1988) 527–554.
58. E. Witten, *Quantum field theory and the Jones polynomial*, Communications in Mathematical Physics **121**, no. 3 (1989) 351–399.

Contents

Examples of Finite-Dimensional Pointed Hopf Algebras in Positive Characteristic

Nicolás Andruskiewitsch, Iván Angiono, and István Heckenberger

To Nikolai Reshetikhin on his 60th birthday with admiration.

Abstract We present new examples of finite-dimensional Nichols algebras over fields of positive characteristic. The corresponding braided vector spaces are not of diagonal type, admit a realization as Yetter-Drinfeld modules over finite abelian groups, and are analogous to braidings over fields of characteristic zero whose Nichols algebras have finite Gelfand-Kirillov dimension.

We obtain new examples of finite-dimensional pointed Hopf algebras by bosonization with group algebras of suitable finite abelian groups.

2010 Mathematics Subject Classification 16T20, 17B37

1 Introduction

1.1 Overview

This is a contribution to the classification of finite-dimensional pointed Hopf algebras in positive characteristic. Beyond the classical theme of cocommutative Hopf algebras—see, for instance, [CF] and the references therein—the problem was considered in several recent works [CLW, HW, NW, NWW1, NWW2, W]. As in various of these papers, the focus of our work is on finite-dimensional Nichols algebras over finite abelian groups. Let $\Bbbk$ be an algebraically closed field

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of characteristic $p \geq 0$. When $p = 0$, such Nichols algebras are necessarily of diagonal type and their classification was achieved in [H]. When $p > 0$, finite-dimensional Nichols algebras of diagonal type of rank 2 and 3 were classified in [HW, W]. Notice that there are more examples than in characteristic 0: indeed, 1 in the diagonal is no longer excluded.

Example 1.1 Assume that $p > 0$. Given $\theta \in \mathbb{N}$, we set $\mathbb{I}_\theta = \{1, 2, \dots, \theta\}$. Let $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta} \in \Bbbk^{\theta \times \theta}$ be a matrix with $q_{ii} = 1 = q_{ij}q_{ji}$ for all $i \neq j \in \mathbb{I}_\theta$. Let (V, c) be a braided vector space of dimension θ, of diagonal type with matrix $\mathbf{q}$ with respect to a basis $(x_i)_{i \in \mathbb{I}_\theta}$, that is $c : V \otimes V \to V \otimes V$ is given by $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$. Then the corresponding Nichols algebra is

$$\mathscr{B}(V) = \mathfrak{s}_{\mathbf{q}}(V) := T(V)/\langle x_i^p,\ i \in \mathbb{I}_\theta, \quad x_i x_j - q_{ij} x_j x_i,\ i < j \in \mathbb{I}_\theta \rangle.$$

Clearly, $\dim \mathfrak{s}_{\mathbf{q}}(V) = p^\theta$.

Furthermore, if $p > 0$, then there are finite-dimensional Nichols algebras over abelian groups that are *not* of diagonal type, a remarkable example being the Jordan plane that has dimension p^2 [CLW] (it gives rise to pointed Hopf algebras of order p^3, see [NW]), in contrast with characteristic 0, where it has Gelfand-Kirillov dimension 2. In fact, Nichols algebras over abelian groups *with finite Gelfand-Kirillov dimension and assuming* $p = 0$ were the subject of the recent papers [AAH1, AAH2]. Succinctly, the main relevant results in loc. cit. are the following:

- It was conjectured in [AAH1] that finite GK-dimensional Nichols algebras of diagonal type have arithmetic root system; the conjecture is true in rank 2 and also in affine Cartan type [AAH2].
- A class of braided vector spaces arising from abelian groups was introduced in [AAH1]; they are decomposable with components being points and blocks. Assuming the validity of the above Conjecture, the finite GK-dimensional Nichols algebras from this class were classified in [AAH1].

Beware that there are finite GK-dimensional Nichols algebras over abelian groups that do not belong to the referred class, see [AAH1, Appendix].

The braided vector spaces in the class alluded to above can be labeled with flourished Dynkin diagrams. The main result of [AAH1] says that the Nichols algebra of a braided vector space in the class has finite Gelfand-Kirillov dimension if and only if its flourished Dynkin diagram is admissible.

From now on we assume that $p > 2$. (The case $p = 2$ has to be treated separately.) In the present paper, we show, adapting arguments from [AAH1], that the Nichols algebras of many braided vector spaces of admissible flourished Dynkin diagrams are finite-dimensional. This result extends Example 1.1 and the Jordan plane [CLW] and is reminiscent of a familiar phenomenon in Lie algebras in positive characteristic. By bosonization we obtain many new examples of finite-dimensional pointed Hopf algebras.

1.2 The Main Result

To describe more precisely our main Theorem we need first to discuss blocks.

For $k < \ell \in \mathbb{N}_0$, we set $\mathbb{I}_{k,\ell} = \{k, k+1, \dots, \ell\}$, $\mathbb{I}_\ell = \mathbb{I}_{1,\ell}$.

