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Frédéric Barbaresco
Frank Nielsen *Editors*

Geometric Structures of Statistical Physics, Information Geometry, and Learning

SPIGL'20, Les Houches, France, July 27–31

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Preface

Geometric Structures of statistical Physics, Information Geometry, and Learning

Ecole de Physique des Houches SPIGL'20 Summer Week

SPRINGER Proceedings in Mathematics & Statistics, 2021

Subject

This book is proceedings of Les Houches Summer Week SPIGL'20 (Joint Structures and Common Foundation of **S**tatistical **P**hysics, **I**nformation Geometry and Inference for **L**earning) organized from July 27–31, 2020, at L'Ecole de Physique des Houches:

Website https://franknielsen.github.io/SPIG-LesHouches2020/
Videos: https://www.youtube.com/playlist?list=PLo9ufcrEqwWExTBPgQPJwAJhoUChMbROr
The conference SPIGL'20 has developed the following topics:

Geometric Structures of Statistical Physics and Information

- Statistical mechanics and geometric mechanics
- Thermodynamics, symplectic and contact geometries
- Lie groups thermodynamics
- Relativistic and continuous media thermodynamics
- Symplectic integrators

Physical Structures of Inference and Learning

Stochastic gradient of Langevin's dynamics
Information geometry, Fisher metric, and natural gradient
Monte Carlo Hamiltonian methods
Variational inference and Hamiltonian controls
Boltzmann machine

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Scientific Rational

In the middle of the last century, Léon Brillouin in “The Science and The Theory of Information” or André Blanc-Lapierre in “Statistical Mechanics” forged the first links between the theory of information and statistical physics as precursors.

In the context of artificial intelligence, machine learning algorithms use more and more methodological tools coming from the physics or the statistical mechanics. The laws and principles that underpin this physics can shed new light on the conceptual basis of artificial intelligence. Thus, the principles of maximum entropy, minimum of free energy, Gibbs–Duhem’s thermodynamic potentials and the generalization of François Massieu’s notions of characteristic functions enrich the variational formalism of machine learning. Conversely, the pitfalls encountered by artificial intelligence to extend its application domains question the foundations of statistical physics, such as the construction of stochastic gradient in large dimension, the generalization of the notions of Gibbs densities in spaces of more elaborate representation like data on homogeneous differential or symplectic manifolds, Lie groups, graphs, and tensors.

Sophisticated statistical models were introduced very early to deal with unsupervised learning tasks related to Ising–Potts models (the Ising–Potts model defines the interaction of spins arranged on a graph) of statistical physics and more generally the Markov fields. The Ising models are associated with the theory of mean fields (study of systems with complex interactions through simplified models in which the action of the complete network on an actor is summarized by a single mean interaction in the sense of the mean field).

The porosity between the two disciplines has been established since the birth of artificial intelligence with the use of Boltzmann machines and the problem of robust methods for calculating partition function. More recently, gradient algorithms for neural network learning use large-scale robust extensions of the natural gradient of Fisher-based information geometry (to ensure reparameterization invariance), and stochastic gradient based on the Langevin equation (to ensure regularization), or their coupling called “natural Langevin dynamics”.

Concomitantly, during the last fifty years, statistical physics has been the object of new geometrical formalizations (contact or symplectic geometry, ...) to try to give a new covariant formalization to the thermodynamics of dynamic systems. We can mention the extension of the symplectic models of geometric mechanics to statistical mechanics, or other developments such as random mechanics, geometric mechanics in its stochastic version, Lie groups thermodynamics, and geometric modeling of phase transition phenomena.

Finally, we refer to computational statistical physics, which uses efficient numerical methods for large-scale sampling and multimodal probability measurements (sampling of Boltzmann–Gibbs measurements and calculations of free energy, metastable dynamics and rare events, ...) and the study of geometric integrators (Hamiltonian dynamics, symplectic integrators, ...) with good properties of covariances and stability (use of symmetries, preservation of invariants, ...).

Machine learning inference processes are just beginning to adapt these new integration schemes and their remarkable stability properties to increasingly abstract data representation spaces.

Artificial intelligence currently uses only a very limited portion of the conceptual and methodological tools of statistical physics. The purpose of this conference is to encourage constructive dialog around a common foundation, to allow the establishment of new principles and laws governing the two disciplines in a unified approach. However, it is also about exploring new chemins de traverse.

Joint Structures and Common Foundations of Statistical Physics, Information Geometry and Inference for Learning
(SPIGL July 27-31th 2020, Les Houches, France)

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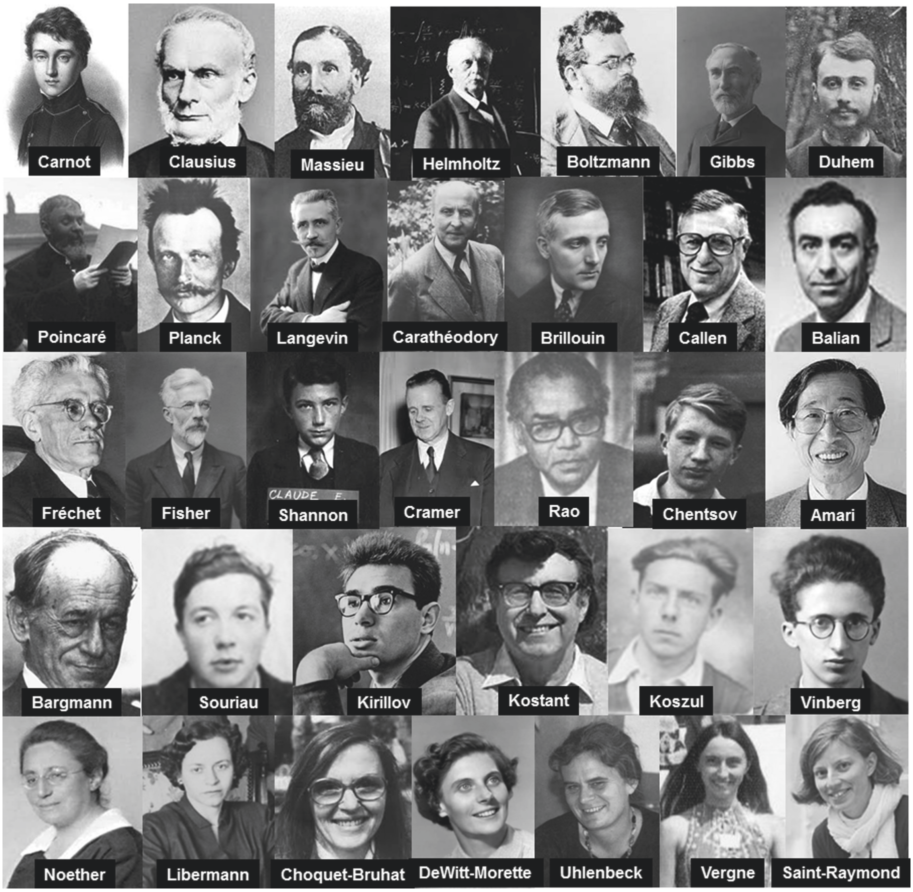

April 2021

Frédéric Barbaresco
Frank Nielsen

Contents

Part I: Tribute to Jean-Marie Souriau Seminal Works

Part II: Lie Group Geometry and Diffeological Model of Statistical Physics and Information Geometry

Part III: Advanced Geometrical Models of Statistical Manifolds in Information Geometry

Part IV: Geometric Structures of Mechanics, Thermodynamics and Inference for Learning

Part V: Hamiltonian Monte Carlo, HMC Sampling and Learning on Manifolds

Part I: Tribute to Jean-Marie Souriau
Seminal Works

Structure des Systèmes Dynamiques Jean-Marie Souriau's Book 50th Birthday

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Abstract. Jean-Marie Souriau's book "Structure des systèmes dynamiques", published in 1970, republished recently by Gabay, translated in English and published under the title "Structure of Dynamical Systems, a Symplectic View of Physics", is a work with an exceptional wealth which, fifty years after its publication, is still topical. In this paper, we give a brief description of its content and we intend to highlight the ideas that to us, are the most creative and promising.

1 A Few Introductory Words

Graduated of the Ecole Normale Supérieure in 1942, Jean-Marie Souriau obtains the aggregation of mathematics in 1945, ranked second. After a brief passage at the CNRS, he joins as an engineer the ONERA recently created and prepares his Ph.D thesis on the airplane stability. Defended in 1952, this work was used for the design of several airplanes, in particular the Concorde. Next he obtains a position as Professor at the Tunis Faculty in 1952 and at Aix-Marseille University in 1964, where he published "Structure des Systèmes Dynamique". For more details on Jean-Marie Souriau's life and personality from his friends and persons who have known him very well, you can read the paper paying hommage to him in [9].

Jean-Marie Souriau's book "Structure des systèmes dynamiques" [6], was published in 1970 in a book collection for students in the first year of master's degree in Mathematics, directed in fact to mathematicians, beginners or experienced, wishing to know the applications of mathematics to physical sciences, and to physicists concerned with knowing certain mathematical tools useful for their researches. The author was very aware of this since, in his Introduction, he gives reading recommendations adapted to both reader categories. The book is illustrated by many figures which mostly are very meaningful schematic representations of the geometric constructions used by the author. The notations are often very non standard but they are totally consistent.

The original edition is out of print, although it is possible to find copies of second hand. Nevertheless, it was republished recently by Gabay. It was also translated in English and published under the title "Structure of Dynamical

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Systems, a Symplectic View of Physics" [7]. The traduction was supervised by Richard Cushman and Gijs Tuynman who have known very well the author. It is also possible to download a scanned version from Jean-Marie's Souriau official website [8], and a lot of other interesting thinks.

2 Introduction

The book begins by a large introduction of 20 pages in which Souriau highlights the guiding ideas. He considers the classical analytical mechanics, that originated in the book by Jean-Louis Lagrange *Mécanique analytique* [3] and remains an essential ingredient of the current physical theories, is not outdated although certain used concepts have become so because they have not the required covariance, in other words because they are in contradiction with Galilean relativity. He wishes to show in his book that a better consideration of Lagrange's thought allows giving to this theory a form compatible with the more recent discoveries of the physical sciences.

3 Chapter I: Differential Geometry

In the first chapter, Souriau presents in less than 70 pages numerous tricky notions from differential manifolds to Lie groups and calculus of variation. The author presents the concept of differential manifold in a rather original way, bypassing the one of topological space, undoubtedly to make this notion easily accessible to beginning students. The differential manifolds are not supposed Hausdorff. Examples of non Hausdorff manifolds such as the space of motions of certain mechanical systems will be indeed encountered in Sect. 5.

Note author's language particularity that could cause confusion: he calls embedding what most of geometers call injective immersion. This was corrected in the English version. This choice is very reasonable because injective immersions are much more frequently encountered than embeddings. For example, orbits of a Lie group action, as well as leaves of a foliation, always are immersed in the manifold in which they are contained, and much more rarely embedded.

4 Chapter II: Symplectic Geometry

The second chapter is also essentially mathematical but shorter than the previous one (about 45 pages). It starts with algebraic notions concerning the skew-symmetric forms, then 2-forms and denoted σ. If it is non degenerate, it is called symplectic, otherwise presymplectic.

Next, Souriau defines the symplectic and presymplectic manifolds, and studies their properties. He shows that under certain conditions the quotient of a presymplectic manifold by its kernel is a symplectic manifold, result that he will use latter on to define the space of motions of a dynamical system that plays a central role in his theory. The elements of the quotient are leaves of a characteristic foliation.

Let u be a differentiable function defined on a symplectic manifold. Its differential du is a 1-form. If the form σ is symplectic, there exists a unique vector associated to du that he denotes $grad\, u$ and call symplectic gradient, such that:

$$-du \equiv \sigma(grad\, u)$$

The flow of this vector field let the symplectic form invariant. Let us remark in passing the non standard notation for the interior product $\sigma(grad\, u)$ of the vector $grad\, u$ and the form σ.