A *block* $\mathcal{V}(\epsilon, \ell)$, where $\epsilon \in \Bbbk^\times$ and $\ell \in \mathbb{N}_{\geq 2}$, is a braided vector space with a basis $(x_i)_{i \in \mathbb{I}_\ell}$ such that for $i, j \in \mathbb{I}_\ell$, $1 < j$:

$$c(x_i \otimes x_1) = \epsilon x_1 \otimes x_i, \qquad c(x_i \otimes x_j) = (\epsilon x_j + x_{j-1}) \otimes x_i. \tag{1.1}$$

In characteristic 0, the only Nichols algebras of blocks with finite GKdim are the Jordan plane $\mathscr{B}(\mathcal{V}(1, 2))$ and the super Jordan plane $\mathscr{B}(\mathcal{V}(-1, 2))$; both have GKdim $= 2$. In our context with $p > 2$, the Jordan plane $\mathscr{B}(\mathcal{V}(1, 2))$ has dimension p^2 [CLW]; see Lemma 3.1. Our starting result is that the super Jordan plane $\mathscr{B}(\mathcal{V}(-1, 2))$ has dimension $4p^2$, see Proposition 3.2. For simplicity a block $\mathcal{V}(\epsilon, 2)$ of dimension 2 is called an ϵ-block. We also prove that a block $\mathcal{V}(\epsilon, 2)$ has finite-dimensional Nichols algebra only when $\epsilon = \pm 1$, see Proposition 3.3.

The braided vector spaces in this paper belong to the class analogous to the one considered in [AAH1]. Briefly, (V, c) belongs to this class if

$$V = V_1 \oplus \cdots \oplus V_t \oplus V_{t+1} \oplus \cdots \oplus V_\theta, \tag{1.2}$$

$$c(V_i \otimes V_j) = V_j \otimes V_i, \ i, j \in \mathbb{I}_\theta, \tag{1.3}$$

where V_h is a ϵ_h-block, with $\epsilon_h^2 = 1$, for $h \in \mathbb{I}_t$; and $\dim V_i = 1$ with braiding determined by $q_{ii} \in \Bbbk^\times$ (we say that i is a point), $i \in \mathbb{I}_{t+1,\theta}$; the braiding between points i and j is given by $q_{ij} \in \Bbbk^\times$ while the braiding between a point and block, respectively two blocks, should have the form as in (4.1), respectively (6.1). For convenience, we attach to (V, c) a flourished graph $\mathcal{D}$ with θ vertices, those corresponding to a 1-block decorated with $\boxplus$, those to -1-block decorated with $\boxminus$ and the point i with $\overset{q_{ii}}{\circ}$. If $i \neq j$ are points, and there is an edge between them decorated by $\widetilde{q}_{ij} := q_{ij}q_{ji}$ when this is $\neq -1$, or no edge if $\widetilde{q}_{ij} = 1$. If h is a block and j is a point, then there is an edge between h and j decorated either by $\mathscr{G}_{hj}$ if the interaction is weak and $\mathscr{G}_{hj} \neq 0$ is the ghost, cf. (4.2), or by $(-, \mathscr{G}_{hj})$ if the interaction is mild and $\mathscr{G}_{hj}$ is the ghost; but no edge if the interaction is weak and $\mathscr{G}_{hj} = 0$. There are no edges between blocks and we assume that the diagram is connected by a well-known reduction argument.

This class of braided vector spaces together with those of diagonal type does not exhaust that of Yetter-Drinfeld modules arising from abelian groups; there are still those containing a pale block as in [AAH1, Chapter 8]. Synthetically our main result is the following.

Theorem 1.2 *Let V be a braided vector space as in the following list, then* $\dim \mathscr{B}(V) < \infty$.

(a) *V has braiding* (4.1) *and is listed in Table 1, or*
(b) *V has braiding* (5.1) *and is listed in Table 2, or*
(c) *V has braiding* (6.1)*, or*
(d) *V has braiding* (7.1) *and is listed in Table 3.*

By bosonization with suitable abelian groups, we get examples of finite-dimensional pointed Hopf algebras in positive characteristic.

Concrete examples of such Hopf algebras are described in Sects. 3.3, 4.5, 5.3, 6.1, and 7.3. We also give a presentation by generators and relations of the Nichols algebras; references to this information and the dimensions are also given in the Tables.

All braided vector spaces in this Theorem belong to the class described above except those in (d) that contain a pale block.