Souriau calls dynamical group (that other authors call symplectic group) of a symplectic or presymplectic manifold V a Lie group G acting on it by canonical transformations. He calls moment of the dynamical group G a differentiable map ψ from a point x of V onto an element μ of the vector space $\mathcal{G}^*$, dual of the Lie algebra $\mathcal{G}$ of G, such that for every generator $Z \in \mathcal{G}$ its infinitesimal action is the symplectic gradient of the function linking, at every $x \in V$ the real $\mu \cdot Z$:

$$\sigma(Z_V(x)) \equiv -d\,[\mu \cdot Z]$$

In the terminology used by most of geometers, this infinitesimal generator is the Hamiltonian vector space, the corresponding Hamiltonian being this function of value $\mu \cdot Z$.

The author gives several examples of dynamical groups, indicates suffisant conditions for a dynamical group admitting a moment and studies its properties. This deals him to propose a generalization of Noether's theorem.

We arrive to one of the most original Souriau's theories, the symplectic cohomology of Lie Groups.

In one hand, we let the element a act on x, next we take the value of the moment map. On the other hand, we take the value μ of the moment map ψ, next we apply the coadjoint representation of the group for the element a of G. Souriau proved that the difference between the two previous results does not depends on the current point x on the manifold, then only on a. This defines a map:

$$\theta(a) \equiv \psi(\underline{a}_V(x)) - \underline{a}_{\mathcal{G}^*}(\psi(x))$$

from the group to the dual of its Lie algebra. It verifies an identity that means that the group acts on the moment space by affine representation with linear part the coadjoint action $\underline{a}_{\mathcal{G}^*}$ and translation part $\theta(a)$:

$$\theta(a \times b) \equiv \theta(a) + \underline{a}_{\mathcal{G}^*}(\theta(b))$$

The differential of θ at the identity, denoted f, is a bilinear form on the dual and, even less obvious, it is skew-symmetric, then a 2-form and it verifies an identity that generalizes Jacobi's one:

$$f(Z)([Z', Z'']) + f(Z')([Z'', Z]) + f(Z'')([Z, Z']) = 0$$

The map θ measures the defect of symplectic cohomology of the dynamical group. It is called symplectic cocycle. It is possible to define also symplectic coboundaries, and passing to the quotient to define classes of symplectic cohomology.

We arrive to another fundamental result called now Kirillov-Kostant-Souriau theorem. In fact, it is not necessary to study case-by-case the action a dynamical group on every symplectic manifold. You can disregard the manifold and take interest on the action of the group onto the dual of its Lie algebra. Then, on each orbit U, there exists a canonical symplectic structure and the identity map of the orbit is a moment map. In other words, we may say that each point μ of the orbit U is its own moment.

5 Chapter III: Mechanics

The starting point is the following 2-form that Souriau attributes to Lagrange. The Lagrange form of a system of material points is the sum of the Lagrange forms of all points of the system, where m_j is the mass of particle j, $\boldsymbol{r}_j$ its position, $\boldsymbol{v}_j$ its velocity and $\boldsymbol{F}_j$ the resultant of the forces acting upon it:

$$\sigma = \sum_j \left(\left\langle m_j \delta \boldsymbol{v}_j - \boldsymbol{F}_j \delta t, \delta' \boldsymbol{r}_j - \boldsymbol{v}_j \delta' t \right\rangle - \left\langle m_j \delta' \boldsymbol{v}_j - \boldsymbol{F}_j \delta' t, \delta \boldsymbol{r}_j - \boldsymbol{v}_j \delta t \right\rangle \right)$$

that could written:

$$\sigma = \sum_j \left(m_j d\boldsymbol{v}_j - \boldsymbol{F}_j dt \right) \dot{\wedge} \left(d\boldsymbol{r}_j - \boldsymbol{v}_j dt \right)$$

where the symbol $\dot{\wedge}$ denotes the operator combining the dot product and the exterior product. You can observe here one of the peculiarities of Souriau's notations. He does not use the symbol $\wedge$ for the exterior product of differential forms, which has some advantages but also drawbacks. Then, the well-known equations of motion can be written in a very compact manner: the tangent vector to the trajectory in the evolution space of the positions, velocities and time belongs to the kernel of the presymplectic form.

Next, Souriau defines the vector $\boldsymbol{E}_j$ in terms of the force $\boldsymbol{F}_j$ occurring in Lagrange 2-forms and a vector $\boldsymbol{B}_j$ freely chosen:

$$\boldsymbol{E}_j \equiv \boldsymbol{F}_j + \boldsymbol{B}_j \times \boldsymbol{v}_j$$

This leads him to formulate what he calls Maxwell's principle in the following form: the vector fields $\boldsymbol{B}_j$ may be chosen in such a way that the Lagrange form is closed. Hence this condition determines these fields in a unique manner. Stating this principle, the author proves that:

1. the fields $\boldsymbol{E}_j$ and $\boldsymbol{B}_j$ must not depend on the velocities of the material points
2. $\boldsymbol{B}_j$ does not depends on the position of other particles than j
3. They verify a condition in which we recognize Maxwell reciprocity principle (in fact a version of the action and reaction principle for forces acting at distance)

4. And they verify two equations in which we recognize Faraday equation and the law of induction (or absence of magnetic monopoles). By contrast, Ampere and Gauss equations are relativistic effects which do not appear.

Souriau adopts Maxwell's principle as a new law of the mechanics (not only of the electromagnetism) and presents various examples: the N-body problem of Celestial mechanics, the gravitation, where $\boldsymbol{E}$ is the usual gravity force and $\boldsymbol{B} \times \boldsymbol{v}$ is Coriolis' force, and of course the electromagnetic field. Without denying the importance of the principle of least action or even the usefulness of these formalisms, the author declares these concepts seem to him less fundamental as the Maxwell principle.

One of the originally of Souriau's work is to emphasize the role played by the space of motions. Starting from the evolution space which is equipped with a presymplectic structure, the space of motions is a symplectic manifold obtained by quotient of the foliation. The space of motions of a dynamical system is not very often considered by modern authors, though it appeared as soon as 1808 in the works of Lagrange. This very natural concept has a nice mathematical property: the space of motions is always endowed with a smooth manifold structure.

One may wonder why the concept of space of motions is not used more by modern authors. Maybe it is because for some dynamical systems, the space of motions is a non-Hausdorff manifold. Another possible explanation is that some scientists are interested in the thorough description of particular motions of a system, rather than by the study of the set of all possible motions. By showing that many important results can be deduced from the symmetries of the space of motions of a system, the author proves that this reluctance is unfounded.

Next Souriau introduce Galilei group, the symmetry group of Mechanics, a Lie group of dimension 10. defined as the set of matrices :

$$a \equiv \begin{bmatrix} A & \boldsymbol{b} & \boldsymbol{c} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix},$$

where, with his notations, A is a rotation, $\boldsymbol{b}$ the Galilean boost, $\boldsymbol{c}$ a space translation ans e a clock change. When the system is isolated, the Galilei group acts onto the evolution space and the space of motions. On the evolution space, the moment of this action is a first integral of the motion. The author details its 10 components, which can be regrouped in three vectors $\boldsymbol{p}$, $\boldsymbol{l}$, $\boldsymbol{g}$ and a scalar E. He gives their physical meaning:

1. $\boldsymbol{p}$ is the total linear momentum;
2. $\boldsymbol{l}$ is the total angular momentum;
3. the equality $\boldsymbol{g} =$ Constant conveys the fact that the center of mass moves on a straight line at constant velocity;
4. the scalar E, defined modulo an additive constant, is the total energy.

By a calculation similar to Bargmann's one, Souriau proved that the symplectic cohomology of the Galilei group is of dimension 1. He considers the number

m that spots the class of cohomology of the action of the Galilei group on the space of motions as being the mass.

In the paragraph The principles of symplectic mechanics, the author first takes place in the frame of the classical mechanics, non relativistic. He is no longer limited to the systems of material points and he adopts the three following assertions as new axioms of the mechanics:

I. The space of the motions of a dynamical system is a connected symplectic manifold.
II. If several dynamical systems evolve independently, the manifold of motions of the composite system is the symplectic direct product of the spaces of motions of the component systems.
III. If a dynamical system is isolated, its manifold of motions admits the Galilei group as a dynamical group.

It is an extension of the principles generally admitted of the classical mechanics, which will allow to the author considering new dynamical systems having a physical interest.

In special relativity, the passage of a Lorentz frame to another one is made by the action of an element of the restricted Poincaré group. The transition to the Relativistic case consists in a simple change of the symmetry group in the third axioms:

III. If a dynamical system is isolated, its manifold of motions admits the restricted Poincaré group as a dynamical group.

In a long section, the author proposes a mechanistic description of elementary particles. In the framework of relativistic Mechanics, an isolated dynamical system is said to be elementary when the Poincaré group acts transitively on the space of its motions. The moment map of its action is then a symplectic diffeomorphism of this space onto a coadjoint orbit of the Poincaré group. For him, so defined elementary systems are mathematical models for the elementary particles of physicists. Besides the classical energy-momentum P, Souriau introduces the polarization 4-vector $W = *(M) \cdot P$, built from P and the moment M associated to the infinitesimal Lorentz transformation, through a linear map $*$ transforming an anti-hermitian operator into another anti-hermitian operator. As example let us consider a particle with spin. It is when P is timelike and when W (which, being orthogonal to P, is spacelike) is non-zero. It is characterized by two invariants, the mass m stemming from the energy-momentum and the spin s from the polarization.

The classification of the coadjoint orbits of Poincaré group provides us the table of elementary particles:

I. A particule with spin (previously described)
II. A particle without spin. It is when P is timelike and $W = 0$
III. A massless particle. It is when both P and W are non-zero and lightlike. The author defines three real numbers, the sign of the energy η, the helicity χ and the spin s

The Nonrelativistic particles are obtained by scaling of lengths and times, next considering the limit when the velocity approaches zero.

6 Chapter IV: Statistical Mechanics

It is devoted to Statistical Mechanics. It contains two sections. The first one is a very condensed course in Measure Theory and Integration, with some notions in Probability. The second one presents the principles of Statistical Mechanics in a very original way based on the Lie group theory. Souriau calls generalized Gibbs probability law any completely continuous probability law of density:

$$f(x) \equiv e^{-[z+Z(\Psi(x))]}$$

where z is a scalar, Z an infinitesimal generator living in the Lie algebra of the group and Ψ is the moment map such that the law is integrable. Then, he establishes the relation between the entropy s, the Planck potential z and the mean value M of the moment Ψ:

$$s = z + Z(M)$$

The author explains that the entropy of the system increases with time, so assuming that the natural equilibria of the gas are elements of the Gibbs set of the group of time translations is a very reasonable assumption. Each Gibbs state is determined by an element Z of the one-dimensional Lie algebra of this group, which is a way of measuring the gas temperature. Since the group of time translations is a subgroup, but not a normal subgroup, of the Galilei group, a dynamical system conservative in some inertial reference frame is not conservative in a different inertial reference frame. This important remark leads the author to introduce the new concept of covariant statistical mechanics by proposing the following principle:

When a dynamical system is invariant by the action of some Lie subgroup G' of the Galilei group, its natural equilibria are the elements of the Gibbs set of the action of G' .

He then discusses in greater detail several examples, among them a gas in a centrifuge (this relative motion is now a rotation around an axis at a constant angular velocity). He considers also a system made by particles with spin, and finds that the most probable orientation of the particles spin is parallel to the rotation axis.

7 Chapter V: A Method of Quantization

The theory of geometric quantization was developed in the early 70' independently by Souriau and Kostant [2] in order to bring into focus some vaguely perceived analogies between representation theory and quantum mechanics that had for a long time been part of the "flolklore" of modern physics. The author defines a quantum manifold as a smooth manifold Y endowed with a contact 1-form ϖ, a presymplectic structure for the exterior differential σ of ϖ, such that all integral curves are the orbits of an action on Y of the 1-dimensional torus. The set of these curves, in other words the quotient of Y by this action, is a symplectic manifold U, called by the author the base of the quantum manifold Y.