Table 1 Finite-dimensional Nichols algebras of a block and a point

V	diagram	q_{22}	$\mathscr{G}$	$\mathscr{B}(V)$	$\dim K$	$\dim \mathscr{B}(V)$
$\mathfrak{L}(1, \mathscr{G})$	$\boxplus \overset{\mathscr{G}}{\rule{1cm}{0.4pt}} \overset{1}{\bullet}$	1	discrete	§4.3.1	$p^{\mathtt{r}+1}$	$p^{\mathtt{r}+3}$
$\mathfrak{L}(-1, \mathscr{G})$	$\boxplus \overset{\mathscr{G}}{\rule{1cm}{0.4pt}} \overset{-1}{\bullet}$	-1	discrete	§4.3.2	$2^{\mathtt{r}+1}$	$2^{\mathtt{r}+1} p^2$
$\mathfrak{L}(\omega, 1)$	$\boxplus \overset{1}{\rule{1cm}{0.4pt}} \overset{\omega}{\bullet}$	$\in \mathbb{G}'_3$	1	§4.3.5	3^3	$3^3 p^2$
$\mathfrak{L}_-(1, \mathscr{G})$	$\boxminus \overset{\mathscr{G}}{\rule{1cm}{0.4pt}} \overset{1}{\bullet}$	1	discrete	§4.3.3	$2^{\frac{\mathtt{r}}{2}} p^{\frac{\mathtt{r}}{2}+1}$	$2^{\frac{\mathtt{r}}{2}+2} p^{\frac{\mathtt{r}}{2}+3}$
$\mathfrak{L}_-(-1, \mathscr{G})$	$\boxminus \overset{\mathscr{G}}{\rule{1cm}{0.4pt}} \overset{-1}{\bullet}$	-1	discrete	§4.3.4	$2^{\frac{\mathtt{r}}{2}+1} p^{\frac{\mathtt{r}}{2}}$	$2^{\frac{\mathtt{r}}{2}+3} p^{\frac{\mathtt{r}}{2}+2}$
$\mathfrak{C}_1$	$\boxminus \overset{(-1,1)}{\rule{1cm}{0.4pt}} \overset{-1}{\bullet}$	-1	1	§4.4	16	$64p^2$

1.3 Contents of the Paper

Section 2 is devoted to preliminaries. The next Sections contain the examples of finite-dimensional Nichols algebras and some realizations over abelian groups; each Section describes a family of braided vector spaces with a certain decomposition as we describe now. In Sect. 3 we compute Nichols algebras of a block. In Sect. 4 we present examples of Nichols algebras corresponding to one block and one point, while in Sect. 5 we consider the case one block and several points. Section 6 is devoted to examples of several blocks and one point. Finally in Sect. 7 we give examples of finite-dimensional Nichols algebras whose braided vector spaces decompose as one pale block and one point.

Table 2 Finite-dimensional Nichols algebras of a block and several points, $\omega \in \mathbb{G}'_3$

V	diagram	$\mathscr{B}(V)$	$dim\mathscr{B}(V)$
$\mathfrak{L}(A_{\theta-1})$, $\theta > 2$	$\boxplus \overset{1}{\text{——}} \overset{-1}{\bullet} \overset{-1}{\text{——}} \overset{-1}{\circ} \dots \overset{-1}{\circ} \overset{-1}{\text{——}} \overset{-1}{\circ}$ $\theta - 1$ vertices	§5.2.7	$p^2 2^6$ $p^2 2^{(\theta-1)(\theta-2)}$
$\mathfrak{L}(A_2, 2)$	$\boxplus \overset{2}{\text{——}} \overset{-1}{\bullet} \overset{-1}{\text{——}} \overset{-1}{\circ}$	§5.2.6	$p^2 2^{12}$
$\mathfrak{L}(A(1\vert 0)_2; \omega)$	$\boxplus \overset{1}{\text{——}} \overset{-1}{\bullet} \overset{\omega}{\text{——}} \overset{-1}{\circ}$	§5.2.2	$p^2 2^7 3^4$
$\mathfrak{L}(A(1\vert 0)_1; \omega)$	$\boxplus \overset{1}{\text{——}} \overset{-1}{\bullet} \overset{\omega^2}{\text{——}} \overset{\omega}{\circ}$	§5.2.1	$p^2 2^4 3^2$
$\mathfrak{L}(A(1\vert 0)_3; \omega)$	$\boxplus \overset{1}{\text{——}} \overset{\omega}{\bullet} \overset{\omega^2}{\text{——}} \overset{-1}{\circ}$	§5.2.3	$p^2 2^7 3^4$
$\mathfrak{L}(A(1\vert 0)_1; r)$	$\boxplus \overset{1}{\text{——}} \overset{-1}{\bullet} \overset{r^{-1}}{\text{——}} \overset{r}{\circ}$, $r \in \mathbb{G}'_N, N > 3$	§5.2.1	$p^2 2^4 N^2$
$\mathfrak{L}(A(2\vert 0)_1; \omega)$	$\boxplus \overset{1}{\text{——}} \overset{-1}{\bullet} \overset{\omega}{\text{——}} \overset{\omega^2}{\circ} \overset{\omega}{\text{——}} \overset{\omega^2}{\circ}$	§5.2.4	$p^2 2^8 3^9$
$\mathfrak{L}(D(2\vert 1); \omega)$	$\boxplus \overset{1}{\text{——}} \overset{-1}{\bullet} \overset{\omega}{\text{——}} \overset{\omega^2}{\circ} \overset{\omega^2}{\text{——}} \overset{\omega}{\circ}$	§5.2.5	$p^2 2^8 3^9$