In practice, the Physicist only knows the space of motion U of a classical system and he has to address the following issue: is there a quantum manifold Y and a symplectomorphism from the base of Y onto the space of motion? This is the problem of the quantization. The author proves that the manifold of motions of a non-relativistic particle with spin is quantizable if and only if the spin of the particle is integer or half integer, when expressed with $\hbar$ as unit.

Isomorphisms of quantum manifolds are called by the author quantomorphisms. Any quantomorphism between two quantum manifolds projects onto a symplectomorphism between their bases. A group Γ of quantomorphisms of a quantum manifold Y projects onto a group of symplectomorphisms of its basis U, and its projection is a group homomorphism. Conversely, a group G of symplectomorphisms of the basis U is said to be liftable if there exists a group Γ of quantomorphisms of Y which projects onto it. He then discusses the quantization of a dynamical group of a quantizable symplectic manifold U, with the quantum manifold Y as quantization. He proves that when a dynamical group of U is quantizable, its symplectic cohomology is zero. He gives examples which prove that this necessary condition is not sufficient. A dynamical group of U which is liftable, but not quantizable, may have an extension which still is a dynamical group of U and is quantizable.

All this deep analysis on geometric basis allow to give solid foundations to the well known correspondance principle. Let u be a smooth function on the space of motions U, representing an observable. Then Souriau explains how to associate to any observable u defined on the base U an operator $\hat{u}$ on the Hilbert space $\mathcal{H}(Y)$ of the state vectors Ψ of the quantum manifold Y, defined by this relation :

$$\hat{u}(\Psi)(\xi) \equiv -i \underset{u}{\delta} [\Psi(\xi)]$$

where in the right hand member, the notation $\underset{u}{\delta}$ means the derivative of Ψ in the direction of the symplectic gradient of u Then Souriau proved that:

1. The operator $\hat{u}$ is hermitian,
2. The map from u to $\hat{u}$ is linear and injective,
3. For $u = 1$, $\hat{u}$ is the identity of the Hilbert space,

and he proved the correspondance between quantum commutators and Poisson brackets (originally proposed by Dirac).

8 Conclusions

In this brief presentation, we have just skimmed over the book content of exceptional depth. A more detailled presentation of the book will be published soon in [1].

When reading this book, we cannot fail to be impressed by the extent and the thoroughness of the author's knowledge, as well in Mathematics as in Mechanics or in Physics, and by the originality and the depth of his thoughts.

We would like also to bring our attention to the fact that Jean-Marie Souriau is the author of two other very remarkable books in French, both published in 1964, « Géométrie et Relativité" [4] and "Calcul linéaire" [5]. In that respect, we would like to recall the origin of the last book. From 1958, while Souriau is still engineer at ONERA, his taste for the teaching drives him on to create a free course (and free of charge) entitled "New methods of the Mathematical physics", as shown in this poster. It is a great success, the amphitheater, even able to contain 200 persons, is full and he has to teach twice. The program of the linear algebra gave rise to the book "Calcul Linéaire". "Géométrie et Relativité" and "Calcul linéaire", which too are very rich and original, deserve —as "Structure des systèmes dynamiques"— to be read, ... and read again.

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Jean-Marie Souriau's Symplectic Model of Statistical Physics: Seminal Papers on Lie Groups Thermodynamics - Quod Erat Demonstrandum

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Abstract. The objective of this chapter is to make better known Jean-Marie Souriau works, more particularly his symplectic model of statistical physics, called "Lie groups thermodynamics". This model was initially described in chapter IV "Statistical Mechanics" of his book "Structure of dynamical systems" published in 1969. We have translated in English some parts of three Souriau's publications which provide more details about this geometric model of Thermodynamics. Entropy acquires a geometric foundation as a function parameterized by mean of moment map in dual Lie algebra, and in term of foliations. Souriau established the generalized Gibbs laws when the manifold has a symplectic form and a connected Lie group G operates on this manifold by symplectomorphisms. Souriau Entropy is invariant under the action of the group acting on the homogeneous symplectic manifold. As quoted by Souriau, these equations are universal and could be also of great interest in Mathematics.

Keywords: Symplectic geometry · Statistical physics · Moment map · Thermodynamics · Lie groups · Representation theory

1 Preamble

«Il est évident que l'on ne peut définir de valeurs moyennes que sur des objets appartenant à un espace vectoriel (ou affine); donc - si bourbakiste que puisse sembler cette affirmation - que l'on n'observera et ne mesurera de valeurs moyennes que sur des grandeurs appartenant à un ensemble possédant physiquement une structure affine. Il est clair que cette structure est nécessairement unique - sinon les valeurs moyennes ne seraient pas bien définies.» - ***Jean-Marie Souriau***

Jean-Marie Souriau has introduced his model of "Lie Groups Thermodynamics" in the chapter IV "Statistical Mechanics" of his book "Structure of Dynamical Systems" published in 1969 [1, 2]. This chapter IV remained little known for a long time, because the first readers of the work were more interested in the symplectic model introduced for classical and quantum mechanics, than its extension for Statistical Physics.

This chapter is a translation of some part from three following Jean-Marie Souriau papers where this Lie Groups Thermodynamics is developed:

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- ***Mécanique statistique, groupes de Lie et cosmologie*** [3]
- ***Géométrie Symplectique et Physique Mathématique*** [4]
- ***Mécanique Classique et Géométrie Symplectique*** [5]

These papers by Jean-Marie Souriau are little known because they were only published in French in CNRS publications. In these papers, Souriau consolidated the bedrock of what he called "***Lie Groups Thermodynamics***". This model is of great importance in the context of the geometrization of information sciences and geometric theory of heat. Souriau built a thermodynamics in which the density of Gibbs is completely covariant. He introduced there a new Riemannian metric invariant under the action of the group, from the "moment map" (geometrization of Emmy Noether Theorem [17, 19]) and a cocycle which translates the lack of equivariance of the coadjoint operator on the moment map and linked to the cohomology of Lie algebra. We have discovered that this metric was in fact a generalization of the Koszul-Fisher metric in Information Geometry. More recently, we have observed [20–25] that the invariance of the Entropy defined on the coadjoint orbits made it possible to identify the Entropy with a generalized Casimir function in coadjoint representation. This definition of Entropy only depends on symmetries, and is an alternative to Shannon and von Neumann approaches who had only given an axiomatic definition. To allow access to a larger number, we have made a translation of these three Souriau's seminal papers, which retains all of the author's original notations. We have only translated parts concerning the symplectic model of statistical Mechanics and his model of Lie Groups Thermodynamics.

From Lagrange's in Analytic Mechanics [18], Jean-Marie Souriau, based on previous works of F. Gallisot [15] and A. Blanc-Lapierre [16], conceived his Symplectic model of Statistical Physics [1–5] and other geometric models in Physics [6–12]. This model was studied by his PhD student P. Iglesias [13, 14], and rediscovered by G. de Saxcé [31–34], C.M. Marle [25–29] and F. Barbaresco [20–24, 45–47]. Symplectic model used by Souriau, especially the non-null cohomology case, was deepened by Jean-Louis Koszul in [30]. New fruitful results have been established [20–24] making links with Maximum Entropy Theory studied by D. Dacunha-Castelle and F. Gamboa [36], Information Geometry applied in Physics by R. Balian [37–41] and the Theory of Integrable Systems synthetized by P. Cartier in [35]. Main objective of Souriau's project in Physics was summarized in his sentence "*There is nothing more in physical theories than symmetry groups except the mathematical construction which allows precisely to show that there is nothing more*" [*Il n'y a rien de plus dans les théories physiques que les groupes de symétrie si ce n'est la construction mathématique qui permet précisément de montrer qu'il n'y a rien de plus*]. In [43], Jean-Marie Souriau summarized his contribution to thermodynamics: "*The second principle of thermodynamics is independent: it indicates that the entropy S increases during a dissipation; here we mean entropy in the sense of Clausius-Boltzmann, which is a function of the statistical state ρ. If therefore a state possesses, for a given mean value of the moment, greatest entropy, it will not be subject to dissipation. These states, if they exist, thus represent the terminal state of dissipation. They are indexed by a parameter $\beta$with values in the Lie algebra of the Lorentz-Poincaré group; they generalize the Gibbs equilibrium states, βplaying the role of temperature. The invariance with respect to the group, and the fact that the entropy S is a convex function of β, imposes very strict, universal conditions – i.e.,* *independent of the system considered. For a large class of systems, for example, there exist necessarily a critical temperature beyond which no equilibrium can exist. In the cases where an equilibrium exists, it generally consists of a rigid rotation about the barycenter, etc. These purely theoretical results are evidently confirmed by numerous astronomical examples: the Earth and the starts rotating about themselves; dissipative evolution imposes a solid rotation on the central regions of the galaxies, which itself can lead to a gravitational instability of the "quasar" type; the Clapeyron relations extend to the geometrical-dynamical quantities, etc. One can, if one wishes, interpret* β*as a space-time vector (the temperature vector of Planck), giving to the metric tensor* g *a null Lie derivative. This suggests describing the dissipative processes by a temperature vector* β*which is no longer compelled by this condition; the corresponding Lie derivative of* g*, the "friction tensor," becomes the source of the dissipation. One obtains in this way a phenomenological model of continuous media which presents some interesting properties: the temperature vector and entropy flux are in duality; the positive entropy production is a consequence of Einstein's equations; the Onsager reciprocity relations are generalized; in the case of a fluid in the non-relativistic approximation, the model unifies heat conduction and viscosity (equations of Fourier and Navier)."*

2 Jean-Marie Souriau Biography

The name Souriau means "mouse" in "the Perche". In the "Vendomois", the Souriau from 1490 to 1819 were all "Master Plowmen" or "Master Millers". Jean-Marie Souriau was born June 3, 1922 in Paris in the 6th arrondissement. Jean-Marie Souriau comes from a family of Philosopher all graduated from ENS Paris. His father, Michel Souriau joined the ENS Paris in 1910 and obtained the aggregation in philosophy in 1914 and wrote in 1938 an article on "Introduction to mathematical symbolism" before being mobilized as a battalion commander. His uncle, Etienne Souriau, is a French philosopher, specialized in aesthetics, entered the ENS Paris in 1912, received first in the aggregation in philosophy in 1920. In 1958, Etienne Souriau was elected member of the Academy of moral sciences and policies by a committee in which Charles de Gaulle appears, and will be the director of the thesis of the filmmaker Éric Rohmer. Etienne Souriau published a book on "Structure of the Work of Art (Structure de l'Oeuvre d'Art)". His grandfather, Paul Souriau, is a French philosopher known for his work on the theory of invention and aesthetics, who entered the ENS Paris in 1873 and aggregated in philosophy in 1876. It can be noted that his grand-father, Paul Souriau, composed a thesis titled "Theory of Invention" (théorie de l'invention), published in 1881, and also a Latin thesis titled "De motus perceptione", which aimed to determine the importance of vision for the perception of movements (the initial thesis title was "De visione motus" and was a precursor to his future work on the perception of movement). We can assume that Jean-Marie Souriau read his grandfather's thesis and was influenced by it for his own work. In 1889, Paul Souriau published a book on "The Aesthetics of Movement (L'Esthétique du mouvement)" which describes two levels of aesthetics for movement: mechanical beauty (the adaptation of movement to fulfill its purpose) and meaning of movement (the meaning that the movement communicates to an outside observer). In doing so, Paul Souriau distinguished movement from perception of movement, concepts that would later become the subject of motor cognition and psychophysics. It is interesting to note that Etienne Souriau studied the structures of aestheticism, Paul Souriau developed the aestheticism of the movement and Jean-Marie Souriau founded the structures of the movement. This triptych will remain an important element of French philosophy at the hinge of this 1900 Spirit (Esprit 1900).