Table 3 Finite-dimensional Nichols algebras of a pale block and a point

V	ϵ	$\widetilde{q}_{12}$	q_{22}	$\mathscr{B}(V)$	dim K	dim $\mathscr{B}(V)$
$\mathfrak{E}_p(q)$	1	1	-1	Sect. 7.1	2^p	$2^p p^2$
$\mathfrak{E}_+(q)$	-1	1	1	Sect. 7.2	$2p$	$2^3 p$
$\mathfrak{E}_-(q)$	-1	1	-1	Sect. 7.2	$2p$	$2^3 p$
$\mathfrak{E}_\star(q)$	-1	-1	-1	Sect. 7.2	$2^4 p^2$	$2^6 p^2$

2 Preliminaries

2.1 Conventions

The $\mathtt{q}$-numbers are the polynomials

$$(n)_\mathtt{q} = \sum_{j=0}^{n-1} \mathtt{q}^j, \qquad (n)^!_\mathtt{q} = \prod_{j=1}^{n} (j)_\mathtt{q}, \qquad \binom{n}{i}_\mathtt{q} = \frac{(n)^!_\mathtt{q}}{(n-i)^!_\mathtt{q} (i)^!_\mathtt{q}} \in \mathbb{Z}[\mathtt{q}],$$

$n \in \mathbb{N}$, $0 \le i \le n$. If $q \in \Bbbk$, then $(n)_q$, $(n)^!_q$, $\binom{n}{i}_q$ denote the evaluations of $(n)_\mathtt{q}$, $(n)^!_\mathtt{q}$, $\binom{n}{i}_\mathtt{q}$ at $\mathtt{q} = q$.

Let $\mathbb{G}_N$ be the group of N-th roots of unity, and $\mathbb{G}'_N$ the subset of primitive roots of order N; $\mathbb{G}_\infty = \bigcup_{N \in \mathbb{N}} \mathbb{G}_N$. All the vector spaces, algebras, and tensor products are over $\Bbbk$.

All Hopf algebras have bijective antipode.

2.2 Yetter-Drinfeld Modules

Let Γ be an abelian group. We denote by $\widehat{\Gamma}$ the group of characters of Γ. The category ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ of Yetter-Drinfeld modules over the group algebra $\Bbbk\Gamma$ consists of Γ-graded Γ-modules, the Γ-grading being denoted by $V = \oplus_{g\in\Gamma} V_g$; that is, $hV_g = V_g$ for all $g, h \in \Gamma$. If $g \in \Gamma$ and $\chi \in \widehat{\Gamma}$, then the one-dimensional vector space $\Bbbk_g^\chi$, with action and coaction given by g and χ, is in ${}^H_H\mathcal{YD}$. Let $W \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ and $(w_i)_{i\in I}$ a basis of W consisting of homogeneous elements of degree g_i, $i \in I$, respectively. Then there are skew-derivations ∂_i, $i \in I$, of $T(W)$ such that for all $x, y \in T(W)$, $i, j \in I$

$$\partial_i(w_j) = \delta_{ij}, \qquad \partial_i(xy) = \partial_i(x)(g_i \cdot y) + x\partial_i(y). \tag{2.1}$$

For a definition of Yetter-Drinfeld modules over arbitrary Hopf algebras we refer, e.g., to [R, 11.6].

2.3 Nichols Algebras

Nichols algebras are graded Hopf algebras $\mathscr{B} = \oplus_{n\geq 0}\mathscr{B}^n$ in ${}^H_H\mathcal{YD}$ coradically graded and generated in degree one. They are completely determined by $V := \mathscr{B}^1 \in {}^H_H\mathcal{YD}$ and it is customary to denote $\mathscr{B} = \mathscr{B}(V)$. If $W \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ as in Sect. 2.2, then the skew-derivations ∂_i induce skew-derivations on $\mathscr{B}(W)$. Moreover, an element $w \in \mathscr{B}^k(W)$, $k \geq 1$, is zero if and only if $\partial_i(w) = 0$ in $\mathscr{B}(W)$ for all $i \in I$. A pre-Nichols algebra of V is a graded Hopf algebra in ${}^H_H\mathcal{YD}$ generated in degree one, with the one-component isomorphic to V.

Example 2.1 Let V be of dimension 1 with braiding $c = \epsilon\,\mathrm{id}$. Let N be the smallest natural number such that $(N)_\epsilon = 0$. Then $\mathscr{B}(V) = \Bbbk[T]/\langle T^N\rangle$, or $\mathscr{B}(V) = \Bbbk[T]$ if such N does not exist.

A braided vector space V is of diagonal type if there exists a basis $(x_i)_{i\in\mathbb{I}_\theta}$ of V and $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta} \in \Bbbk^{\theta\times\theta}$ such that $q_{ij} \neq 0$ and $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all $i, j \in \mathbb{I} = \mathbb{I}_\theta$. Given a braided vector space V of diagonal type with a basis (x_i), we denote in $T(V)$, or $\mathscr{B}(V)$, or any intermediate Hopf algebra,

$$x_{ij} = (\mathrm{ad}_c\, x_i)\, x_j, \qquad x_{i_1 i_2 \dots i_M} = (\mathrm{ad}_c\, x_{i_1})\, x_{i_2\dots i_M}, \tag{2.2}$$

for $i, j, i_1, \dots, i_M \in \mathbb{I}$, $M \geq 2$. A braided vector space V of diagonal type is of Cartan type if there exists a generalized Cartan matrix $\mathbf{a} = (a_{ij})$ such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$ for all $i \neq j$.

Theorem 2.2 *If V is of Cartan type with matrix $\mathbf{a}$ that is not finite, then* $\dim \mathscr{B}(V) = \infty$.

Proof The argument in [AAH2, Proposition 3.1] is characteristic-free and applies here because there are infinite real roots in the root system of **a**. □

3 Blocks

We consider braided vector spaces $\mathcal{V}(\epsilon, 2)$ with braiding (1.1), $\epsilon^2 = 1$.

3.1 The Jordan Plane

Here we deal with $\mathcal{V} = \mathcal{V}(1, 2)$. In characteristic 0, $\mathscr{B}(\mathcal{V})$ is the well-known algebra presented by x_1 and x_2 with the relation (3.1). In positive characteristic, $\mathscr{B}(\mathcal{V})$ is a truncated version of that algebra.

Lemma 3.1 ([CLW]) *$\mathscr{B}(\mathcal{V})$ is presented by generators x_1, x_2 and relations*

$$x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, \tag{3.1}$$

$$x_1^p, \tag{3.2}$$

$$x_2^p. \tag{3.3}$$

Also $\dim \mathscr{B}(\mathcal{V}) = p^2$ *and* $\{x_1^a x_2^b : 0 \le a, b < p\}$ *is a basis of* $\mathscr{B}(\mathcal{V})$. □

In characteristic 2, the relations of $\mathscr{B}(\mathcal{V})$ are different.

Let $\Gamma = \mathbb{Z}/p = \langle g \rangle$. We realize $\mathcal{V}$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by $g \cdot x_1 = x_1$, $g \cdot x_2 = x_2 + x_1$, $\deg x_i = g$, $i \in \mathbb{I}_2$. Thus the Hopf algebra $\mathscr{B}(\mathcal{V})\#\Bbbk\Gamma$ has dimension p^3.

3.2 The Super Jordan Plane

Let $\mathcal{V} = \mathcal{V}(-1, 2)$ be the braided vector space with basis x_1, x_2 and braiding

$$c(x_i \otimes x_1) = -x_1 \otimes x_i, \quad c(x_i \otimes x_2) = (-x_2 + x_1) \otimes x_i, \quad i \in \mathbb{I}_2. \tag{3.4}$$

Let g be a generator of the cyclic group $\mathbb{Z}$. We realize $\mathcal{V}(-1, 2)$ in ${}^{\Bbbk\mathbb{Z}}_{\Bbbk\mathbb{Z}}\mathcal{YD}$ by $g \cdot x_1 = -x_1$, $g \cdot x_2 = -x_2 + x_1$, $\deg x_i = g$, $i \in \mathbb{I}_2$. As in (2.2),

$$x_{21} = (\operatorname{ad}_c x_2)\, x_1 = x_2x_1 + x_1x_2. \tag{3.5}$$

The Nichols algebra $\mathscr{B}(\mathcal{V}(-1,2)) = T(\mathcal{V}(-1,2))/\mathcal{J}(\mathcal{V}(-1,2))$ (called the super Jordan plane) was studied in [AAH1, 3.3] over fields of characteristic 0. Assuming $p > 2$, the basic features of $\mathscr{B}(\mathcal{V}(-1,2))$ are summarized here:

Proposition 3.2 *The defining ideal $\mathcal{J}(\mathcal{V}(-1,2))$ is generated by*

$$x_1^2, \tag{3.6}$$

$$x_2x_{21} - x_{21}x_2 - x_1x_{21}, \tag{3.7}$$

$$x_{21}^p, \tag{3.8}$$

$$x_2^{2p}. \tag{3.9}$$

The set $B = \{x_1^a x_{21}^b x_2^c : a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,p-1}, c \in \mathbb{I}_{0,2p-1}\}$ is a basis of $\mathscr{B}(\mathcal{V})$ and $\dim \mathscr{B}(\mathcal{V}) = 4p^2$.