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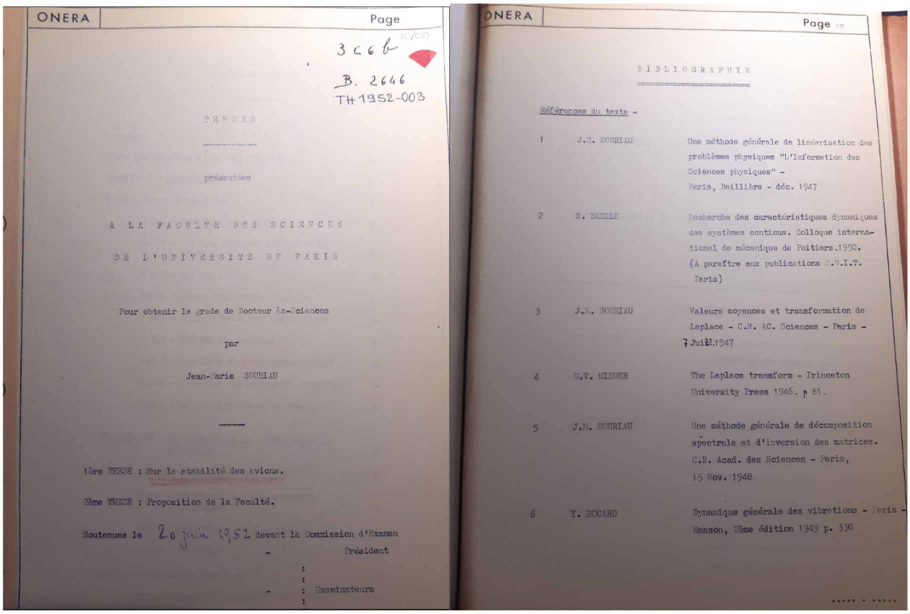
Fig. 1. Jean-Marie Souriau, a student at the Ecole Normale Supérieure in Paris in 1942, with Jacques Dixmier and René Deheuvels among others

Jean-Marie Souriau from 1932 to 1942 did his secondary studies in Nancy, Nîmes, Grenoble and Versailles. Jean-Marie Souriau married Christianne Hoebrechts, who died prematurely in 1985 and with whom he had five children Isabelle, Catherine, Yann, Jérôme and Magali. He entered the ENS Paris in 1942, passing twice in the unoccupied zone in Lyon and a second time in Paris. Also received at the Ecole Polytechnique, he resigned to join the ENS Paris. During his studies at the ENS, he took courses at the Sorbonne from the physicist Yves Rocard and the mathematician Elie Cartan.

He volunteered for “La France Libre” in 1944. On his return in 1946, he passed the mathematics aggregation, and the same year joined a laboratory working on the scanning electron microscope and then entered as a researcher in a “theoretical physics” session at the CNRS (Fig. 1).

He finally opted for a career as an aeronautical engineer at ONERA by becoming head of research groups and defending his thesis in June 1952 on the theme of “aircraft stability “which was supervised by André Lichnerowicz (professor at the College of France) and Joseph Pérès (collaborator of Vito Volterra), which was useful for the design of the “Caravelle” and “Concorde” aircrafts (ONERA obtains royalties from Souriau patents). In this thesis, he refers to the book by Yves Rocard on “General dynamics of vibrations”. On his thesis, he wrote [43] “*I studied the problems of vibrations and stability which arise in aeronautics and in some other techniques; this work allowed me to develop stability criteria which are presented in the form of algorithms which can be easily calculated from theoretical data or from tests; they have since been regularly used in various fields (subsonic and supersonic airplanes, navigation instruments, etc.)*”. I obtained a copy of this thesis through colleagues at ONERA, whose cover I am reproducing below (Fig. 2).



ONERA Page

THESES

présentées

A LA FACULTE DES SCIENCES

DE L'UNIVERSITE DE PARIS

Pour obtenir le grade de Docteur ès-Sciences

par

Jean-Marie SOURIAU

1ère THESE : Sur la stabilité des avions.

2ème THESE : Proposition de la Faculté.

Soutenues le 20 juin 1952 devant la Commission d'Examen

\- Président

\- Examinateurs

ONERA Page

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6	Y. ROCARD	Dynamique générale des vibrations - Paris - Masson, 2ème édition 1949 p. 330

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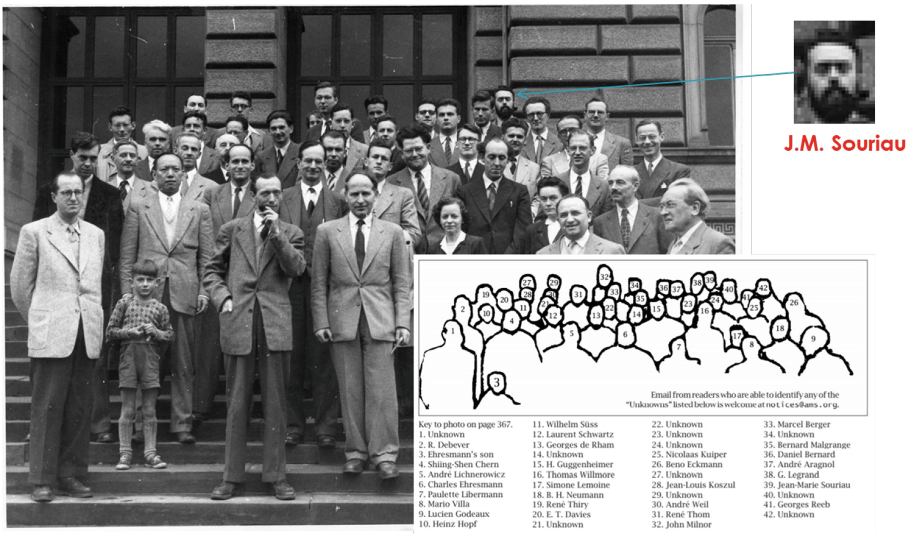
Fig. 2. Cover page of the thesis of Jean-Marie Souriau “On the stability of planes” defended on June 20, 1952, as well as the bibliographic page which refers to the work of Yves Rocard.