Proof Since $\partial_2(x_{21}) = 0$, we have that $\partial_2(x_{21}^n) = 0$ for every $n \in \mathbb{N}$. Also $\partial_1(x_{21}) = x_1$ and $g \cdot x_{21} = x_{21}$. Both (3.6) and (3.7) are 0 in $\mathscr{B}(\mathcal{V})$ being annihilated by ∂_1 and ∂_2, cf. (2.1). From (3.6) and (3.7) we see that in $\mathscr{B}(\mathcal{V})$

$$x_2^2x_1 = x_1(x_2^2 + x_{21}), \tag{3.10}$$

$$x_{21}x_1 = x_1x_2x_1 = x_1x_{21}. \tag{3.11}$$

By the preceding, we have

$$\partial_1(x_{21}^n) = \sum_{1\le i\le n} x_{21}^{i-1}\partial_1(x_{21})x_{21}^{n-i} = nx_1x_{21}^{n-1}.$$

Hence $x_{21}^p = 0$, i.e., (3.8) holds. Next we prove (3.9). Clearly $\partial_1(x_2^n) = 0$ for every $n \in \mathbb{N}$. We observe that

$$g \cdot x_2^2 = (-x_2 + x_1)^2 = x_2^2 - x_{21}, \qquad \partial_2(x_2^2) = g \cdot x_2 + x_2 = x_1.$$

Setting for simplicity $a := x_{21}$ and $b := x_2^2$, we have for any $n \in \mathbb{N}$:

$$\partial_2(x_2^{2n}) = \sum_{1\le i\le n} x_2^{2(i-1)}\partial_2(x_2^2)\left(g \cdot x_2^{2(n-i)}\right) = \sum_{1\le i\le n} b^{i-1}x_1\,(b-a)^{n-i}.$$

By (3.7), (3.10), and (3.11) we have

$$ax_1 = x_1a, \qquad bx_1 = x_1(b+a), \qquad ba = a(a+b), \tag{3.12}$$

hence

$$\partial_2(x_2^{2n}) = x_1 \sum_{1\le i\le n} (b+a)^{i-1}\,(b-a)^{n-i}\,.$$

We prove recursively that for all $n \in \mathbb{N}$

$$(b-a)^n = b^n - nab^{n-1}, \quad A_n := \sum_{i\in\mathbb{I}_n} (b+a)^{i-1}\,(b-a)^{n-i} = nb^{n-1}. \tag{3.13}$$

The case $n = 1$ is evident. We start with the first identity:

$$\begin{aligned}(b-a)^{n+1} &= (b-a)\,(b-a)^n = (b-a)(b^n - nab^{n-1})\\ &= b^{n+1} - nbab^{n-1} - ab^n + na^2b^{n-1} = b^{n+1} - (n+1)ab^n\end{aligned}$$

as desired. For the second identity we use the first:

$$A_{n+1} = (b-a)^n + (b+a)A_n = b^n - nab^{n-1} + (b+a)nb^{n-1} = (n+1)b^n.$$

The claim is proved; summarizing we have

$$\partial_2(x_2^{2n}) = nx_1x_2^{2(n-1)}. \tag{3.14}$$

In particular, this implies (3.9).

We now argue as in [AAH1, 3.3.1]. The quotient $\widetilde{\mathcal{B}}$ of $T(\mathcal{V})$ by (3.6)–(3.9) projects onto $\mathscr{B}(\mathcal{V})$ and the subspace I spanned by B is a left ideal of $\widetilde{\mathcal{B}}$, by (3.7), (3.11). Since $1 \in I$, $\widetilde{\mathcal{B}}$ is spanned by B. To prove that $\widetilde{\mathcal{B}} \simeq \mathscr{B}(\mathcal{V})$, we just need to show that B is linearly independent in $\mathscr{B}(\mathcal{V})$. We claim that this is equivalent to prove that $B' = \{x_2^c x_{21}^b x_1^a : a \in \{0,1\}, b \in \mathbb{I}_{0,p-1}, c \in \mathbb{I}_{0,2p-1}\}$ is linearly independent. Indeed, $\widetilde{\mathcal{B}}$ is spanned by B' since the subspace spanned B' is also a left ideal; if B' is linearly independent, then the dimension of $\widetilde{\mathcal{B}}$ is $4p^2$, so B should be linearly independent and vice versa. Suppose that there is a non-trivial linear combination of elements of B' in $\mathscr{B}(\mathcal{V})$ of minimal degree. As

$$\partial_1(x_2^c x_{21}^b) = b\, x_2^c x_{21}^{b-1} x_1, \qquad \partial_1(x_2^c x_{21}^b x_1) = x_2^c x_{21}^b, \tag{3.15}$$

such linear combination does not have terms with a or b greater than 0. We claim that the elements x_2^c, $c \in \mathbb{I}_{0,2p-1}$, are linearly independent, yielding a contradiction. By homogeneity it is enough to prove that they are $\neq 0$. If c is even this follows from (3.14). If $c = 2n + 1$ with $n < p$, then

$$\partial_2(x_2^{2n+1}) = \partial_2(x_2^{2n})g \cdot x_2 + x_2^{2n} = -nx_1x_2^{2n-1} + x_2^{2n}.$$

Again a degree argument gives the desired claim. □

Let $\Gamma = \mathbb{Z}/2p$. We may realize $\mathcal{V}(-1,2)$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by the same formulas as above; thus $\mathscr{B}(\mathcal{V})\#\Bbbk\Gamma$ is a pointed Hopf algebra of dimension $8p^3$.

3.3 Realizations

Let H be a Hopf algebra. A *YD-pair* for H is a pair $(g, \chi) \in G(H) \times \mathrm{Hom}_{\mathrm{alg}}(H, \Bbbk)$ such that

$$\chi(h)\, g = \chi(h_{(2)})h_{(1)}\, g\, \mathcal{S}(h_{(3)}), \qquad h \in H. \tag{3.16}$$

Let $\Bbbk_g^\chi$ be a one-dimensional vector space with H-action and H-coaction given by χ and g respectively; then (3.16) says that $\Bbbk_g^\chi \in {}^H_H\mathcal{YD}$.

If $\chi \in \mathrm{Hom}_{\mathrm{alg}}(H, \Bbbk)$, then the space of (χ, χ)-derivations is

$$\mathrm{Der}_{\chi,\chi}(H, \Bbbk) = \{\eta \in H^* : \eta(h\ell) = \chi(h)\eta(\ell) + \chi(\ell)\eta(h)\ \forall h, \ell \in H\}.$$

A *YD-triple* for H is a collection (g, χ, η) where (g, χ) is a YD-pair for H, $\eta \in \mathrm{Der}_{\chi,\chi}(H, \Bbbk)$, $\eta(g) = 1$ and

$$\eta(h)g_1 = \eta(h\_2)h\_1g_2\mathcal{S}(h\_3), \qquad h \in H. \tag{3.17}$$

Given a YD-triple (g, χ, η) we define $\mathcal{V}_g(\chi, \eta) \in {}^H_H\mathcal{YD}$ as the vector space with a basis $(x_i)_{i\in\mathbb{I}_2}$, whose H-action and H-coaction are given by

$$h \cdot x_1 = \chi(h)x_1, \quad h \cdot x_2 = \chi(h)x_2 + \eta(h)x_1, \quad \delta(x_i) = g \otimes x_i, \quad h \in H,\ i \in \mathbb{I}_2;$$

the compatibility is granted by (3.16), (3.17). As a braided vector space, $\mathcal{V}_g(\chi, \eta) \simeq \mathcal{V}(\epsilon, 2)$, $\epsilon = \chi(g)$.

Consequently, if H is finite-dimensional and $\epsilon^2 = 1$, then $\mathscr{B}(\mathcal{V}_g(\chi, \eta))\#H$ is a Hopf algebra satisfying

$$\dim\left(\mathscr{B}(\mathcal{V}_g(\chi, \eta))\#H\right) = \begin{cases} p^2 \dim H, & \text{when } \epsilon = 1, \\ 4p^2 \dim H, & \text{when } \epsilon = -1. \end{cases} \tag{3.18}$$

3.4 Exhaustion in Rank 2

We recall some facts from [AAH1, §3.4].

Let H be a Hopf algebra with bijective antipode and $V \in {}^H_H\mathcal{YD}$. Let $0 = V_0 \subsetneq V_1 \cdots \subsetneq V_d = V$ be a flag of Yetter-Drinfeld submodules with $\dim V_i = \dim V_{i-1} + 1$ for all i. Then $V^{\mathrm{diag}} := \mathrm{gr}\, V$ is of diagonal type. If $\mathscr{B}$ is a pre-Nichols algebra of V, then it is a graded filtered Hopf in ${}^{H}_{H}\mathcal{YD}$ and $\mathcal{B}^{\text{diag}} := \operatorname{gr}\mathscr{B}$ is a pre-Nichols algebra of V^{diag}.

Proposition 3.3 *Let* $\epsilon \in \Bbbk^{\times}$. *If* $\dim \mathscr{B}(\mathcal{V}(\epsilon, 2)) < \infty$, *then* $\epsilon^2 = 1$.

Proof Let $\mathcal{V} = \mathcal{V}(\epsilon, 2)$; it has a flag as above and $\mathcal{V}^{\text{diag}}$ is the braided vector space of diagonal type with matrix $(q_{ij})_{i,j\in\mathbb{I}_2}$, $q_{ij} = \epsilon$ for all $i, j \in \mathbb{I}_2$. Hence

$$\dim \mathscr{B}(\mathcal{V}^{\text{diag}}) \leq \dim \mathscr{B}(\mathcal{V}(\epsilon, 2)). \tag{3.19}$$

Step 1 If $\epsilon \notin \mathbb{G}_\infty$, then $\dim \mathscr{B}(\mathcal{V}(\epsilon, 2)) = \infty$.

Proof Here $\dim \mathscr{B}(\mathcal{V}^{\text{diag}}) = \infty$ by Example 2.1 and (3.19) applies. □

Step 2 If $\epsilon \in \mathbb{G}'_N$, $N \geq 4$, then $\dim \mathscr{B}(\mathcal{V}(\epsilon, \ell)) = \infty$ for all $\ell \geq 2$.

Proof Here $\mathcal{V}^{\text{diag}}$ is of Cartan type with Cartan matrix $\begin{pmatrix} 2 & 2-N \\ 2-N & 2 \end{pmatrix}$. Thus Theorem 2.2 and (3.19) apply. □

Step 3 Let $\epsilon \in \mathbb{G}'_3$. Then $\dim \mathscr{B}(\mathcal{V}(\epsilon, 2)) = \infty$.