During this period, he also invented in 1948 an algorithm named Le Verrier-Souriau algorithm [44] which allows to compute the characteristic polynomial of a matrix and which was used on the first IBM computers in the United States. From 1948 to 1952, he also provided continuing education at the Special School of Aeronautical Works (ESTA, Paris) under the general title “New Methods of Mathematical Physics”. From 1951 to

1952, he created and ran the Mechanics course in the third year of the École Normale Supérieure de l'Enseignement Technique (ÉNSET, Paris). From 1952, he also had a university education in the following disciplines: Mathematics, Mechanics, Relativity, Mathematical Methods of Physics and Computer Science.

After his thesis in 1954, he joined the "Institut des Hautes Etudes", rue de Rome in Tunis, and moved with his wife to Carthage. It is during this period that he rereads and deepens the work of Lagrange in Analytical Mechanics and discovers the symplectic structures that he will formalize in his book "Structure of dynamic systems". It is by thinking of his discussions with ONERA engineers that he invents his masterpiece, the "moment map" (application moment). We can read in the interview by Patrick Iglesias [14] "*It was with the memory of discussions with engineers who asked themselves the following question: what is essential in mechanics. I remember very well an engineer who asked me: is mechanics simply the principle of conservation of energy? This is fine for a one-parameter system, but once there are two, it is not enough. I had learned of course the Lagrange equations and all the analytical principles of mechanics, but it was all a cookbook; we did not see any real principles*". He remained in Tunis from 1952 to 1958, as Lecturer, then as Full Professor at the Institut des Hautes Études. In 1953, he participated in Strasbourg in the conference on Differential Geometry (Fig. 3).



Email from readers who are able to identify any of the "Unknowns" listed below is welcome at notices@ams.org.

Key to photo on page 367.
1. Unknown
2. R. Debever
3. Ehresmann's son
4. Shiing-Shen Chern
5. André Lichnerowicz
6. Charles Ehresmann
7. Paulette Libermann
8. Mario Villa
9. Lucien Godeaux
10. Heinz Hopf
11. Wilhelm Süss
12. Laurent Schwartz
13. Georges de Rham
14. Unknown
15. H. Guggenheimer
16. Thomas Willmore
17. Simone Lemoine
18. B. H. Neumann
19. René Thiry
20. E. T. Davies
21. Unknown
22. Unknown
23. Unknown
24. Unknown
25. Nicolaas Kuiper
26. Beno Eckmann
27. Unknown
28. Jean-Louis Koszul
29. Unknown
30. André Weil
31. René Thom
32. John Milnor
33. Marcel Berger
34. Unknown
35. Bernard Malgrange
36. Daniel Bernard
37. André Aragnol
38. G. Legrand
39. Jean-Marie Souriau
40. Unknown
41. Georges Reeb
42. Unknown

Fig. 3. Jean-Marie Souriau at the Conference on "Differential Geometry" in Strasbourg in 1953. In the same picture, Jean-Louis Koszul, André Weil, Shiing-Shen Chern, Georges de Rham, Charles Ehresmann, Lucien Godeaux, Heinz Hopf, André Lichnerowicz (the director of the thesis of Jean-Marie Souriau), Bernard Malgrange, John Milnor, Georges Reeb, Laurent Schwartz, René Thom, Paulette Libermann.

In 1958, he became Professor at the University of Aix-Marseille. He remained in Marseille throughout his career, and from 1978 to 1985 became Director of the Center for Theoretical Physics of Marseille (CNRS laboratory) in charge of the teams in Theoretical

Mechanics, Geometry and Quantification, Astronomy and Cosmology. He was also professor of Mathematics at the University of Provence (Aix-Marseille I) and ended up as an exceptional Professor with second echelon. He was also a member of the "Société Mathématique de France" and of the French Society of Specialists in Astronomy. For five years, he also taught the course of the third interuniversity cycle of Pure Mathematics in Marseille and the third interuniversity cycle of Theoretical Physics in Marseille-Nice. He was a member of the Editorial Board of the Journal of Geometry and Physics in Florence. He organized two International Colloquiums of the CNRS in 1968 and 1981 and Days of the "Société Mathématique de France". Honored by the Academic Palms and of the National Order of Merit, he obtained the Prize on the subject "Vibrations "put up for competition by the Association for Aeronautical Research in 1952, the Prize on the subject "Cosmology" put out for competition by the Foundation Louis Jacot in 1978, the Grand Prix Jaffe of the French Academy of Sciences in 1981 and the Great Scientist prize of the City of Paris in 1986. Jean-Marie Souriau died in 2012 in its 90th year.

3 1st Souriau Paper: "Statistical Mechanics, Lie Group and Cosmology - 1st Part: Symplectic Model of Statistical Mechanics"

The classical notion of Gibbs' canonical ensemble is extended to the case of a symplectic manifold on which a Lie group has a symplectic action ("dynamic group"). The rigorous definition given here makes it possible to extend a certain number of classical thermodynamic properties (temperature is here an element of the Lie algebra of the group, heat an element of its dual), notably inequalities of convexity. In the case of non-commutative groups, particular properties appear: the symmetry is spontaneously broken, certain relations of cohomological type are verified in the Lie algebra of the group. Various applications are considered (rotating bodies, covariant or relativistic statistical Mechanics). [These results specify and complement a study published in an earlier work [8], which will be designated by the initials SSD].

3.1 Distribution Functions

The initial concept of classical statistical mechanics is the distribution function; it is a real function ρ, defined on the phase space of a dynamic system, such as the integral

$$\int_D \rho dp_1 \ldots dp_n dq_1 \ldots dq_n \tag{1}$$

is equal to the probability that the point $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ representing the state of the system at time t is contained in a domain D. This obviously requires that ρ is ≥ 0 and that the integral (1), extended to the entire phase space, be equal to 1.

It is further assumed that ρ verifies Liouville equation

$$\frac{\partial \rho}{\partial t} + \sum_j \frac{\partial \rho}{\partial q_j}\frac{\partial h}{\partial p_j} - \frac{\partial \rho}{\partial p_j}\frac{\partial h}{\partial q_j} = 0 \tag{2}$$

h being hamiltonian function.