Proof The proof of [AAH1, §3.5 – Step 3] holds verbatim. □

The Proposition is proved. □

4 One Block and One Point

4.1 The Setting and the Statement

Let $(q_{ij})_{1\leq i,j\leq 2}$ be a matrix of invertible scalars and $a \in \Bbbk$. We assume that $\epsilon := q_{11}$ satisfies $\epsilon^2 = 1$. Let V be a braided vector space with a basis $(x_i)_{i\in\mathbb{I}_3}$ and a braiding given by

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} = \begin{pmatrix} \epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 & q_{12}x_3 \otimes x_1 \\ \epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 & q_{12}x_3 \otimes x_2 \\ q_{21}x_1 \otimes x_3 & q_{21}(x_2 + ax_1) \otimes x_3 & q_{22}x_3 \otimes x_3 \end{pmatrix}. \tag{4.1}$$

Let $V_1 = \langle x_1, x_2 \rangle$ (the block) and $V_2 = \langle x_3 \rangle$ (the point). Let $\Gamma = \mathbb{Z}^2$ with canonical basis g_1, g_2. We realize (V, c) in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ as $\mathcal{V}_{g_1}(\chi_1, \eta) \oplus \Bbbk^{\chi_2}_{g_2}$ with suitable χ_1, χ_2, and η, where $V_1 = \mathcal{V}_{g_1}(\chi_1, \eta)$, while $V_2 = \Bbbk^{\chi_2}_{g_2}$. Thus $V_1 \simeq \mathcal{V}(\epsilon, 2)$; thus we use the notations and results from Sect. 3.2.

The *interaction* between the block and the point is $q_{12}q_{21}$; it is

weak if $q_{12}q_{21} = 1$, mild if $q_{12}q_{21} = -1$, strong if $q_{12}q_{21} \notin \{\pm 1\}$.

In characteristic 0, we introduced a normalized version of a called the ghost, which is discrete when it belongs to $\mathbb{N}$. In our context, $p > 2$, we need a variant of this notion. First we say that V has *discrete ghost* if $a \in \mathbb{F}_p^\times$. When this is the case, we pick a representative $\mathtt{r} \in \mathbb{Z}$ of $2a$ by imposing

$$\mathtt{r} \in \begin{cases} \{1-p, \dots, -1\}, & \epsilon = 1, \\ \{1, \dots, 2p-1\} \cap 2\mathbb{Z}, & \epsilon = -1; \end{cases} \qquad \text{set } \mathscr{G} := \begin{cases} -\mathtt{r}, & \epsilon = 1, \\ \mathtt{r}, & \epsilon = -1. \end{cases} \tag{4.2}$$

Then $\mathscr{G}$ is called the *ghost*. In this Section we consider the following braided vectors spaces with braiding (4.1), where the ghost is discrete and $q_{22} \in \mathbb{G}_\infty$:

$$\begin{aligned} &\mathfrak{L}(q_{22}, \mathscr{G}): && \text{weak interaction}, && \epsilon = 1; \\ &\mathfrak{L}_-(q_{22}, \mathscr{G}): && \text{weak interaction}, && \epsilon = -1; \\ &\mathfrak{C}_1: && \text{mild interaction}, && \epsilon = q_{22} = -1, \quad \mathscr{G} = 1. \end{aligned}$$

In this Section, we shall prove part (a) of Theorem 1.2.

Theorem 4.1 *Let V be a braided vector space with braiding* (4.1)*. If V is as in Table 1, then* $\dim \mathscr{B}(V) < \infty$.

To prove the Theorem, we consider $K = \mathscr{B}(V)^{\operatorname{co} \mathscr{B}(V_1)}$. By [HS, Proposition 8.6], $\mathscr{B}(V) \simeq K \# \mathscr{B}(V_1)$ and K is the Nichols algebra of

$$K^1 = \operatorname{ad}_c \mathscr{B}(V_1)(V_2). \tag{4.3}$$

Now $K^1 \in {}^{\mathscr{B}(V_1)\#\Bbbk\Gamma}_{\mathscr{B}(V_1)\#\Bbbk\Gamma}\mathcal{YD}$ with the adjoint action and the coaction given by

$$\delta = (\pi_{\mathscr{B}(V_1)\#\Bbbk\Gamma} \otimes \operatorname{id})\Delta_{\mathscr{B}(V)\#\Bbbk\Gamma}. \tag{4.4}$$

In order to describe K^1, we set

$$z_n := (\operatorname{ad}_c x_2)^n x_3, \qquad n \in \mathbb{N}_0. \tag{4.5}$$

4.2 Weak Interaction

Here $q_{12}q_{21} = 1$. In general,

$$c^2_{|V_1 \otimes V_2} = \operatorname{id} \iff q_{12}q_{21} = 1 \text{ and } a = 0. \tag{4.6}$$